Preprint typeset in JHEP style - HYPER VERSION

Physics 511: Problem Set 2

Gregory W. Moore

ABSTRACT: Due October 24, 2014.

-TOC- Contents

More Instructions:

1. Late problem sets are not accepted. Get started soon.
2. I strongly prefer getting a pdf copy. It would be nice if you could tex your answers. If you cannot do this write very clearly and hand it in by beginning of class Oct. 24.
3. In general (but not always) each problem is graded so that each section counts for 10 points. You get an extra 30 points for sending me a texed pdf version with filename: Yourlastname-Yourfirstname-PS2.pdf

   (Because there will be figures it is probably easiest to draw them and scan to pdf. The NHETC secretary Diane Soyak can help you do that. If you make computer-drawn figures - great!)
4. There are many many more exercises in the notes. I strongly encourage you to do as many as you can! (But I will not grade them.)

1. Variation of the Gauss linking number

Verify explicitly that a small variation $\vec{x}_1(s) \to \vec{x}_1(s) + \delta\vec{x}_1(s)$ in the trajectory of a loop C_1 does not change the integral formula for the Gauss linking number $L(C_1, C_2)$.

2. Spin in the field of two dyons

Prove the formula in the notes for the spin in the electromagnetic field created by a pair of dyons.

3. Point particle action as $0+1$-dimensional "quantum gravity"

a.) Show that the action has a gauge invariance under reparametrizations $s \to f(s)$ where $f(s)$ is a monotonically increasing differentiable function of s.

b.) Introduce a metric on the domain $\mathcal{D}$ $g_{ss}(s)(ds)^2$. Since $g_{ss} > 0$ we can define a positive squareroot so the length element is $e(s)ds$. (This would be called an *einbein* in general relativity.) Suppose $x : \mathcal{D} \to \mathcal{S}$ is a map into a Riemannian manifold with metric $G_{\mu\nu}(x)$. Consider the action:

$$S = \frac{1}{2} m \int_{\mathcal{D}} \left(-\frac{dx^\mu}{ds} \frac{dx^\nu}{ds} G_{\mu\nu}(x(s)) e^{-1}(s) + c^2 e(s) \right) ds \tag{3.1}$$

$G_{\mu\nu}$ has signature mostly plus. Show that the einbein can be eliminated by algebraic equations of motion to produce the action for a particle moving in a spacetime $\mathcal{S}$.

c.) Verify that the action is invariant under diffeomorphisms $s \to f(s)$ provided $e(s)ds$ transforms line a line element.

d.) Show that with a suitable rescaling of e one can take an $m \to 0$ limit preserving a good kinetic term.

4. Gluing

Consider a torus obtained by gluing together opposite sides of a square. Define closed oriented curves A and B as in the notes. They are usually called *A-cycles* and *B-cycles*.

a.)*S-transformation.* Centering the square on the origin, consider the transformation of the torus to itself by making a 90 degree counterclockwise rotation before gluing. Describe what this does to the standard A and B cycles. Describe the action of S^2.

b.) *Dehn twist.* Let $\gamma \subset C$ be a closed curve in a surface. A Dehn twist around γ is a diffeomorphism of C to itself obtained by isolating a small annular neighborhood of γ and twisting one circle boundary of the cylinder by 2π, holding the other circle boundary fixed. Draw pictures illustrating the action of a Dehn twist around the A-cycle on both the A-cycle and the B-cycle.

c.) Consider the operation of first doing a Dehn twist around the A-cycle and then doing an S-transformation. Call this ST. What happens if we perform ST three times?

d.) Consider two solid tori, considered as a product of a disk and a circle $D^2 \times S^1$. What three-dimensional manifold is obtained by gluing together the solid tori along their torus boundary where one first makes an S-transformation on one of the two tori?

5. Abelianization of the fundamental groups of closed surfaces

The *abelianization* of a group G is the quotient of the commutator subgroup N generated by all group commutators $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$, for all pairs of elements of G.

a.) Show N is a normal subgroup and that G/N is an abelian group.

b.) Compute the abelianization of the fundamental groups of closed surfaces. This is a finitely generated abelian group. Give its rank and torsion subgroup.