Physics 511: Problem Set 1

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ABSTRACT: Due Sept. 24, 2014.

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Instructions

1. Late problem sets are not accepted. Get started soon.
2. I strongly prefer getting a pdf copy. It would be nice if you could tex your answers. If you cannot do this write very clearly and put in my mailbox by noon.
3. There are many many more exercises in the notes. I strongly encourage you to do as many as you can! (But I will not grade them.)

1. Classical matrix groups

a.) The orthogonal groups are defined by:

$$\begin{aligned} O(n,\kappa) &:= \{A \in M_n(\kappa) : AA^{tr} = 1\} \\ SO(n,\kappa) &:= \{A \in O(n,\kappa) : \det A = 1\} \end{aligned} \tag{1.1}$$

eq:orthgroup

where $M_n(\kappa)$ is the set of $n \times n$ matrices over the field κ. (You can take $\kappa = \mathbb{R}, \mathbb{C}$.) Show that they are groups.

b.) Similarly, the symplectic groups are defined by:

$$Sp(2n,\kappa) := \{A \in M_n(\kappa) | A^{tr} J A = J\} \tag{1.2}$$

eq:SymplecticGrou

where

$$J = \begin{pmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{pmatrix} \in M_{2n}(\mathbb{R}) \tag{1.3}$$

eq:symplecticform

Show that $Sp(2n,\kappa)$ is a group.

c.) Show that if A is a symplectic matrix then A^{tr} is a symplectic matrix

d.) Decompose a symplectic matrix into $n \times n$ blocks:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.4}$$

where $a, b, c, d \in M_n(\kappa)$. Write necessary and sufficient conditions for A to be a symplectic matrix.

2. Symplectic groups and canonical transformations

Let q^i, p_i $i = 1, \dots n$ be coordinates and momenta for a classical mechanical system.

The **Poisson bracket** of two functions $f(q^1, \dots q^n, p_1, \dots p_n)$, $g(q^1, \dots q^n, p_1, \dots p_n)$ is defined to be

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) \tag{2.1}$$

a.) Show that

$$\{q^i, q^j\} = \{p_i, p_j\} = 0 \qquad \{q^i, p_j\} = \delta^i{}_j \tag{2.2}$$

Suppose we define new coordinates and momenta Q^i, P_i to be linear combinations of the old:

$$\begin{pmatrix} Q^1 \\ \vdots \\ Q^n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2n} \\ \vdots & \ddots & \vdots \\ a_{2n,1} & \cdots & a_{2n,2n} \end{pmatrix} \cdot \begin{pmatrix} q^1 \\ \vdots \\ q^n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \tag{2.3}$$

where $A = (a_{ij})$ is a constant $2n \times 2n$ matrix.

b.) Show that

$$\{Q^i, Q^j\} = \{P_i, P_j\} = 0 \qquad \{Q^i, P_j\} = \delta^i_j \tag{2.4}$$

if and only if A is a symplectic matrix.

3. Decomposing the reverse shuffle

Consider the permutation which takes $1, 2, \dots, n$ to $n, n-1, \dots, 1$.

a.) Write the cycle decomposition.

b.) Write a decomposition of this permutation in terms of the *elementary generators* $\sigma_i = (i, i+1)$. (See the exercise in Section 4.3.)

4. Subgroups of A_4

Write down all the subgroups of A_4.

5. Words in a finite Heisenberg group

As in the notes define P, Q be $N \times N$ "clock" and "shift" matrices:

$$P_{i,j} = \delta_{j=i+1 \text{mod} N} \tag{5.1}$$ eq:ShiftMatrix

$$Q_{i,j} = \delta_{i,j}\omega^j \tag{5.2}$$ eq:ClockMatrix

a.) Show that the word

$$P^{n_1}Q^{m_1}P^{n_2}Q^{m_2}\cdots P^{n_k}Q^{m_k} \tag{5.3}$$

where $n_i, m_i \in \mathbb{Z}$ can be written as $\xi P^x Q^y$ where $x, y \in \mathbb{Z}$ and ξ is an N^{th} root of unity. Express x, y, ξ in terms of n_i, m_i.

b.) Let Heis_N the the group generated by P, Q. Show that there is an exact sequence

$$1 \to \mathbb{Z}_N \to \text{Heis}_N \to \mathbb{Z}_N \times \mathbb{Z}_N \to 1 \tag{5.4}$$

6. p-groups

a.) Show that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

b.) Show more generally that if p is prime $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n-m}} \oplus \mathbb{Z}_{p^m}$ are not isomorphic if $0 < m < n$.

c.) How many nonisomorphic abelian groups have order p^n?