This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration - 1hr 20min. Solve all problems (30 points total)

Euler equations for zero torque
\[
\begin{align*}
\lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2 \omega_3 \\
\lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3 \omega_1 \\
\lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1 \omega_2
\end{align*}
\]

Tensor of Inertia for a cylinder
\[
I = \begin{bmatrix}
\frac{1}{12}Mh^2 + \frac{1}{4}MR^2 & 0 & 0 \\
0 & \frac{1}{12}Mh^2 + \frac{1}{4}MR^2 & 0 \\
0 & 0 & \frac{1}{2}MR^2
\end{bmatrix},
\]

1. **(5 pts)** A rigid body consists of three point masses \( m \) fastened in positions \((3b,0,0), (b,b,0), \) and \((0,b,-b)\). Find the tensor of inertia.

2. **(10 pts)** A long cylinder of mass \( M \), length \( h \), and radius \( R \) (\( R = 0.2 \) \( h \)) is spun along its axis with a frequency \( \omega \) and is thrown in the air. What is the angular frequency of the small oscillations of its axis of rotation?

3. Two rigid rods separated by distance \( L \) connect two immovable walls as shown in the figure below. Masses \( 2m \) and \( m \) move along the rods without friction and are connected to the walls by springs with Hooke’s coefficients of \( 2k \) and \( k \) respectively. When these springs are relaxed, the masses are underneath each other as shown in the figure. The masses are also connected by a rubber band with coefficient \( k \). It’s relaxed length is much smaller then \( L \) and can be neglected.

   a) **(7 pts)** write down Hamiltonian equations for the system, using deviations from equilibrium positions \( x_1 \) and \( x_2 \) as generalized coordinates

   b) **(8 pts)** Find and describe the normal modes and frequencies of small oscillations
(1) \[ m_1: (3b, 0, 0), \quad m_2: (b, b, 0), \quad m_3: (0, b, -b) \]

\[
\begin{align*}
I_{xx} &= \sum_i m_i (y_i^2 + z_i^2) = m (b^2 + b^2 + b^2) = 3mb^2 \\
I_{yy} &= m (9b^2 + b^2 + b^2) = 11mb^2 \\
I_{zz} &= m (9b^2 + b^2 + b^2 + b^2) = 12mb^2 \\
I_{xy} &= -\sum m x_i y_i = -m (b^2) \\
I_{xz} &= 0 \\
I_{yz} &= -m (-b^2) = mb^2
\end{align*}
\]

\[
I = mb^2 \begin{pmatrix} 3 & -1 & 0 \\ -1 & 11 & 1 \\ 0 & 1 & 12 \end{pmatrix}
\]
(2) For cylinder

\[ \lambda = \lambda_1 = \lambda_2 = \frac{1}{12} M h^2 + \frac{1}{4} M R^2 = \left( \frac{1}{12} + \frac{1}{100} \right) M h^2 \]

\[ \lambda_3 = \frac{1}{2} M R^2 = \frac{1}{2} \cdot 0.2^2 M h^2 = \frac{1}{50} M h^2 \]

\[ \lambda_3 < \lambda \]

\[ \lambda - \lambda_3 = \left( \frac{1}{12} - \frac{1}{100} \right) M h^2 = \Lambda \]

\[ \begin{cases} 
\lambda \dot{\omega}_1 = (\lambda - \lambda_3) \omega_2 \omega_3 \\
\lambda \dot{\omega}_2 = (\lambda_3 - \lambda) \omega_1 \omega_3 > 0 \\
\lambda_3 \ddot{\omega}_3 = 0 
\end{cases} \]

\[ \omega_3 = \text{const} \]

\[ \begin{cases} 
\ddot{\omega}_1 = -\frac{\Lambda^2}{\lambda^2} \omega_3^2 \omega_1 \\
\ddot{\omega}_2 = -\frac{\Lambda^2}{\lambda^2} \omega_3^2 \omega_2 
\end{cases} \]

\[ \sum = \omega_3 \frac{\Lambda}{\lambda} = \omega_3 \frac{\lambda - \lambda_3}{\lambda} \]
\[ \frac{1}{12} - \frac{1}{100} = \frac{300}{300} \frac{25 - 3}{25 + 3} = \frac{22}{28} = \frac{11}{14} \]

\[ S_2 = \frac{11}{14} \omega_3 \]
\[ T = \frac{1}{2} 2m \dot{x}_1^2 + \frac{1}{2} 2m \dot{x}_2^2 \]
\[ U = \frac{1}{2} 2k x_1^2 + \frac{1}{2} 2k x_2^2 + \]
\[ + \frac{1}{2} k \left( L^2 + (x_1 - x_2)^2 \right) \]
\[ L = m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - k x_1 - \frac{1}{2} k x_2^2 - \]
\[ - \frac{1}{2} k (x_1 - x_2)^2 \]

\[ p_1 = \frac{\partial L}{\partial \dot{x}_1} = 2m \dot{x}_1 \]
\[ \Rightarrow \quad \dot{H} = p_1 \dot{x}_1 + p_2 \dot{x}_2 - L = \]
\[ = \frac{p_1^2}{2m} + \frac{p_2^2}{m} - L = \]
\[ \dot{H} = \frac{p_1^2}{4m} + \frac{p_2^2}{2m} + k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_1 - x_2)^2 \]
\[ \frac{\partial E}{\partial P_1} = \frac{p_1}{2m} = x_1 \]

\[ \frac{\partial E}{\partial P_2} = \frac{p_2}{m} = x_2 \]

\[ \frac{\partial E}{\partial x_1} = 2kx_1 + k(x_1 - x_2) = 3kx_1 - kx_2 = -\dot{p}_1 \]

\[ \frac{\partial E}{\partial x_2} = kx_2 - k(x_1 - x_2) = -kx_1 + 2kx_2 = -\dot{p}_2 \]
\[ \begin{align*}
2m \ddot{x}_1 &= -\left(3Kx_1 - Kx_2\right) \\
2m \ddot{x}_2 &= -\left(-Kx_1 + 2Kx_2\right)
\end{align*} \]

\[
\begin{pmatrix}
2m & 0 \\
0 & m
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix}
= -
\begin{pmatrix}
3K & -K \\
-K & 2K
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[
\begin{vmatrix}
2K - mw^2 & -K \\
-K & 2K - mw^2
\end{vmatrix}
= m^2
\begin{vmatrix}
3w_0^2 - 2w^2 & -w_0^2 \\
-w_0^2 & 2w_0^2 - w^2
\end{vmatrix}
= 0
\]

\[
(3w_0^2 - 2w^2)(2w_0^2 - w^2) - w_0^4 = 0
\]
\[(3\omega_0^2 - 2\omega^2)(2\omega_0^2 - \omega^2) - \omega_0^4 = 0\]

\[2\omega^4 - 7\omega_0^2\omega^2 + 5\omega_0^4 = 0\]

\[\mathcal{D} = 49\omega_0^4 - 4 \cdot 2.5 \cdot \omega_0^4 = 9\omega_0^4\]

\[\omega^2 = \frac{1}{4}(7 \pm 3)\omega_0^2 = \begin{cases} \omega_0^2 \\ \frac{5}{2}\omega_0^2 \end{cases}\]

New eigenvectors:\n
\[\omega = \omega_0:\]

\[
\begin{pmatrix}
3\omega_0^2 - 2\omega^2 \\
-\omega^2 \\
2\omega_0^2 - \omega_0^2
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =
\begin{pmatrix}
\omega_0^2 - \omega^2 \\
-\omega^2 + \omega_0^2 \\
-x_0^2 + \omega_0^2 + \omega_0^2
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0
\]

\[X_1 = X_2\]

\[\rightarrow \text{ bodies 1 and 2 oscillate in phase on rubber band at constant length.}\]
\[ \omega^2 = \frac{5}{2} \omega_0^2 : \begin{pmatrix} 3 \omega_0^2 - 5 \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2 \omega_0^2 - \frac{5}{2} \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \]

\[ = \begin{pmatrix} -2 \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\frac{1}{2} \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \]

\[ x_1 = -\frac{1}{2} x_2 \]

bodies 1 and 2 oscillate in counter-phase