Physics 511: Math Methods for Physicists, 2014

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ABSTRACT: General information and administrative matters September 10, 2014

Contents

1. Prerequisites

This course is primarily intended for beginning graduate students. It covers some basic mathematical subjects used in physics. I will try to keep prerequisites to a minimum.

2. Grand Plan

This course does *not* follow the list of topics in the blue book. If you want a standard dry math methods course please read the standard textbooks:

- Mathematical Methods for Physicists, George Arfken and Hans Weber

- Mathematical Methods of Physics, Mathews & Walker.

- Physical Mathematics, Cahill

I prefer to tie together various topics showing the unity of mathematics and to stress topics which have proven to be more relevant in research in particle physics and string theory, and some aspects of condensed matter theory, during recent decades.

The course is roughly divided into three broad sections:

1. Group theory, linear analysis, and group representation theory.

2. Complex analysis, some special functions, asymptotic analysis, saddle points and integrals.

3. Differential equations.

At the end we will tie together all three subjects by discussing the beautiful Riemann-Hilbert problem, the complete solution of ODE's with three regular singular points by the Gauss hypergeometric functions, and more broadly the relation between representations of surface groups and the monodromy of ordinary differential equations in one complex variable.

3. Tentative Detailed Plan

Remarks:

1. If there is any topic you would particularly like to hear about please feel free to make a request and, if I know anything about it, I'll try to work it in.

2. I also might drop some topics if I feel we are getting too far behind. This is just a sketch of topics and I might well add/delete topics.

3. I will proceed at a pace so that I can explain things clearly and I will not rush to cover all the offered material. We might well not get to the end of the list of topics.

3.1 Abstract Group Theory

1. Basic definitions

2. Homomorphism and isomorphism

3. The symmetric group

4. Cosets and conjugacy: Lagrange theorem; Conjugacy classes in S_n

5. Kernel, image, exact sequence, and group extension

6. Central extensions. Heisenberg groups. (Maybe.)

7. Comments on classification. Kronecker's theorem.

3.2 Linear Algebra User's Manual

1. Modules, vector spaces, algebras. The quaternion algebra.

2. Constructions on vector spaces: Direct sum, quotient, tensor product, dual space, tensor algebras.

3. Real, complex, and quaternion vector spaces

4. Eigenvalues, characteristic polynomial, and Jordan canonical form

5. Inner product spaces, Hilbert space, projection operators, and a little bit about the theory of operators (spectral theorem...)

6. The Dirac-von Neumann axioms of quantum mechanics

7. Canonical forms of antisymmetric, symmetric, and orthogonal matrices

8. Gram-Schmidt and orthogonal polynomials

9. Quadratic forms and lattices (maybe)

3.3 A little bit about topology

1. Some motivating examples: Gauss linking number; the angular momentum of a two-dyon system; Dirac quantization.

2. Homotopy and homotopy groups. Homology groups.

3. Surface groups and the braid group

4. Overview of some uses of homotopy in physics

3.4 A little bit about manifolds

1. Definitions

2. Implicit function theorem and local picture of a sub-manifold

3. Codimension and linking

4. Differential forms and Maxwell's equations (maybe)

3.5 Group actions, orbits and symmetries of objects

1. Transformation groups. Left and right actions.

2. Orbits. Stabilizer-orbit theorem. Spaces of orbits.

3. Symmetries of regular polygons and the platonic solids

4. Symmetries of lattices

5. Theory of symmetric functions

3.6 Implementation of symmetry in quantum mechanics: Wigner's theorem

3.7 Introduction to representation theory

1. Reducible and irreducible (ϕ-) representations. Schur's lemma and Dyson's 3-fold way.

2. Decomposition of the regular representation of a compact group: The Peter-Weyl theorem

3. Fourier analysis and representation theory. Poisson summation formula. Bloch theory

4. Characters and projection operators. Decomposition into irreps

3.8 Survey of matrix groups

3.9 Lie algebras

1. Baker, Campbell, Hausdorff formula

2. Lie algebras in general. Jacobi formula

3. Lie algebras of the classical matrix groups

4. Super-Lie algebras

5. Supersymmetric quantum mechanics

3.10 The wonderful $2{\times}2$ matrix groups, rotations and boosts in low dimensions, and spinor formalism

3.11 Induced representations and special functions

1. Induced representations

2. Wigner functions, Legendre functions and $SU(2)$. The Hopf fibration. Coherent states.

3.12 Holomorphic functions and conformal maps

3.13 Holomorphic functions and geometry

3.14 Cauchy's theorem and its uses

1. Cauchy's theorem and Cauchy integral formula

2. Zeroes of analytic functions and analytic continuation

3. Schwarz-Christoffel formula

3.15 Winding numbers, Laurent expansions, and classification of singularities

3.16 Residue theorem and the evaluation of sums and integrals

1. Evaluation of some standard types of integrals using the Cauchy theorem

2. Analytic continuation of the integral formula for the Bessel function

3. Evaluation of sums: Sommerfeld-Watson transformation. Euler-MacLaurin series.

4. Branch cuts and asymptotics

5. Laplace transforms

3.17 Infinite products

Weierstrass factorization theorem.

3.18 Elliptic functions and modular forms

Eisenstein series and the ring of modular forms
    Dedekind eta function

3.19 Γ, β, and ζ

1. Gamma function, its analytic continuation, and its functional equations.

2. Riemann zeta function. Polylogarithms. Bose-Einstein and Fermi-Dirac distributions.

3. Functional equation and analytic continuation of the Riemann zeta function. The Casimir effect.

4. The Riemann hypothesis

5. Summing up periodic arrays of potentials and point sources

3.20 Principal value and Hilbert transform

1. Kramers-Kronig relations.

2. Eigenvalue distributions. Derivation of the Wigner semicircle law.

3. Analyticity and causality.

4. The analytic S-matrix.

3.21 Asymptotic Expansions

1. General properties of asymptotic expansions

2. Phragmen-Lindelof theorem and Borel summability

3. The Stirling formula (to all orders).

3.22 Saddle point estimates of one-dimensional integrals

1. Stationary phase and steepest descent

2. Derivation of the Hardy-Ramanujan formula

3. The Airy function and Stokes' phenomenon

4. Expansion to all orders: Feynman diagrams in zero dimensions

3.23 Saddle point methods of multi-dimensional integrals

Moduli. Lefshetz thimbles. Matrix models. Path integrals.

3.24 Solutions of differential equations: Existence, Uniqueness, and Boundary Conditions

3.25 Qualitative behavior of solutions to ODEs

Gradient flows. Symplectic flows. Behavior near fixed points.

3.26 Linear ODE's and Green's functions

3.27 Analytic points and singular points of ODE's

1. Regular singular points. Regular singular point at infinity.

2. Method of Frobenius and Fuchsian differential equations

3. Monodromy. Flat gauge fields

4. Irregular singular points: Stokes' phenomenon and Stokes matrices

3.28 The Riemann differential equation and the Gauss hypergeometric functions

1. Fuchsian differential equations. Statement of the connection problem

2. Action of conformal transformations: Reduction of the RDE with three regular singular points to the Gauss hypergeometric equation

3. Melin-Barnes transformation and the connection problem for the Gauss hypergeometric function

4. The Riemann-Hilbert problem

4. Administrative

1. Notes for all the lectures will be handed out or, possibly, posted on the web page for the course:

   http://www.physics.rutgers.edu/~gmoore/511Fall2014/Physics511Fall2014.html

2. The grade for those taking the course for credit will be based on homework problem sets. There might or might not be a final exam.

3. As a courtesy to others, PLEASE DO NOT EAT OR DRINK DURING CLASS. You may bring a water bottle.

4. Please note: Attendance at the lectures IS NOT OPTIONAL. If you cut class it will affect your grade.

5. We need a time for making up the classes supposed to be held when I'm out of town. Options: Extend class time: (Would need to add 27 minutes to each of 21 classes). Double up on some Fridays or Wednesdays. Add some lectures on Mondays. Add lectures on December 11 and 12.

6. Even if you have sent me email before, please send an email to gmoore@physics.rutgers.edu with Subject: Physics 511 Student. Please tell me your name, how you like to be addressed, and your background and interests. Please be brief! Two or three lines at most.