13.20 $\quad \vec{F} = \text{const.} \qquad U = -\int_0^{\vec{r}} \vec{F}\, d\vec{l} = -\vec{F}\,\vec{r}$

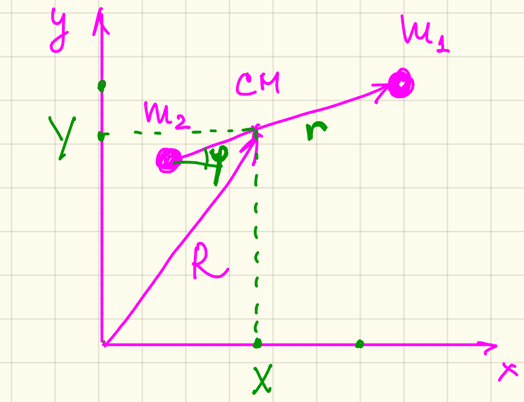
a) $\quad \mathcal{H} = \dfrac{p^2}{2m} - \vec{F}\cdot\vec{r} \quad , \quad p^2 = m\dot{\vec{r}}^2$

b) $\quad$ if $\vec{F} = F\hat{x}$, $\quad \vec{F}\cdot\vec{r} = Fx \quad , \quad \mathcal{H} = \dfrac{p_x^2}{2m} + \dfrac{p_y^2}{2m} - Fx = \mathcal{H}(x, p_x, p_y)$

c) $\quad$ if $\vec{F} = F_x\hat{x} + F_y\hat{y} \quad , \quad \mathcal{H} = \dfrac{p_x^2}{2m} + \dfrac{p_y^2}{2m} - F_x x - F_y y = \mathcal{H}(x, y, p_x, p_y)$

13.21

Coordinates: X, Y of CM, r, φ



$$\vec{r}_1 = \vec{R} + \vec{r}\,\frac{m_2}{m_1+m_2} \qquad \vec{r}_2 = \vec{R} - \vec{r}\,\frac{m_1}{m_1+m_2}$$

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \dot{\vec{r}}\,\frac{m_2}{M} \qquad \dot{\vec{r}}_2 = \dot{\vec{R}} - \dot{\vec{r}}\,\frac{m_1}{M}$$

$$\dot{\vec{r}}_1^{\,2} = \dot{\vec{R}}^2 + \dot{r}^2\left(\frac{m_2}{M}\right)^2 + 2\,\dot{\vec{r}}\,\dot{\vec{R}}\,\frac{m_2}{M}$$

$$\dot{\vec{r}}_2^{\,2} = \dot{\vec{R}}^2 + \dot{r}^2\left(\frac{m_1}{M}\right)^2 - 2\,\dot{\vec{r}}\,\dot{\vec{R}}\,\frac{m_1}{M}$$

$$T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}(m_1+m_2)\cdot\dot{\vec{R}}^2 + \frac{1}{2}(m_1+m_2)\frac{\dot{\vec{r}}^{\,2}}{M^2}\,m_1\cdot m_2$$

$$= \frac{1}{2}M\left(\dot{X}^2+\dot{Y}^2\right) + \frac{1}{2}\underbrace{\frac{m_1 m_2}{M}}_{=\mu}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

$$U = \frac{1}{2}k\,(r-l_0)^2$$

$p_x = M\dot{x}$, $p_y = M\dot{Y}$, $p_r = \mu\dot{r}$, $p_\varphi = \mu r^2\dot{\varphi}$ $\quad (r\dot{\varphi})^2 = \dfrac{p_\varphi^2}{\mu^2 r^2}$

$$\mathcal{H} = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{p_r^2}{2\mu} + \frac{p_\varphi^2}{2\mu r^2} + \frac{1}{2}k(r - l_0)^2$$

$\hookrightarrow$ X, Y, and φ are ignorable; $\mathcal{H}$ only depends on r

$$\frac{\partial \mathcal{H}}{\partial x} = -\dot{p}_x = 0 \qquad \frac{\partial \mathcal{H}}{\partial r} = -\dot{p}_r = -\frac{p_\varphi^2}{\mu}\frac{1}{r^3} + k(r - l_0)$$

$$\frac{\partial \mathcal{H}}{\partial y} = -\dot{p}_y = 0$$

$$\frac{\partial \mathcal{H}}{\partial p_\varphi} = \dot{\varphi} = \frac{p_\varphi}{\mu r^2}$$

$$\frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{p}_\varphi = 0$$

$$\frac{\partial \mathcal{H}}{\partial p_r} = \dot{r} = \frac{p_r}{\mu}$$

$$\frac{\partial \mathcal{H}}{\partial p_x} = \dot{x} = \frac{p_x}{M}$$

$$\frac{\partial \mathcal{H}}{\partial p_y} = \dot{y} = \frac{p_y}{M}$$

c) if $p_\varphi = 0$ (i.e. $L = 0$)

$$\left.\begin{aligned} -\dot{p}_r &= k(r - l_0) \\ \dot{r} &= p_r/\mu \end{aligned}\right\} \quad \mu \ddot{r} = -k(r - l_0)$$

$$\boxed{r = l_0 + A \cdot \cos\left(\sqrt{\frac{k}{\mu}}\, t + \delta\right)}$$

harmonic oscillations around l_0

d) $p_\varphi = L \neq 0$

$$\left\{\begin{aligned} -\dot{p}_r &= -\frac{L^2}{\mu r^3} + k(r - l_0) \\ \dot{r} &= p_r/\mu \end{aligned}\right. \quad \mu \ddot{r} = \frac{L^2}{\mu r^3} - k(r - l_0)$$

non-linear eqn!

can not be solved in terms of elementary functions. Non-harmonic oscillations (unless $r - l_0 \ll 1$)

13.25

$$\mathcal{H} = \mathcal{H}(q, p)$$

$$q = \sqrt{2P}\,\sin Q \;, \quad p = \sqrt{2P}\,\cos Q$$

$$\frac{\partial q}{\partial Q} = \sqrt{2P}\cdot\cos Q = p$$

$$\frac{\partial p}{\partial Q} = -\sqrt{2P}\cdot\sin Q = -q$$

$$\frac{\partial q}{\partial P} = \sqrt{2}\,\frac{1}{2}\,\frac{1}{\sqrt{P}}\sin Q = \frac{1}{\sqrt{2P}}\sin Q \;; \quad \frac{\partial p}{\partial P} = \frac{1}{\sqrt{2P}}\cdot\cos Q$$

$$\frac{\partial \mathcal{H}}{\partial Q} = \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} = -\dot{p}\,p - \dot{q}\,\dot{q} = -\frac{1}{2}\frac{d}{dt}\left(p^2 + q^2\right) =$$

$$= -\frac{1}{2}\cdot\frac{d}{dt}\left(2P\sin^2 Q + 2P\cos^2 Q\right) = -\dot{P}$$

$$\frac{\partial \mathcal{H}}{\partial Q} = -\dot{P}$$

$$q = \sqrt{2P}\sin Q \quad , \quad p = \sqrt{2P}\cos Q$$

$$\frac{\partial q}{\partial P} = \sqrt{2}\,\frac{1}{2}\,\frac{1}{\sqrt{P}}\sin Q = \frac{1}{\sqrt{2P}}\sin Q \quad ; \quad \frac{\partial p}{\partial P} = \frac{1}{\sqrt{2P}}\cdot\cos Q$$

$$\dot{q} = \sqrt{2}\,\frac{1}{2}\,\frac{\dot{P}}{\sqrt{P}}\sin Q + \sqrt{2P}\cos Q\cdot\dot{Q} = \frac{\partial q}{\partial P}\cdot\dot{P} + p\cdot\dot{Q} = \dot{q} \quad \Big\{ \frac{\partial p}{\partial P}$$

$$+$$

$$\dot{p} = \frac{\dot{P}}{\sqrt{2P}}\cos Q - \sqrt{2P}\sin Q\cdot\dot{Q} = \frac{\partial p}{\partial P}\cdot\dot{P} - q\cdot\dot{Q} = \dot{p} \quad \Big\} -\frac{\partial q}{\partial P}$$

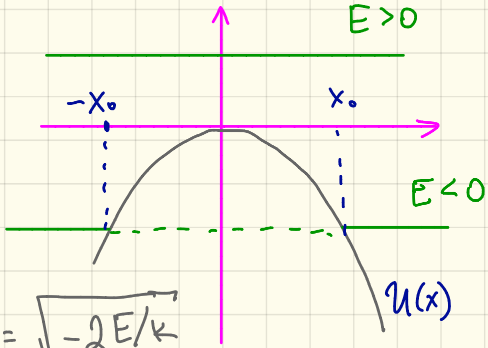
$$\left(p\frac{\partial p}{\partial P} + q\frac{\partial q}{\partial P}\right)\dot{Q} = \dot{q}\frac{\partial p}{\partial P} - \dot{p}\frac{\partial q}{\partial P}$$

$$\frac{\partial \mathcal{H}}{\partial P} = \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} = -\dot{p}\frac{\partial q}{\partial P} + \dot{q}\frac{\partial p}{\partial P} = \dot{Q}\,\frac{1}{2}\frac{\partial}{\partial P}\left(p^2+q^2\right) = \dot{Q}\,\cancel{\frac{1}{2}\frac{\partial}{\partial P}(2P)}$$

$$\frac{\partial \mathcal{H}}{\partial P} = \dot{Q}$$

13.28 $\quad F = kx, \quad k > 0 \qquad U = -\frac{1}{2}kx^2$

a)



$x_0 = \sqrt{-2E/k}$

if $E > 0$ particle moves from $-\infty$ to $+\infty$ or from $+\infty$ to $-\infty$

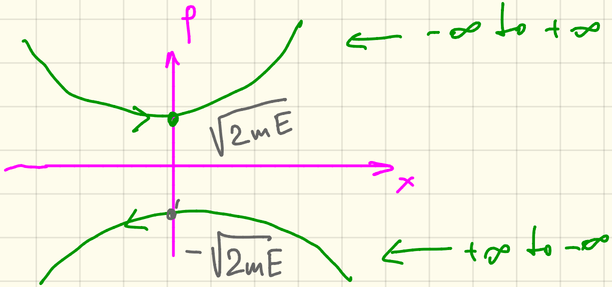
if $E < 0$ low x are excluded.

- comes from $-\infty$, reaches $-x_0$ and returns to $-\infty$.
- or comes from $+\infty$, reflects at x_0 and returns to $+\infty$.

b) $$\mathcal{H} = \frac{p^2}{2m} - \frac{1}{2}kx^2 = E$$

for $E > 0$

$$\frac{p^2}{a^2} - \frac{x^2}{b^2} = c^2 \rightarrow \text{hyperbola } p(x)$$



$$\frac{p^2}{2m} - \frac{1}{2} k x^2 = E$$

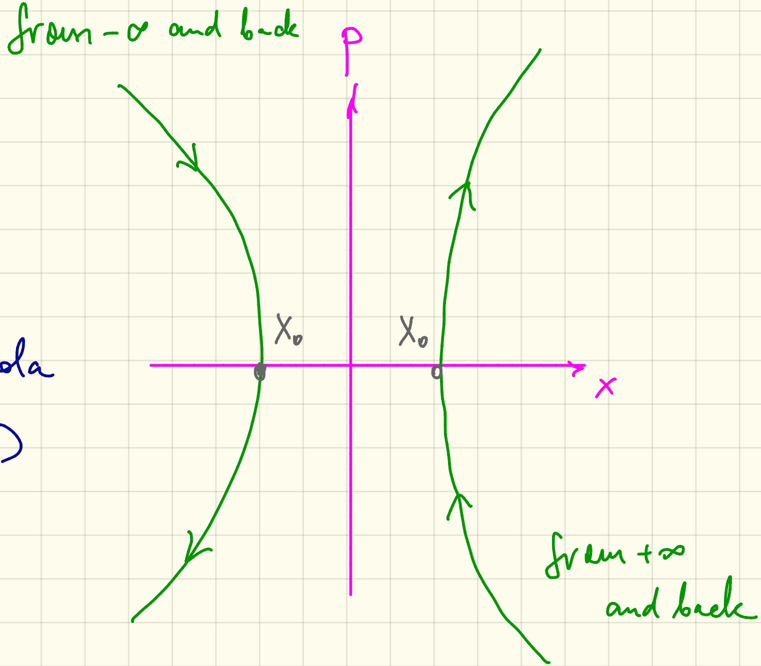
$$x_0 = \sqrt{-2E/k}$$

for $E < 0$

$$\frac{p^2}{a^2} - \frac{x^2}{b^2} = -c^2$$

$$\frac{x^2}{b^2} - \frac{p^2}{a^2} = c^2 \longrightarrow \text{hyperbola}$$

$x(p)$



13.35 initial phase space volume.

$$\pi R_0^2 L_0 \cdot 2 \Delta p_z \cdot \pi \Delta p_T^2$$

transformed to a cylinder of radius R, L_0, Δp_z

$$R_0^2 \cdot \Delta p_T^2 = R^2 \Delta p_{T_1}^2$$

$$\Delta p_{T_1} = \Delta p_T \cdot \frac{R_0}{R}$$

p_T spread is enlarged, therefore the beam is going to expand in R