

①

$$\vec{B} = \text{const}$$

$$\vec{A} = -\frac{1}{2} [\vec{r} \times \vec{B}]$$

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{2} \left[\vec{B} \cdot (\vec{\nabla} \times \vec{r}) - \vec{r} \cdot (\vec{\nabla} \times \vec{B}) \right]$$

$$\vec{\nabla} \times \vec{r} = 0 \Rightarrow \vec{\nabla} \cdot \vec{A} = 0.$$

$\vec{\nabla} \times \vec{B} = 0$, since $\vec{B} = \text{const}$

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2} \left((\vec{B} \cdot \vec{\nabla}) \cdot \vec{r} - (\vec{r} \cdot \vec{\nabla}) \cdot \vec{B} + \vec{r} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{r}) \right) = \dots$$

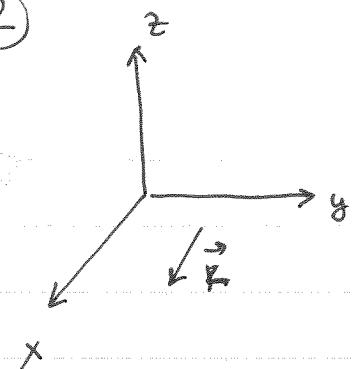
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since $B = \text{const.}$

$$\vec{B} (\vec{\nabla} \cdot \vec{r}) = \vec{B} \cdot \left(\frac{\partial}{\partial x} \cdot x + \frac{\partial}{\partial y} \cdot y + \frac{\partial}{\partial z} \cdot z \right) = 3 \vec{B}$$

$$\begin{aligned}
 (\vec{B} \vec{\nabla}) \cdot \vec{r} &= \hat{i} \cdot \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) \cdot x + \\
 &+ \hat{j} \cdot \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) y + \\
 &+ \hat{k} \cdot \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) z = \hat{i} B_x + \hat{j} B_y + \hat{k} B_z = \vec{B}
 \end{aligned}$$

$$\dots = -\frac{1}{2} (\vec{B} - 3\vec{B}) = \vec{B}$$

(2)

HW12

$$\vec{K} = K \hat{z}$$

$$\vec{A} = ?$$

The magnetic field \vec{B} , as we showed in class using Ampere law is

$$\vec{B} = \begin{cases} -\frac{1}{2} \mu_0 K \hat{y}, & z > 0 \\ +\frac{1}{2} \mu_0 K \hat{y}, & z < 0. \end{cases}$$

There are 2 ways to get \vec{A} from this, and they are extremely informative to compare.

A) symmetry: $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, so since \vec{J} is always parallel to \hat{x} , \vec{A} should also be parallel to x :

$$\vec{A} = A(x, y, z) \cdot \hat{x}$$

Now, $A(x, y, z)$ should not depend on x or y \Rightarrow the problem is symmetric in x - y plane, so

$$\vec{A} = A(z) \cdot \hat{x}$$

So:

$$\nabla \times (\vec{A}(z) \cdot \hat{x}) = \vec{B} \Rightarrow \text{for } z > 0: -\frac{1}{2} \mu_0 K \hat{y} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(z) & 0 & 0 \end{vmatrix} = \hat{y} \frac{\partial A}{\partial z}$$

$$\frac{dA}{dz} = -\frac{1}{2} \mu_0 K \Rightarrow \vec{A} = -\frac{1}{2} \mu_0 K z \cdot \hat{x}, \quad z > 0 \quad \left. \right\}$$

(dropped a constant since A defined up to a curve) $+ \frac{1}{2} \mu_0 K z \hat{x}, \quad z < 0 \quad \left. \right\}$

(2)
B)

From the previous problem, if B is uniform,

$$\vec{A} = -\frac{1}{2} [\vec{r} \times \vec{B}]$$

$$\text{For } z > 0: \quad \vec{B} = -\frac{1}{2} \mu_0 K \hat{y}$$

$$\vec{A} = \frac{1}{4} \mu_0 K [\vec{r} \times \hat{y}] = \frac{1}{4} \mu_0 K \left(x[\hat{x} \times \hat{y}] + y[\hat{y} \times \hat{y}] + z[\hat{z} \times \hat{y}] \right) =$$

$$\boxed{\vec{A} = \frac{1}{4} \mu_0 K \cdot x \cdot \hat{z} - \frac{1}{4} \mu_0 K \cdot z \cdot \hat{x}}$$

This is different from the A) solution, $\vec{A} = -\frac{1}{2} \mu_0 K z \cdot \hat{x}$!!

Plus, it has \hat{z} component!!

What is going on?

Remember, we can always add to A a field A_0 , as long as $\nabla \times \vec{A}_0 = 0$. Let's compute the curl of the difference between solutions A) and B):

$$\vec{A}_0 = \left(\frac{1}{4} \mu_0 K x \hat{z} - \frac{1}{4} \mu_0 K z \hat{x} \right) - \left(-\frac{1}{2} \mu_0 K z \hat{x} \right) =$$

$$\vec{A}_0 = \frac{1}{4} \mu_0 K \left(x \hat{z} + z \hat{x} \right)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & 0 & x \end{vmatrix} = \hat{x} \frac{\partial}{\partial y} x + \hat{y} \left(\frac{\partial}{\partial z} z - \frac{\partial}{\partial x} x \right) + \hat{z} \left(\frac{\partial}{\partial y} z \right) = \\ = \hat{y} (1-1) = 0 !$$

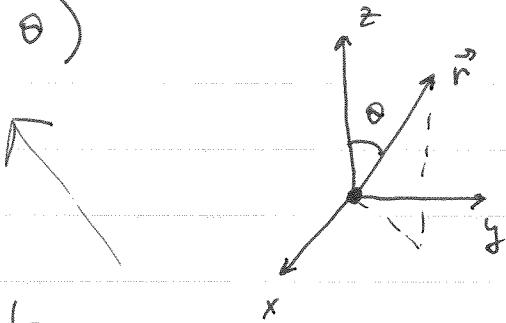
So both A) and B) solutions are correct

Note, that $\nabla \cdot \vec{A} = 0$ for both solutions !!.

(3)

$$\vec{B}_{\text{dir}}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$

$$\vec{m} = m \cdot \hat{z}$$



need to show that one can write

$$\vec{B}_{\text{dir}}(\vec{r}) = \frac{\mu_0}{4\pi r^3} (3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m}) = \dots$$

$$\vec{m} \cdot \hat{r} = m \cdot \cos\theta$$

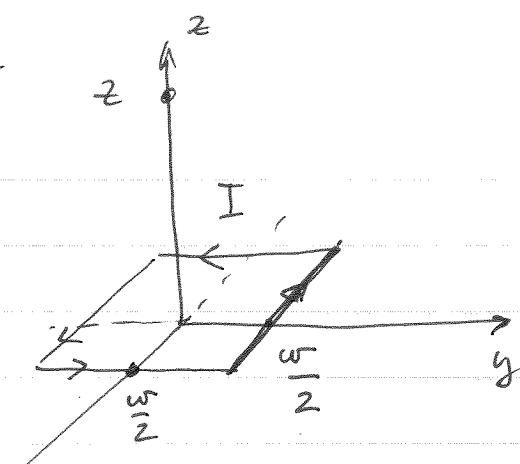
$$\vec{m} = m \hat{z} = m (\cos\theta \hat{r} - \sin\theta \hat{\theta})$$

$$\dots = \frac{\mu_0}{4\pi r^3} (3m \cdot \cos\theta \cdot \hat{r} - m \cos\theta \hat{r} + m \cdot \sin\theta \hat{\theta}) =$$

$$= \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$

(4)

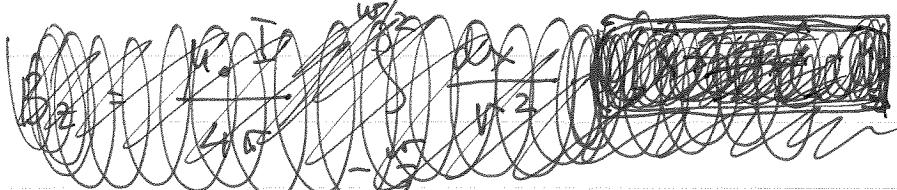
HW12



$$\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{[dl \times \hat{\mu}]}{\mu^2}$$

Only z component is non-zero

Contribution of each side to z -component have the same



for side 1:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{[(dx \cdot \hat{x}) \times (\mu_x \hat{x} + \mu_y \hat{y} + \mu_z \hat{z})]}{\mu^3}$$

z -component is

$$B_z = \frac{\mu_0 I}{4\pi} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx \cdot \mu_y}{\mu^3}, \text{ where } \left\{ \begin{array}{l} \mu = \sqrt{z^2 + \frac{w^2}{4} + x^2} \\ \mu_y = \frac{w}{2} \end{array} \right.$$

Total field is 4 times that, so

$$B_z = \frac{\mu_0 I}{4\pi} \frac{w}{2} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx}{\left(\sqrt{z^2 + \frac{w^2}{4} + x^2} \right)^3} = \frac{\mu_0 I w}{2\pi} \cdot \frac{1}{z^2 + \frac{w^2}{4}} \int \frac{du}{(1+u^2)^{3/2}} =$$

$$u = \frac{x}{\sqrt{z^2 + \frac{w^2}{4}}}$$

$$\int \frac{du}{(1+u^2)^{3/2}} = \frac{u}{\sqrt{1+u^2}} + C$$

(4)

-2-

HW12

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx}{\sqrt{z^2 + \frac{w^2}{4} + x^2}} = \frac{1}{z^2 + \frac{w^2}{4}} \cdot 2 \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx}{\sqrt{z^2 + \frac{w^2}{4}}} \cdot \frac{1}{\sqrt{1 + \frac{w^2/4}{z^2 + w^2/4}}} =$$

$$= \frac{1}{z^2 + \frac{w^2}{4}} \cdot \frac{w}{\sqrt{z^2 + \frac{w^2}{2}}}$$

$$B = \frac{\mu_0 I}{2\pi} \cdot \frac{w^2}{(z^2 + \frac{w^2}{4}) \sqrt{z^2 + \frac{w^2}{2}}}$$

for $z \gg w$,

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{w^2}{z^3} \hat{z}$$

Field of a dipole $\vec{B} = \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$,

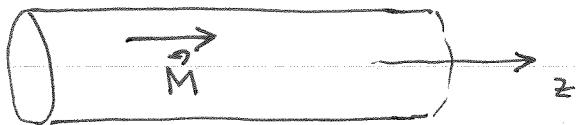
$$\text{for } \theta = 0, \vec{B} = \frac{\mu_0 m}{4\pi z^3} \cdot 2 \hat{z} = \frac{\mu_0 m}{2\pi z^3} \hat{z},$$

so for the square loop

$$m = Iw^2, \text{ as expected.}$$

(5)

Kw12



Bound currents: $\vec{J}_b = 0$ ($\vec{M} = \text{const}$)

$K_b = M$, flowing around the cylinder,

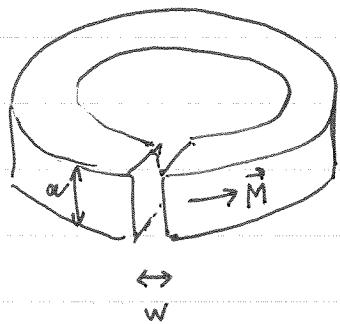
i.e. if $\vec{M} = M \cdot \hat{z}$ } in cylindrical coordinates.
 $\vec{K}_b = M \cdot \hat{\phi}$

Equivalent to a solenoid!

$B = 0$ outside.

$B = \mu_0 K_b = \mu_0 M$ inside $\vec{B} = \mu_0 \vec{M}$ (parallel to \vec{M})

(6)

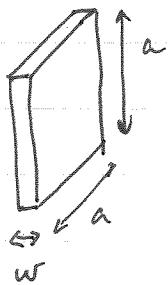
HW12

Let's consider the field as a superposition of a field magnetized toroid (no gap) and a small square of width w with magnetization $-\vec{M}$.

Field of a full toroid:

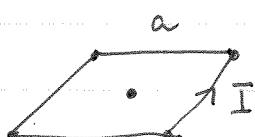
$$K_b = M, \quad B = \mu_0 M$$

Field of a slice of the magnetized square bar



For $w \ll a$ we can view this as a field of a square wire frame that carries a current

$$I = -K_b \cdot w = -Mw$$



field at the center of the frame is, according to problem 4):

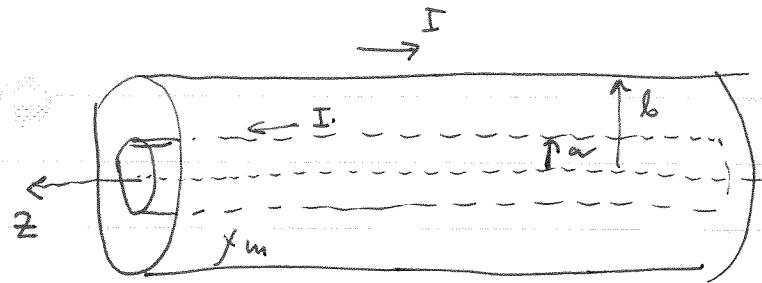
$$B = \frac{\mu_0 I}{2\pi} \frac{a^2}{a^2/4 \cdot \sqrt{a^2/2}} = \frac{2\sqrt{2}}{\pi} \frac{\mu_0 I}{a} = -\frac{2\sqrt{2}}{\pi} \frac{\mu_0 w M}{a}$$

Total field in the center of the gap is then:

$$B = \mu_0 M \left(1 - \frac{2\sqrt{2} w}{\pi a} \right)$$

(7)

HW12



Cylindrical symmetry, can use Ampere law for H :

a)

$$\oint \vec{H} d\vec{l} = I_f^{\text{enc}} \quad \vec{H} = H \hat{z} \rightarrow \text{Field lines are circles.}$$

I in the $+z$ direction for $r=a$

For $a < r < b$:

$$2\pi r \cdot H = I \Rightarrow H = \frac{I}{2\pi r}$$

in linear media

$$B = \mu_0 H (1 + \chi_m) = \boxed{\frac{\mu_0 I (1 + \chi_m)}{2\pi r}}$$

b) Bound currents:

$$\vec{M} = \chi_m \vec{H} = \frac{I \chi_m}{2\pi r} \cdot \hat{z}$$

$$\vec{j}_b = [\vec{\nabla} \times \vec{M}] = \left[\frac{1}{r} \frac{\partial M_z}{\partial \phi} - \frac{\partial M_r}{\partial z} \right] \hat{r} + \left[\frac{\partial M_r}{\partial z} - \frac{\partial M_z}{\partial r} \right] \hat{\phi} +$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r M_\phi) - \frac{\partial M_r}{\partial \phi} \right] \hat{z} = - \frac{\partial M_\phi}{\partial z} \hat{r} + \frac{1}{r} \frac{\partial}{\partial r} (r M_\phi) \hat{z} =$$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left(\kappa \cdot \frac{\mu_0 I (1 + \chi_m)}{2\pi r} \right) \hat{z} = \boxed{0 = \vec{j}_b}$$

$$\vec{K}_b = [\vec{M} \times \hat{n}] =$$

$$\text{for } r=a \quad \hat{n} = -\hat{r}, \text{ so } \vec{K}_b = -[\hat{\phi}, \hat{r}] \cdot \frac{I \chi_m}{2\pi a} = \frac{I \chi_m}{2\pi a} \hat{z}$$

Total current along $r=a$ is then

$$I_{tot} = I + 2\pi a \cdot K_b = I (1 + \chi_m) \Rightarrow B = \frac{\mu_0 I_{tot}}{2\pi r} = \boxed{\frac{\mu_0 I (1 + \chi_m)}{2\pi r}}$$