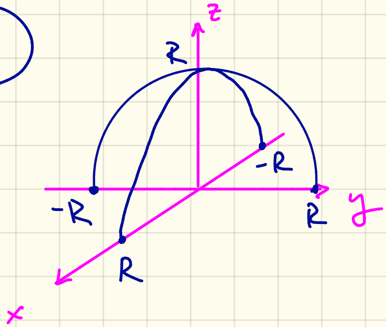
10.5

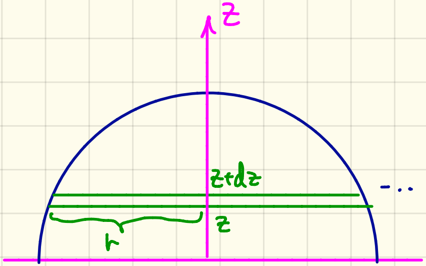


Find C.O.M:

$x_{cm} = y_{cm} = 0 \leftarrow$ symmetry.

$$z_{cm} = \frac{1}{M}\int_0^R z \cdot dm = \ldots$$

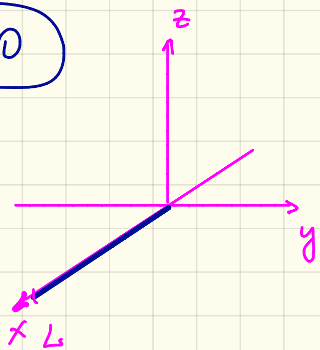
$$dm = \rho \cdot \pi r^2 \cdot dz = \rho \cdot \pi \cdot (R^2 - z^2) \cdot dz$$



$$\ldots = \frac{\cancel{\pi \rho}}{\frac{4}{6}\cancel{\pi} R^3 \cancel{\rho}} \cdot \int_0^R z(R^2 - z^2)\, dz = \frac{6}{4R^3}\left(\frac{1}{2}R^2 z^2 - \frac{1}{4}z^4\right)\Bigg|_0^R =$$

$$= \frac{3}{2R^3}\left[\frac{1}{2}R^4 - \frac{1}{4}R^4\right] = \frac{3}{8}R$$

10.10



Rod length L, mass M

a)

$$I = \int_0^L x^2\,dm = \int_0^L x^2 \cdot \frac{M}{L}\,dx = \frac{M}{L} \cdot \int_0^L x^2\,dx =$$

$$= \frac{M}{L} \cdot \frac{1}{3} L^3 = \boxed{\frac{1}{3} M L^2 = I}$$

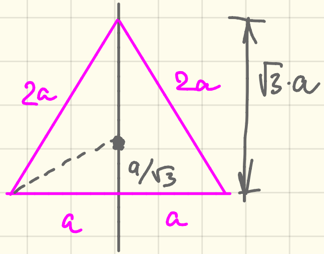
b) if center of the rod is @ origin

$$I_0 = \frac{M}{L} \int_{-L/2}^{L/2} x^2\,dx = \frac{M}{L} \cdot \frac{1}{3} \cdot 2\left(\frac{L}{2}\right)^3 = \boxed{\frac{1}{12} M L^2 = I_0}$$

Sanity check: $\underbrace{I = I_0 + M\left(\frac{L}{2}\right)^2}_{\text{paralell axis theorem}} = M L^2 \left(\frac{1}{12} + \frac{1}{4}\right) = \frac{1}{3} M L^2$

(10.12)

$I_{xz} = I_{yz} = 0$: reflection symmetry in xy plane



I_{zz}: $\quad u = x + \frac{a}{\sqrt{3}}$ $\qquad \rho = \frac{M}{\sqrt{3}\,a^2}$

$$I = \int_0^{\sqrt{3}a} \int_{-a+\frac{u}{\sqrt{3}}}^{a-\frac{u}{\sqrt{3}}} \left(\left(u - \frac{a}{\sqrt{3}}\right)^2 + y^2 \right) \rho \, du\, dy =$$

$$= \frac{2M}{\sqrt{3}\,a^2} \left[\int_0^{\sqrt{3}a} \int_0^{a-\frac{u}{\sqrt{3}}} \left(u - \frac{a}{\sqrt{3}}\right)^2 dy\, du + \int_0^{\sqrt{3}a} \int_0^{a-\frac{u}{\sqrt{3}}} y^2 \, dy\, du \right] =$$

$$= \frac{2M}{\sqrt{3}\,a^2} \left[I_1 + I_2 \right] =$$

$$I_1 = \int_0^{\sqrt{3}a} \int_0^{a-\frac{u}{\sqrt{3}}} \left(u - \frac{a}{\sqrt{3}}\right)^2 dy\,du = \int_0^{\sqrt{3}a} \left(u - \frac{a}{\sqrt{3}}\right)^2 \cdot \left(a - \frac{u}{\sqrt{3}}\right) du =$$

$$= \int_0^{\sqrt{3}a} \left(u^2 + \frac{a^2}{3} - \frac{2}{\sqrt{3}} a \cdot u\right)\left(a - \frac{u}{\sqrt{3}}\right) du =$$

$$= \int_0^{\sqrt{3}a} \left(u^3\left(-\frac{1}{\sqrt{3}}\right) + u^2\left(a + a\frac{2}{3}\right) + u\left(-\frac{a^2}{3\sqrt{3}} - \frac{2a^2}{\sqrt{3}}\right) + \frac{a^3}{3}\right) du =$$

$$= \int_0^{\sqrt{3}a} \left(-\frac{1}{\sqrt{3}}\right)u^3 + \frac{5}{3}a \cdot u^2 - \frac{7}{3\sqrt{3}} u + \frac{a^3}{3}\Big) du =$$

$$= -\frac{1}{\sqrt{3}} \frac{1}{4} 9 a^4 + \frac{5}{\sqrt{3}} a \cdot \frac{1}{\cancel{3}} \cancel{3} \cancel{\sqrt{3}} \cdot a^3 - \frac{7}{\cancel{3}\sqrt{3}} \frac{1}{2} \cancel{3} a^2 + \frac{a^3}{\sqrt{3}} \cdot \cancel{\sqrt{3}} a =$$

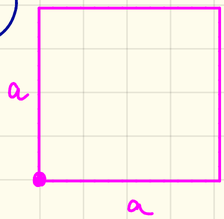
$$= \frac{a^4}{\sqrt{3}}\left[-\frac{9}{4}+5-\frac{7}{2}+1\right] = \frac{a^4}{\sqrt{3}}\left[6-\frac{23}{4}\right] = \boxed{\frac{a^4}{\sqrt{3}}\;\frac{1}{4} = I_1}$$

$$I_2 = \int_0^{\sqrt{3}a}\int_0^{a-\frac{u}{\sqrt{3}}} y^2\,dy\,du = \int_0^{\sqrt{3}a}\frac{1}{3}\left(a-\frac{u}{\sqrt{3}}\right)^3 du = \left|\begin{array}{l} v = a-\frac{u}{\sqrt{3}} \\ dv = -\frac{1}{\sqrt{3}}\,du \end{array}\right| =$$

$$= \frac{1}{3}\left(-\sqrt{3}\right)\int_a^0 v^3\,dv = \boxed{\frac{1}{\sqrt{3}}\cdot\frac{1}{4}\cdot a^4 = I_2}$$

$$I_{zz} = \frac{2M}{\sqrt{3}\,a^2}\cdot\left[\frac{a^4}{\sqrt{3}\cdot 4}+\frac{a^4}{\sqrt{3}\cdot 4}\right] = \boxed{\frac{1}{3}Ma^2 = I_{zz}}$$

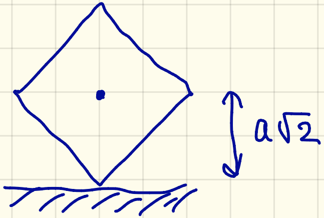
10.15



$$I = \int_0^a \int_0^a (x^2+y^2)\,dx\,dy \cdot \frac{M}{a^2} =$$

$$= \frac{M}{a^2} \int_0^a \left(\frac{1}{3}a^3 + y^2 a\right) dy = \frac{M}{a^2}\left(\frac{1}{3}a^3 \cdot a + \frac{1}{3}a^3 a\right) =$$

$$I_{zz} = \frac{2}{3} M a^2 \qquad I_{zx} = I_{zy} = 0 \quad \text{(symmetry)}$$

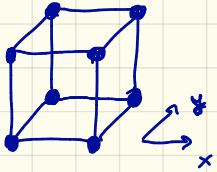


Energy conservation:

$$M g a\sqrt{2} = M g a + \frac{1}{2} I_{zz} \cdot \omega^2$$

$$\omega^2 = \frac{M g a(\sqrt{2}-1)}{2/3\, M a^2} = \boxed{\frac{3}{2}\,\frac{g}{a} \cdot (\sqrt{2}-1) = \omega^2}$$

10.22



a)

$$I_{xx} = \sum m_\alpha \cdot (y_\alpha^2 + z_\alpha^2) = m \cdot 2 \cdot [0 + a^2 + a^2 + 2a^2] = 8a^2 m$$

$$I_{xy} = -\sum m_\alpha x_\alpha y_\alpha = -ma[0+0+a+a] = -2a^2 m$$

$$I_{xz} = -\sum m_\alpha \cancel{x_\alpha} \cancel{z_\alpha} = -ma[0+a+0+a] = -2a^2 m$$

x	y	z
0	0	0
0	a	0
0	0	a
0	a	a
a	0	0
a	a	0
a	0	a
a	a	a

$$\boxed{I = \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix} \cdot ma^2}$$

b) Coordinates in units of $\frac{a}{2}$:

-1	-1	-1
-1	-1	1
-1	1	-1
-1	1	1
1	-1	-1
1	-1	1
1	1	-1
1	1	1

$$I_{xx} = \frac{1}{4} ma^2 [8+8] = 4ma^2$$

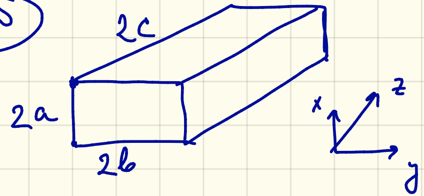
$$I_{xy} = -\frac{1}{4} ma^2 [4-4] = 0$$

reflection symmetry in xy, yz and zx planes

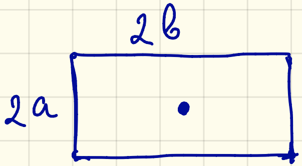
↳ I should be diagonal

$$\boxed{I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot ma^2}$$

(10.25)



I should be diagonal
→ reflection symmetry in all planes (xy, yz, zx)



$$I_{zz} = \int_{-a}^{a}\int_{-b}^{b} (x^2+y^2)\frac{M}{4ab}\cdot dx\,dy =$$

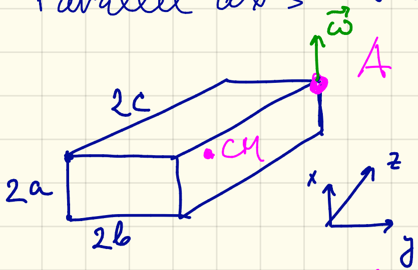
$$= \frac{M}{ab}\int_0^a\int_0^b (x^2+y^2)\,dy\,dx = \frac{M}{ab}\int_0^a\left(x^2 b + \frac{1}{3}b^3\right)dx =$$

$$= \frac{M}{a}\int_0^a\left(x^2+\frac{1}{3}b^2\right)dx = \frac{M}{a}\left(\frac{1}{3}a^3+\frac{1}{3}b^2 a\right) = \frac{1}{3}M(a^2+b^2)$$

Similarly for I_{xx} & I_{yy}:

$$\boxed{\mathbf{I} = \frac{1}{3}M\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}}$$

b) "Parallel axis theorem"



$$I_{xx}^{A} = I_{xx} + M\,(b^2+c^2)$$

$$I_{xy}^{A} = I_{xy} - M\,ab$$

$$\vdots$$

A w.r.t CM is at (a, b, c)

$$\overleftrightarrow{I} = \frac{1}{3}M \begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ab & 4(a^2+c^2) & -3bc \\ -3ac & -3bc & 4(b^2+c^2) \end{bmatrix}$$

c) $\vec{\omega} = (\omega, 0, 0)$

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega} = \frac{1}{3}M\omega\,\left(4(b^2+c^2),\ -3ab,\ -3ac\right)$$