Rutgers Physics 382 Mechanics II (Spring'19/Gershtein)

Class Exam - March 5, 2019

This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration - 1hr 20min. Solve all problems (30 points total)

Euler equations for zero torque

$$\left.\begin{aligned} \lambda_1\dot{\omega}_1 &= (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 &= (\lambda_3-\lambda_1)\omega_3\omega_1 \\ \lambda_3\dot{\omega}_3 &= (\lambda_1-\lambda_2)\omega_1\omega_2 \end{aligned}\right\}$$

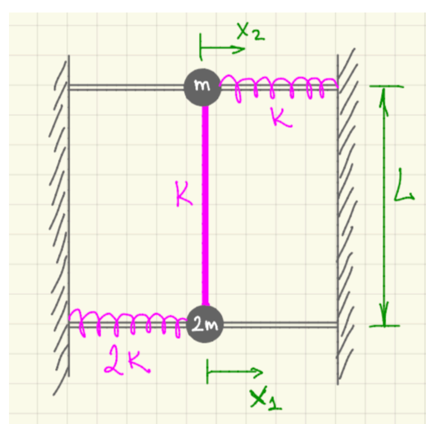
Tensor of Inertia for a cylinder

$$\mathrm{I} = \begin{bmatrix} \frac{1}{12}Mh^2+\frac{1}{4}MR^2 & 0 & 0 \\ 0 & \frac{1}{12}Mh^2+\frac{1}{4}MR^2 & 0 \\ 0 & 0 & \frac{1}{2}MR^2 \end{bmatrix},$$

1. **(5 pts)** A rigid body consists of three point masses **m** fastened in positions (3b,0,0), (b,b,0), and (0,b,-b). Find the tensor of inertia.

2. **(10 pts)** A long cylinder of mass ***M***, length ***h***, and radius ***R (R=0.2 h)*** is spun along its axis with a frequency ω and is thrown in the air. What is the angular frequency of the small oscillations of its axis of rotation?

3. Two rigid rods separated by distance ***L*** connect two immovable walls as shown in the figure below. Masses ***2m*** and ***m*** move along the rods without friction and are connected to the walls by springs with Hooke's coefficients of ***2k*** and ***k*** respectively. When these springs are relaxed, the masses are underneath each other as shown in the figure. The masses are also connected by a rubber band with coefficient ***k***. It's relaxed length is much smaller then ***L*** and can be neglected.



a) (7 pts) write down Hamiltonian equations for the system, using deviations from equilibrium positions x_1 and x_2 as generalized coordinates

b) (8 pts) Find and describe the normal modes and frequencies of small oscillations

① $m_1: (3b, 0, 0)$

$m_2: (b, b, 0)$

$m_3: (0, b, -b)$

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) =$$

$$= m(b^2 + b^2 + b^2) = 3mb^2$$

$$I_{yy} = m(9b^2 + b^2 + b^2) = 11mb^2$$

$$I_{zz} = m(9b^2 + b^2 + b^2 + b^2) = 12mb^2$$

$$I_{xy} = -\sum m x_i y_i = -m(b^2)$$

$$I_{xz} = 0$$

$$I_{yz} = -m(-b^2) = mb^2$$

$$\boxed{\overleftrightarrow{I} = mb^2 \begin{pmatrix} 3 & -1 & 0 \\ -1 & 11 & 1 \\ 0 & 1 & 12 \end{pmatrix}}$$

② For cylinder

$$\lambda = \lambda_1 = \lambda_2 = \frac{1}{12} M h^2 + \frac{1}{4} M R^2 = \left(\frac{1}{12} + \frac{1}{100}\right) M h^2$$

$$\lambda_3 = \frac{1}{2} M R^2 = \frac{1}{2} \cdot 0.2^2 M h^2 = \frac{1}{50} M h^2$$

$$\lambda_3 < \lambda \qquad \lambda - \lambda_3 = \left(\frac{1}{12} - \frac{1}{100}\right) M h^2 = \Lambda$$

$$\begin{cases} \lambda \dot{\omega}_1 = (\lambda - \lambda_3)\, \omega_2 \omega_3 \\ \lambda \dot{\omega}_2 = (\lambda_3 - \lambda)\, \omega_1 \omega_3 \\ \lambda_3 \dot{\omega}_3 = 0 \end{cases} \Rightarrow \quad \omega_3 = \text{const}$$

$$\begin{cases} \ddot{\omega}_1 = -\frac{\Lambda^2}{\lambda^2} \omega_3^2\, \omega_1 \\ \ddot{\omega}_2 = -\frac{\Lambda^2}{\lambda^2} \omega_3^2\, \omega_2 \end{cases} \leftarrow \text{oscillations!}$$

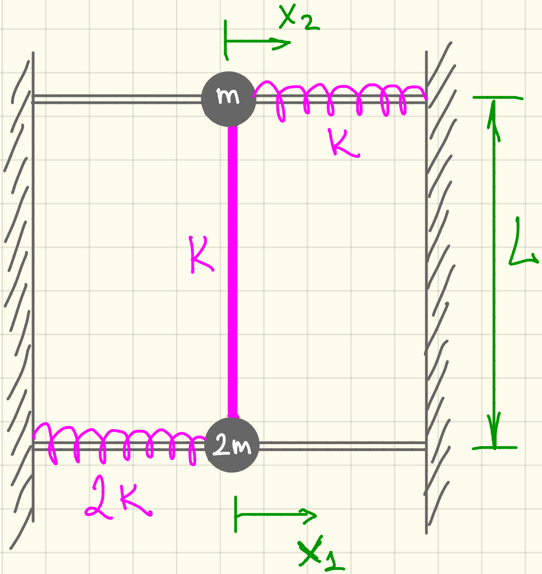
$$\boxed{\Omega = \omega_3 \frac{\Lambda}{\lambda} = \omega_3 \frac{\lambda - \lambda_3}{\lambda}}$$

$$\frac{300}{12} = \frac{100}{4} = 25$$

$$\frac{\frac{1}{12} - \frac{1}{100}}{\frac{1}{12} + \frac{1}{100}} = \frac{300}{300} \; \frac{25 - 3}{25 + 3} = \frac{22}{28} = \frac{11}{14}$$

$$\boxed{\Omega = \frac{11}{14}\,\omega_3}$$

(3)



$$T = \frac{1}{2} 2m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} 2k x_1^2 + \frac{1}{2} k x_2^2 + $$
$$+ \frac{1}{2} k \left(L^2 + (x_1 - x_2)^2 \right)$$

$$\mathcal{L} = m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - k x_1^2 - \frac{1}{2} k x_2^2 -$$
$$- \frac{1}{2} k (x_1 - x_2)^2$$

$$\left. \begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = 2m \dot{x}_1 \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m \dot{x}_2 \end{aligned} \right\} \Rightarrow$$

$$\mathcal{H} = p_1 \dot{x}_1 + p_2 \dot{x}_2 - \mathcal{L} =$$

$$= \frac{p_1^2}{2m} + \frac{p_2^2}{m} - \mathcal{L} =$$

$$m \dot{x}_1^2 = \frac{p_1^2}{4m}$$

$$\frac{1}{2} m \dot{x}_2^2 = \frac{p_2^2}{2m}$$

$$\mathcal{H} = \frac{p_1^2}{4m} + \frac{p_2^2}{2m} + k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$$

$$\mathcal{H} = \frac{p_1^2}{4m} + \frac{p_2^2}{2m} + k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$$

$$\frac{\partial \mathcal{H}}{\partial p_1} = \boxed{\frac{p_1}{2m} = \dot{x}_1}$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m} = \dot{x}_2}$$

$$\frac{\partial \mathcal{H}}{\partial x_1} = 2k x_1 + k(x_1 - x_2) = \boxed{3k x_1 - k x_2 = -\dot{p}_1}$$

$$\frac{\partial \mathcal{H}}{\partial x_2} = k x_2 - k(x_1 - x_2) = \boxed{-k x_1 + 2k x_2 = -\dot{p}_2}$$

$$\begin{cases} 2m\ddot{x}_1 = -(3kx_1 - kx_2) \\ m\ddot{x}_2 = -(-kx_1 + 2kx_2) \end{cases}$$

$$\frac{k}{m} = \omega_0^2$$

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 3k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} 3k - 2m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = m^2 \begin{vmatrix} 3\omega_0^2 - 2\omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$(3\omega_0^2 - 2\omega^2)(2\omega_0^2 - \omega^2) - \omega_0^4 = 0$$

$$(3\omega_0^2 - 2\omega^2)(2\omega_0^2 - \omega^2) - \omega_0^4 = 0$$

$$2\omega^4 - 7\omega_0^2\omega^2 + 5\omega_0^4 = 0$$

$$D = 49\omega_0^4 - 4\cdot 2\cdot 5\cdot\omega_0^4 = 9\omega_0^4$$

$$\omega^2 = \frac{1}{4}(7 \pm 3)\omega_0^2 = \begin{cases} \omega_0^2 \\ \frac{5}{2}\omega_0^2 \end{cases}$$

<u>Now eigenvectors!</u>

$\omega^2 = \omega_0^2$:

$$\begin{pmatrix} 3\omega_0^2 - 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega_0^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & +\omega_0^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\boxed{x_1 = x_2}$$

$\rightarrow$ bodies 1 and 2 oscillate in phase
rubber band @ constant length

$$\omega^2 = \frac{5}{2}\omega_0^2 : \quad \begin{pmatrix} 3\omega_0^2 - 5\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \frac{5}{2}\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= \begin{pmatrix} -2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\frac{1}{2}\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\boxed{x_1 = -\frac{1}{2} x_2}$$

bodies 1 and 2 oscillate in counter-phase