(1) Symmetry: $\vec{E} = E(r)\cdot\hat{r}$

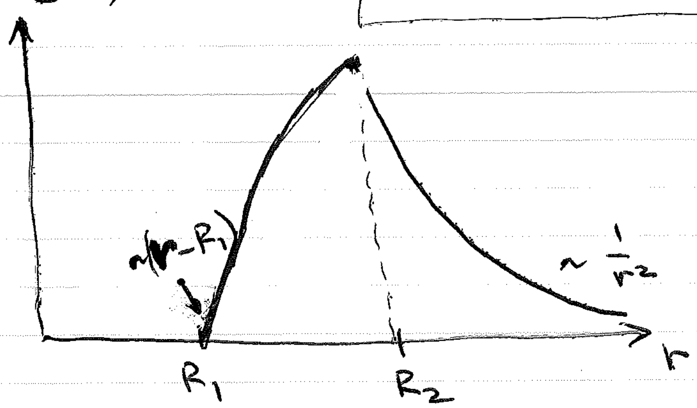
$r < R_1$: $\boxed{E(r) = 0}$

$r > R_2$: $4\pi r^2 E(r) = \dfrac{Q_{enc}}{\varepsilon_0} = \dfrac{\rho}{\varepsilon_0}\dfrac{4}{3}\pi\left(R_2^3 - R_1^3\right)$

$$\boxed{E(r) = \frac{\rho}{3\varepsilon_0}\frac{R_2^3 - R_1^3}{r^2}}$$

$R_1 < r < R_2$: $4\pi r^2 E(r) = \dfrac{\rho}{\varepsilon_0}\dfrac{4}{3}\pi\left(r^3 - R_1^3\right)$

$$\boxed{E(r) = \frac{\rho}{3\varepsilon_0}\left(r - \frac{R_1^3}{r^2}\right) = \frac{\rho}{3\varepsilon_0}(r - R_1)\cdot\frac{r^2 + rR_1 + R_1^2}{r^2}}$$



for $r = R_1 + \varepsilon$

$E(r) \sim \varepsilon$

V:
$$-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} - \cancel{\frac{1}{r}\frac{\partial V}{\partial \theta}}\hat{\theta} - \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}}\hat{\varphi} = E(r)\cdot\hat{r}$$

$r > R_2$: $\dfrac{dV}{dr} = -\dfrac{\rho}{3\varepsilon_0}\dfrac{R_2^3 - R_1^3}{r^2} \Rightarrow \boxed{V = \dfrac{\rho}{3\varepsilon_0}\dfrac{R_2^3 - R_1^3}{r}}$

$R_1 < r < R_2$: $\dfrac{dV}{dr} = -\dfrac{\rho}{3\varepsilon_0}\left(r - \dfrac{R_1^3}{r^2}\right) \Rightarrow V = -\dfrac{\rho}{3\varepsilon_0}\left(\dfrac{1}{2}r^2 + \dfrac{R_1^3}{r}\right) + \text{const.}$

$$V(R_2) = \frac{\rho}{3\varepsilon_0}\left(\frac{R_2^3 - R_1^3}{R_2}\right) = -\frac{\rho}{3\varepsilon_0}\left(\frac{1}{2}R_2^2 + \frac{R_1^3}{R_2}\right) + C$$

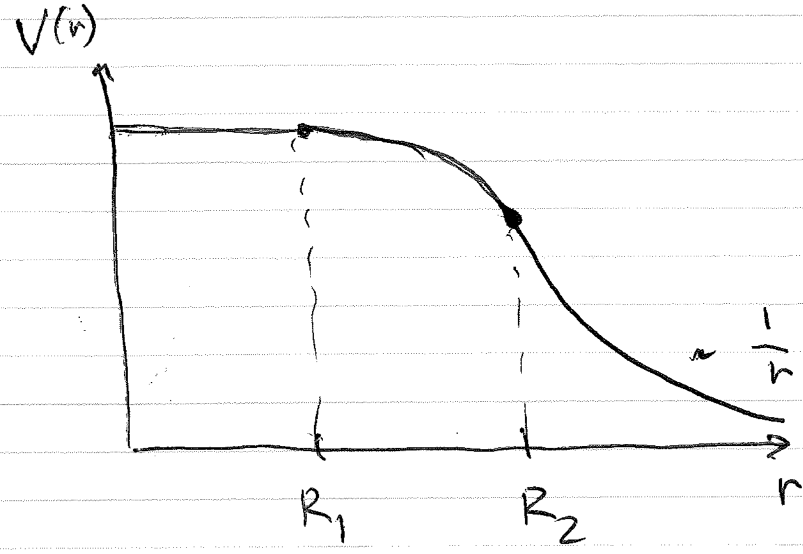
$$C = \frac{\rho}{3\varepsilon_0}\left[R_2^2 - \cancel{\frac{R_1^3}{R_2}} + \frac{1}{2}R_2^2 + \cancel{\frac{R_1^3}{R_2}}\right] = \frac{\rho}{3\varepsilon_0}\frac{3}{2}R_2^2$$

(1) -cont-

$$V = \frac{\rho}{2\varepsilon_0} R_2^2 - \frac{\rho}{3\varepsilon_0}\left(\frac{1}{2}r^2 + \frac{R_1^3}{r}\right) =$$

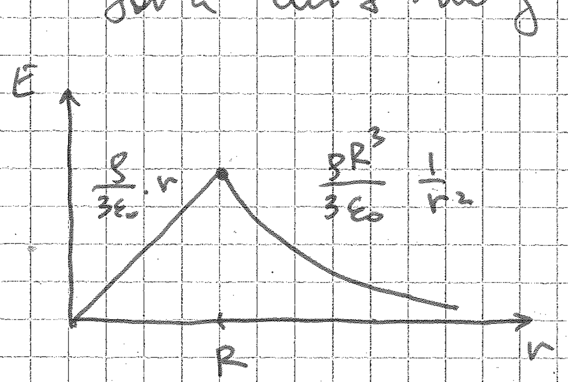
$$= \frac{\rho}{6\varepsilon_0}\left(3R_2^2 - r^2 - \frac{2R_1^3}{r}\right)$$

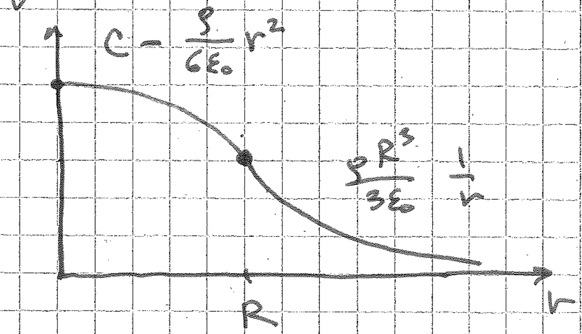
$r < R_1$: $\quad V = \text{const.} = \frac{\rho}{6\varepsilon_0}\left(3R_2^2 - R_1^2 - 2R_1^2\right) = \frac{\rho}{2\varepsilon_0}\left(R_2^2 - R_1^2\right)$



(1) - alternative solution:

for a uniformly charged sphere, according to Gauss law



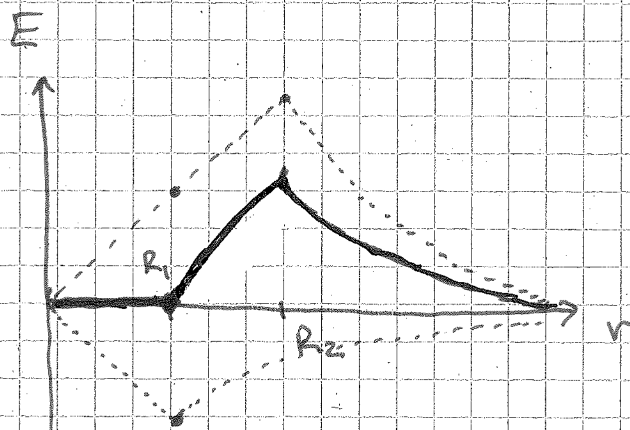


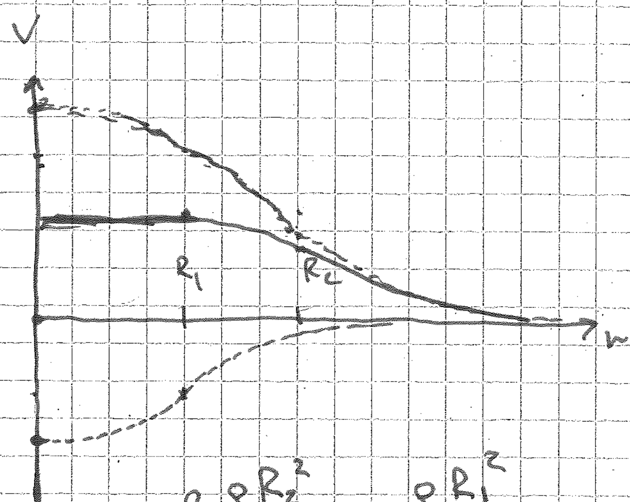
$$C - \frac{\rho}{6\varepsilon_0} R^2 = \frac{\rho}{3\varepsilon_0} R^2$$

$$C = \frac{\rho}{3\varepsilon_0} R^2 \left(1 + \frac{1}{2}\right) = \frac{\rho \cdot R^2}{2\,\varepsilon_0}$$

For two spheres, R_1 and $-\rho$

R_2 and $+\rho$

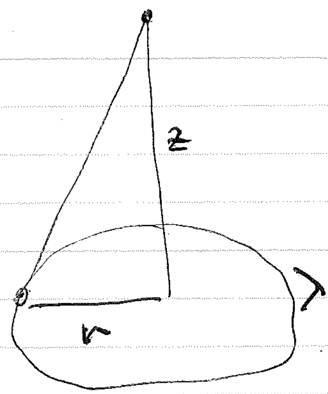




$$E^{tot} = \begin{cases} 0 \\ \dfrac{\rho}{3\varepsilon_0} r - \dfrac{\rho R_1^3}{3\varepsilon_0} \cdot \dfrac{1}{r^2} \\ \dfrac{\rho R_2^3}{3\varepsilon_0} \dfrac{1}{r^2} - \dfrac{\rho R_1^3}{3\varepsilon_0} \dfrac{1}{r^2} \end{cases}$$

$$V^{tot} = \begin{cases} \dfrac{\rho R_2^2}{2\varepsilon_0} - \dfrac{\rho R_1^2}{2\varepsilon_0} \\ \dfrac{\rho R_2^2}{2\varepsilon_0} - \dfrac{\rho R_1^3}{3\varepsilon_0} \dfrac{1}{r} - \dfrac{\rho}{6\varepsilon_0} \cdot r^2 \\ \dfrac{\rho R_2^3}{3\varepsilon_0} \dfrac{1}{r} - \dfrac{\rho R_1^3}{3\varepsilon_0} \dfrac{1}{r} \end{cases}$$

(2)



All points on the loop are equidistant from the point (0, 0, z).

Therefore the potential is

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{r^2+z^2}} = \frac{1}{4\pi\varepsilon_0} \frac{2\pi r \lambda}{\sqrt{z^2+r^2}}$$

$$\boxed{V(0,0,z) = \frac{\lambda}{2\varepsilon_0} \frac{r}{\sqrt{z^2+r^2}}}$$

(3)

Electric field in a sperical capacitor:

$$\vec{E} = \hat{r} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}, \quad R_1 < r < R_2$$

a)
$$W = \frac{\varepsilon_0}{2} \int E^2 d\tau = \frac{\varepsilon_0}{2} \int_{R_1}^{R_2} \int_0^{\pi} \int_0^{2\pi} \frac{Q^2}{16\pi^2 \varepsilon_0^2} \frac{1}{r^4} r^2 dr \cdot \sin\theta \, d\theta \, d\varphi =$$

$$= \frac{\cancel{\varepsilon_0}}{\cancel{2}} \frac{Q^2}{16\pi^2 \varepsilon_0^{\cancel{2}}} \cdot \cancel{2\pi} \cdot \cancel{2} \cdot \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0} \left(-\frac{1}{r}\right)\Bigg|_{R_1}^{R_2} =$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \boxed{\frac{Q^2}{8\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2} = W}$$

b) bringing over charge Δq when capacitor's charge is q:

$$\Delta W = \int_{R_2}^{R_1} \left(-\frac{q \cdot \Delta q}{4\pi\varepsilon_0 \cdot r^2}\right) \cdot dr = \frac{q \Delta q}{4\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2}$$

$$W = \int_0^Q \Delta W = \frac{1}{4\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2} \frac{1}{2} Q^2 = \boxed{\frac{Q^2}{8\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2}}$$

c) Potential difference between plates

$$\Delta V = \int_{R_1}^{R_2} \vec{E} \, d\vec{l} = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2} \Rightarrow \boxed{C = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1}}$$

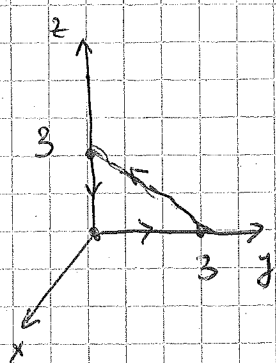
$$W = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{Q}{C}\right)^2 = \frac{1}{2} \frac{Q^2}{C} = \boxed{\frac{Q^2}{8\pi\varepsilon_0} \frac{R_2 - R_1}{R_1 R_2}}$$

4 $\vec{A} = xy\cdot\hat{x} + 2yz\cdot\hat{y} - 3xz\cdot\hat{z}$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \underline{y + 2z - 3x}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}(0-2y) + \hat{y}(0+3z) + \hat{z}(0-x)$$

$$= \boxed{-2y\hat{x} + 3z\hat{y} - x\hat{z} = \vec{B} = \vec{\nabla}\times\vec{A}}$$



$$\Phi = \int_0^3 dy \int_0^{3-y} dz\cdot \vec{B}\cdot(+\hat{x}) = -\int_0^3 dy \int_0^{3-y} dz\,(2y) =$$

$$= -\int_0^3 dy\cdot 2y\cdot(3-y) = -2\int_0^3 (3y - y^2)\,dy = -2\left(\frac{3}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^3 =$$

$$= -2\left(\frac{3}{2}\cdot 9 - \frac{1}{\cancel{3}}\cdot\cancel{3}\cdot 9\right) = -18\left(\frac{3}{2} - 1\right)\boxed{= -9}$$

$$\oint \vec{A}\,d\vec{l} = \int_{(0,0,0)}^{(0,3,0)} \vec{A}\cdot\hat{y}\,dy + \int_{(0,3,0)}^{(0,0,3)} \vec{A}\cdot\frac{\hat{z}-\hat{y}}{\sqrt{2}}\cdot dl + \int_{(0,0,3)}^{(0,0,0)} \vec{A}(-\hat{z})\cdot dz =$$

$$= \int_{(0,0,0)}^{(0,3,0)} \cancel{2yz}\cdot dy + \int_{(0,0,3)}^{(0,0,0)} \cancel{3x\cdot dz} + \int_{(0,3,0)}^{(0,0,3)} \frac{1}{\sqrt{2}}\left(\cancel{-3xz} - 2yz\right) dl =$$

z = 0 along path

x = 0 along path

x = 0 along path

$$= -\frac{2}{\sqrt{2}} \int_{(0,3,0)}^{(0,0,3)} yz\cdot dl = -\frac{2}{\cancel{\sqrt{2}}} \int_0^3 y(3-y)\,\cancel{\sqrt{2}}\,dy = -2\left(\frac{3}{2}y^2 - \frac{1}{3}y^3\right)_0^3 =$$

$$\boxed{= -9}$$