Rutgers Physics 382 Mechanics II (Spring'19/Gershtein)

Final Exam – May 10, 2019

This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration – 3 hours.

1. (5 pts) Spaceship A moves with speed 0.2c (as viewed from some inertial frame) and is chased by spaceship B with speed of 0.5c in the same frame.

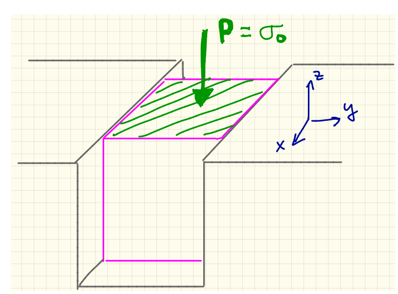
a) what is the speed of spaceship B as observed from spaceship A?
b) what is the speed of spaceship A as observed from spaceship B?

2. (5 pts) Differential cross section of a scattering process is given by $\frac{d\sigma}{d\Omega} = A^2 \cdot (\cos\varphi)^2$. Calculate the total cross section.

3. (5 pts) An infinitely long string is tied to the wall at x=0 and occupies negative x values. It is kept under tension of T and has linear density μ. At x=-∞ the string is vibrated, inducing the wave propagating along the x axis with amplitude A.

a) Determine the amplitude, frequency, and phase of the reflected from the wall wave.
b) Determine the amplitude, frequency, and the wave number of the standing wave resulting from the interference of the incident and reflected waves

4. (7 pts) A rubber cube (with known *BM* and *SM*) is placed into a trough running along the x-axis. The trough dimensions are such that the cube snugly fits (see figure). Pressure of σ_o is applied uniformly to the top of the cube. Assuming there is no friction between the cube and the trough, the stress and strain tensors should be diagonal, i.e.



$$\overleftrightarrow{\Sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & -\sigma_0 \end{bmatrix} \text{ and } \overleftrightarrow{\mathrm{E}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}.$$

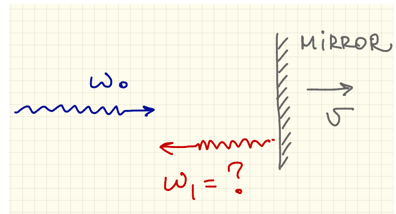
a) Find the five unknowns: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \sigma_x$, and σ_y

(Recall the generalized Hooke's law $\vec{\vec{E}} = \frac{9 \cdot BM \cdot \vec{\vec{\Sigma}} - (3 \cdot BM - 2 \cdot SM) \cdot tr\vec{\vec{\Sigma}} \cdot \vec{\vec{1}}}{18 \cdot BM \cdot SM}$).

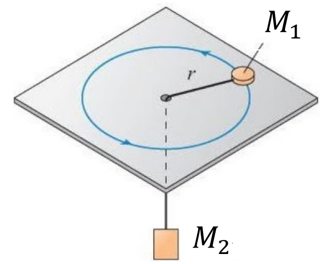
*(**HINT: two out of the five are easily reasoned to be zeroes**, start with figuring out which ones)*

b) What is the change in volume of the cube (ΔV/V)?

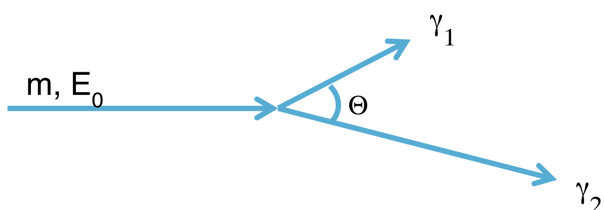
5. ***(6 pts)*** A beam of light has frequency ω_0 and is travelling along the x-axis in some inertial system. It strikes a mirror that also moves along the x-axis with speed v. Find the frequency of the reflected light



6. ***(6 pts)*** Mass M_1 can move without friction on the table. It is attached by a light string to a mass M_2 as shown in the figure.
a) write the Hamiltonian of the system and the Hamilton equations of motion (but do not solve them)
b) find the solution to the Hamilton equation of motion where M_2 is at rest

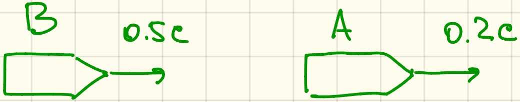


7. ***(6 pts)*** A particle of known mass *m* has energy E_0 in the lab frame. It decays into two photons. The opening angle between the photons in the lab frame is Θ. What are the energies of the two photons?

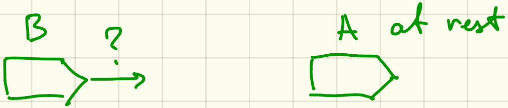


(1)

(K):



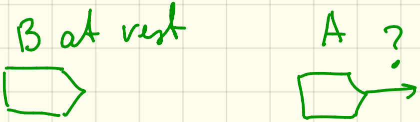
a) (K'):



(K') moves with speed 0.2c in (K)

$$v_{B'} = \frac{0.5 - 0.2}{1 - 0.5 \cdot 0.2} c = \frac{0.3}{0.9} c = \frac{c}{3}$$

b) (K'')



(K'') moves with speed 0.5c in (K)

$$v_{A''} = \frac{0.2 - 0.5}{1 - 0.5 \cdot 0.2} c = -\frac{0.3}{0.9} c = -\frac{c}{3}$$

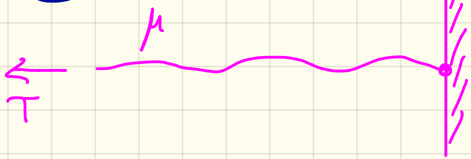
②

$$\frac{d\sigma}{d\Omega} = A^2 \cdot \cos^2\varphi$$

$$\sigma_{tot} = \int A^2 \cos^2\varphi \, d\Omega = A^2 \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^{\pi} \sin\theta \, d\theta =$$

$$= A^2 \cdot \frac{2\pi}{2} \cdot 2 = 2\pi A^2$$

3



Incoming wave:

$$f_I = A\cos(kx - \omega t) \qquad k = \frac{\omega}{c}, \; \omega = \sqrt{\frac{T}{\mu}}$$

reflected wave $\quad f_R = B\cos(kx + \omega t + \delta)$

so full wave

$$f = A\cos(kx - \omega t) + B\cos(kx + \omega t + \delta)$$

Boundary condition: $\quad f(0, t) = 0$

$$A\cos(-\omega t) + B\cos(\omega t + \delta) = 0$$

a)

$$A = B, \quad \delta = \pi$$

Reflected wave $\quad \boxed{f_R = A\cos(kx + \omega t + \pi)}$

same amplitude, frequency, phase $= \pi$

(3)

b) Full wave

$$f = A\cos(kx-\omega t) + A\cos(kx+\omega t+\pi) =$$

$$= A\left(\cos(kx-\omega t) - \cos(kx+\omega t)\right) =$$

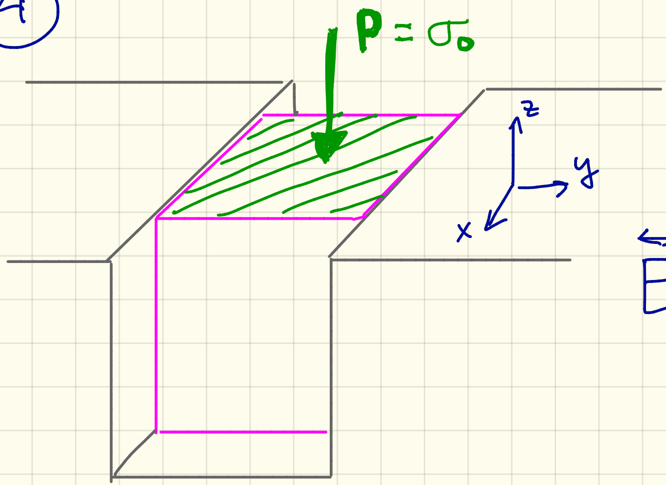
$$= A\left(\cancel{\cos kx \cos\omega t} + \sin kx \sin\omega t - \cancel{\cos kx \cos\omega t} + \sin kx \sin\omega t\right)$$

$$\boxed{f = 2A \sin kx \cdot \sin\omega t}$$

twice the amplitude

same frequency and wave number.

④



$$\overleftrightarrow{\Sigma} = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & -\sigma_0 \end{pmatrix}$$

$$\overleftrightarrow{E} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

$\varepsilon_y = 0 \rightarrow$ can not expand! $\sigma_x = 0 \rightarrow$ no pressure along x

$$\overleftrightarrow{E} = \frac{1}{18\,BM\,SM} \cdot \left(9BM\,\overleftrightarrow{\Sigma} - (3BM - 2SM)(\sigma_y - \sigma_0) \cdot \overleftrightarrow{1} \right)$$

$$\varepsilon_x = \frac{-1}{18\,BM\,SM} (3BM - 2SM)(\sigma_y - \sigma_0)$$

$$0 = \frac{1}{18\,BM\,SM} \left(9BM \cdot \sigma_y - (3BM - 2SM)(\sigma_y - \sigma_0) \right)$$

$$\begin{cases} 0 = \dfrac{1}{18\,BM\,SM}\left(9BM\cdot\sigma_y - (3BM-2SM)(\sigma_y-\sigma_0)\right) \\ \varepsilon_z = \dfrac{1}{18\,BMSM}\left(9BM(-\sigma_0) - (3BM-2SM)(\sigma_y-\sigma_0)\right) \end{cases}$$

$$0 = \sigma_y\left(9BM - 3BM + 2SM\right) + \sigma_0\left(3BM-2SM\right)$$

$$\boxed{\sigma_y = -\sigma_0 \cdot \frac{3BM-2SM}{6BM+2SM}}$$

$$\sigma_y - \sigma_0 = -\sigma_0\left(\frac{3BM-2SM}{6BM+2SM}+1\right) = -\sigma_0\,\frac{9BM}{6BM+2SM}$$

$$\varepsilon_x = \frac{-1}{18\,BMSM}\left(3BM-2SM\right)\cdot(-\sigma_0)\cdot\frac{9BM}{6BM+2SM} = \frac{\sigma_0}{2SM}\,\frac{3BM-2SM}{6BM+2SM}$$

$$\boxed{\varepsilon_x = \frac{\sigma_0}{2SM}\,\frac{3BM-2SM}{6BM+2SM}}$$

$$\varepsilon_z = \frac{-\sigma_0}{18\,BM\cdot SM}\left(9BM - (3BM-2SM)\cdot\frac{9BM}{6BM+2SM}\right) =$$

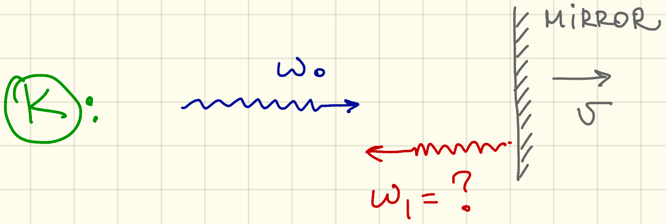
$$= -\frac{\sigma_0}{2SM}\left(1 - \frac{3BM-2SM}{6BM+2SM}\right) = -\frac{\sigma_0}{2SM}\,\frac{3BM+4SM}{6BM+2SM}$$

$$\boxed{\varepsilon_z = -\frac{\sigma_0}{2SM}\cdot\frac{3BM+4SM}{6BM+2SM}}$$

$$\mathrm{tr}\,\overleftrightarrow{E} = \frac{\sigma_0}{2SM}\,\frac{1}{6BM+2SM}\left(\cancel{3BM}-2SM-\cancel{3BM}-4SM\right) = -\frac{\sigma_0\cdot 6\cancel{SM}}{2\cancel{SM}\cdot(6BM+2SM)}$$

$$\boxed{\Delta V/V = -\sigma_0\cdot\frac{3}{6BM+2SM}}$$

5



Boost into the frame of the mirror:

4-vector of incident photon:

in K: $\left(\frac{\omega_0}{c}, \frac{\omega_0}{c}, 0, 0\right)$

in K^*: $\left(\frac{\omega^*}{c}, \frac{\omega^*}{c}, 0, 0\right)$

Lorentz transformation:

$$\frac{\omega_0}{c} = \gamma\left(\frac{\omega^*}{c} + \beta\frac{\omega^*}{c}\right) = \frac{\omega^*}{c}\cdot\gamma\cdot(1+\beta)$$

$$\frac{\omega^*}{c} = \frac{\omega_0}{c}\,\frac{1}{\gamma}\cdot\frac{1}{1+\beta}$$

4-vector of reflected photon:

in K^*: $\left(\frac{\omega^*}{c}, -\frac{\omega^*}{c}, 0, 0\right)$

in K: $\left(\frac{\omega_1}{c}, -\frac{\omega_1}{c}, 0, 0\right)$

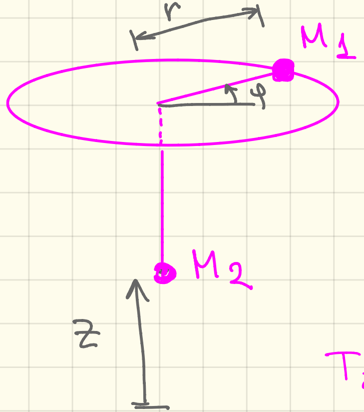
Lorentz transformation:

$$\frac{\omega_1}{c} = \gamma\left(\frac{\omega^*}{c} + \beta\left(-\frac{\omega^*}{c}\right)\right) = \frac{\omega^*}{c}\gamma(1-\beta)$$

$$\frac{\omega_1}{c} = \frac{\omega_0}{c}\,\frac{1}{\gamma}\cdot\frac{1}{1+\beta}\,\gamma\cdot(1-\beta)$$

$$\boxed{\omega_1 = \omega_0\,\frac{1-\beta}{1+\beta}}$$

⑥



2 deg. of freedom: z, φ

choose $z=0$ so that if

$z=0$ then $r=0$.

$T_2 = \frac{1}{2} m_2 \dot{z}^2 \qquad T_1 = \frac{1}{2} m_1 (\dot{z}^2 + z^2 \dot{\varphi}^2) \qquad U = m_2 g z$

$$\mathcal{L} = \frac{1}{2}(m_2+m_1)\dot{z}^2 + \frac{1}{2} m_1 z^2 \dot{\varphi}^2 - m_2 g z$$

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = (m_1+m_2)\dot{z} \qquad \dot{z}^2 = \frac{p_z^2}{(m_1+m_2)^2}$$

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m_1 z^2 \dot{\varphi} \qquad \dot{\varphi}^2 = \frac{p_\varphi^2}{m_1^2 z^4}$$

$$\mathcal{H} = \frac{p_z^2}{2(m_1+m_2)} + \frac{p_\varphi^2}{2 m_1 z^2} + m_2 g z$$

a)

$$\mathcal{H} = \frac{p_z^2}{2(m_1+m_2)} + \frac{p_\varphi^2}{2 m_1 z^2} + m_2 g z$$

z: $$\frac{\partial \mathcal{H}}{\partial p_z} = \dot{z} = \frac{p_z}{m_1+m_2}$$

φ: $$\frac{\partial \mathcal{H}}{\partial p_\varphi} = \dot{\varphi} = \frac{p_\varphi}{m_1 z^2}$$

$$\frac{\partial \mathcal{H}}{\partial z} = -\dot{p}_z = m_2 g - \frac{p_\varphi^2}{m_1 z^3}$$

$$\frac{\partial \mathcal{H}}{\partial \varphi} = -\dot{p}_\varphi = 0.$$

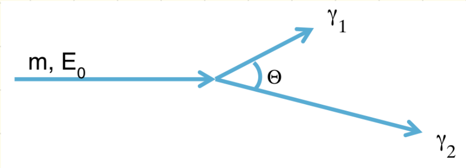
b)

$$\dot{p}_z = 0 \Rightarrow \frac{p_\varphi^2}{m_1 z^3} = m_2 g \Rightarrow \frac{m_1^2 z^4 \cdot \dot{\varphi}^2}{m_1 z^3} = m_2 g$$

$$m_1 z \dot{\varphi}^2 = m_2 g$$

$$\dot{\varphi} = \sqrt{\frac{m_2}{m_1} g}\ \frac{1}{\sqrt{z}}$$

7



$$\underline{P_0} = \underline{P_1} + \underline{P_2}$$

$$m^2 = 2 E_{\gamma_1} E_{\gamma_2} (1 - \cos\theta)$$

$E_{\gamma_1} + E_{\gamma_2} = E_0$, so

$$E_{\gamma_1}(E_0 - E_{\gamma_1}) = \frac{m^2}{2(1-\cos\theta)}$$

$$E_{\gamma_1}^2 - E_0 E_{\gamma_1} + \frac{m^2}{2(1-\cos\theta)} = 0 \quad \leftarrow \text{quad. eqn for } E_{\gamma_1}$$

$$\boxed{E_{\gamma_1} = \frac{E_0}{2} \pm \sqrt{\frac{E_0^2}{4} - \frac{m^2}{2(1-\cos\theta)}}}$$