Relativity Notes

From experiment: speed of light is the same in every inertial frame.

follows: events simultaneous in one frame may not be simultaneous in others.

Science is study of cause and effect.

IS THIS THE END OF SCIENCE?

Event: (ct, x, y, z) ← depends on a choice of frame

Interval between two events:

$$\Delta S^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Suppose Event 1: light emitted at (ct_1, x_1, y_1, z_1)

Event 2: light arrived at (ct_2, x_2, y_2, z_2)

$$\Delta S^2 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 =$$

$$= c^2 \Delta t^2 - \Delta \ell^2 = c^2 \Delta t^2 - (c \Delta t)^2 = 0$$

Same events in another frame: $(ct_1', x_1', y_1', z_1')$, $(ct_2', x_2', y_2', z_2')$

speed of light is universal.

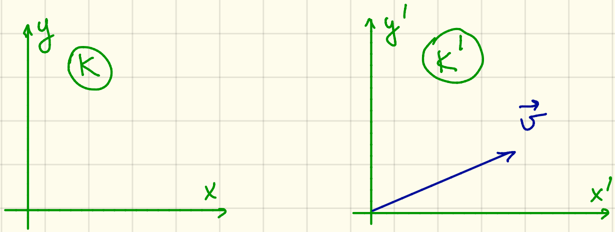
$$\Delta S'^2 = c^2 \Delta t'^2 - \Delta \ell'^2 = c^2 \Delta t'^2 - (c \Delta t')^2 = 0$$

may be different from Δt, $\Delta \ell$

$\boxed{\text{if } \Delta S^2 = 0 \text{ in one inertial frame it is zero in } \underline{\underline{\text{every}}} \text{ inertial frame}}$

What about non-zero intervals?

Take infinitesimal interval ds^2 in frame K



K, K' → inertial frames

K' moves with velocity $\vec{v}$ w.r.t. K

Suppose in frame K' the same interval is ds'^2

How can they be related?

$$ds' = f \cdot ds \quad \longrightarrow \text{as } ds \to 0 \text{ so must } ds'$$

f is some function

f can not depend on x, y, z → momentum conserves!

can not depend on t → energy conserves!

can not depend on direction of $\vec{v}$ → angular momentum conserves!

f can only depend on $|\vec{v}|$

So, $ds' = f(|v|)\,ds$, $f(0) = 1$.

Now take 3 frames: K, K', K''

K' speed in K is v_1

K'' speed in K is v_2

K'' speed in K' is v_{12}

For an interval ds:

$$\left.\begin{aligned} ds' &= f(v_1)\cdot ds \\ ds'' &= f(v_2)\cdot ds \\ ds'' &= f(v_{12})\, ds' \end{aligned}\right\} \rightarrow \quad \begin{aligned} f(v_{12})\cdot f(v_1)\cdot ds &= f(v_2)\cdot ds \\ f(v_{12}) &= \frac{f(v_2)}{f(v_1)} \end{aligned}$$

But! v_{12} depends on the angle between v_1 and v_2!

$\rightarrow$ f has to be constant. $\Rightarrow f \equiv 1$.

Thus:

$$ds^2 = (c\,dt)^2 - dx^2 - dy^2 - dz^2$$

is the same in every frame!

Compare to 3-D space and Newton mechanics:

coordinates of both ends of a ruler may be different in different frames, but its length is the same.

Hypothesis: we live <u>not</u> in 3-D with absolute time and metrics $dl^2 = dx^2 + dy^2 + dz^2$, but in 4-D Minkowski space with metrics

$$ds^2 = (c\,dt)^2 - dx^2 - dy^2 - dz^2$$

Time and length may be relative, but the interval is <u>absolute</u>!

Frames K and K'

Event 1: (ct_1, x_1, y_1, z_1) $\quad$ $(ct_1', x_1', y_1', z_1')$

Event 2: (ct_2, x_2, y_2, z_2) $\quad$ $(ct_2', x_2', y_2', z_2')$

Interval $\Delta s^2 = c^2 \Delta t^2 - \Delta \ell^2 = c^2 \Delta t'^2 - \Delta \ell'^2 = \Delta s'^2$

Suppose $\Delta \ell = 0$ (two events happened at the same point)

then $\Delta s^2 > 0$ in every system

Suppose $\Delta t = 0$ (two events happened at the same time)

then $\Delta s^2 < 0$ in every system

if $\Delta s^2 < 0$ the events can not be causally related

if $\Delta s^2 > 0$, they can be. <u>Causality is safe.</u>

$\Delta s^2 > 0$: time-like interval ← more interesting, since cause-effect relationships are time-like.

$\Delta s^2 < 0$: space-like interval

Note: we defined interval as

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \ell^2$$

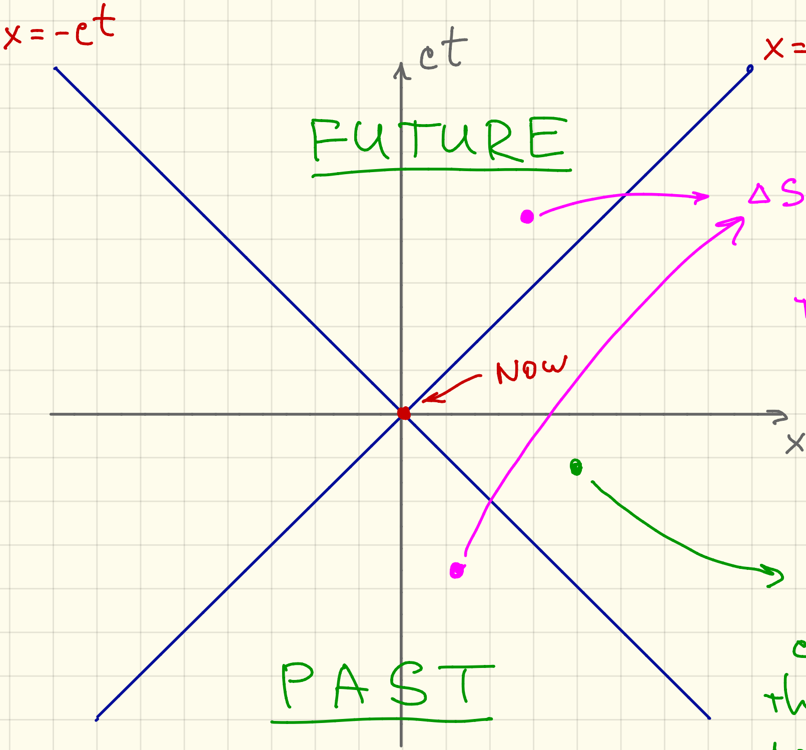
Your text book defined it as

$$\Delta s^2_{BOOK} = \Delta \ell^2 - c^2 \Delta t^2 = - \Delta s^2$$

→ both are fine, but for the latter time like intervals are imaginary numbers.

→ many concepts & derivations are MUCH easier if a time-like interval is a real number

Future and past are absolute



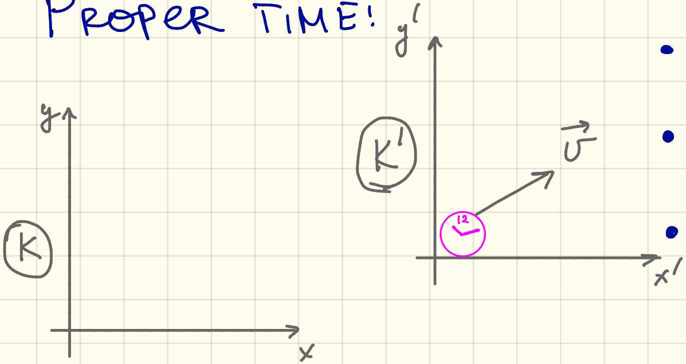
$\Delta s^2 > 0$: time-like

CAN FIND A FRAME where the events happen at the same point, separated by time $\tau = \frac{\Delta s}{c}$

$\Delta s^2 < 0$: space-like

CAN FIND a FRAME where the events happen at the same time, separated by distance $l = -i\,\Delta s$, and therefore can not be causaly related.

PROPER TIME!



- K and K' are inertial frames.
- origin of K' has speed $\vec{v}$ in K.
- a clock is placed at the origin of K'

in frame K, after time dt the clock moves by $v\,dt = \sqrt{dx^2 + dy^2 + dz^2}$

Interval between two positions:

$$ds^2 = (c\,dt)^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

Same interval in K':

$$ds'^2 = ds^2 = c^2 dt'^2$$

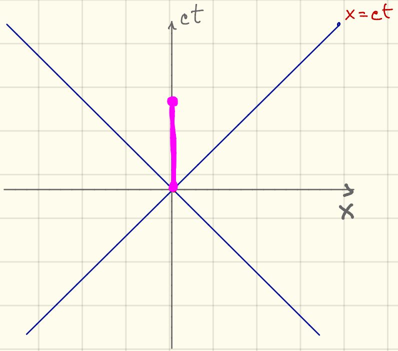
the clock is at rest in K'

Thus: $dt' = dt \cdot \sqrt{1 - \frac{v^2}{c^2}}$

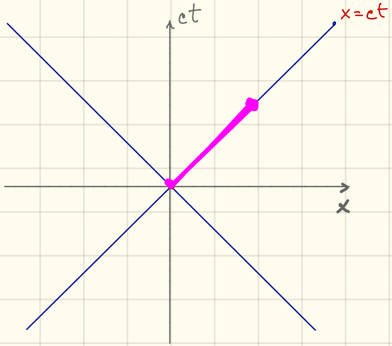
- <u>Moving clocks run slow</u>
- TIME IN REST FRAME OF A BODY IS CALLED <u>PROPER TIME</u>

World lines: trajectories in Minkowski space

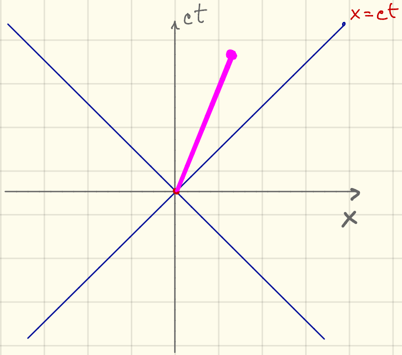
BODY AT REST:

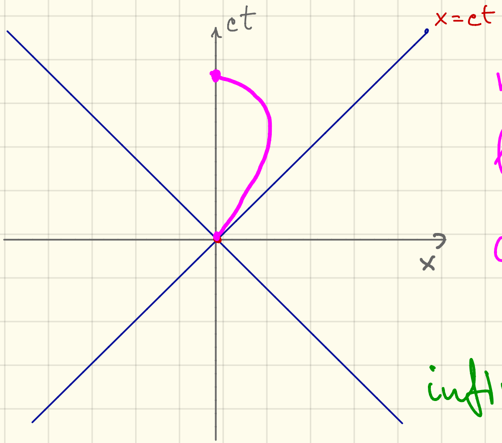


uniform motion with $v=c$



uniform motion, $v<c$

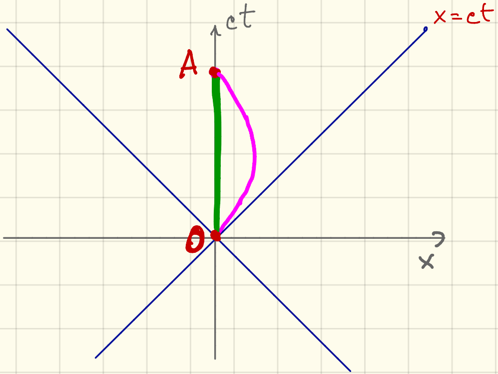




non-uniform motion:

body moves away from the origin and then back

World line can be looked at as infinite sum of infitesimal uniform motions.



If the bodies traveling along green and magenta world lives are clocks:

$$dt' = dt\sqrt{1-\frac{v^2}{c^2}} \qquad v = v(t)$$

moving clock — clock at rest

Integrate along the world line to get time elapsed

$$t'_A - t'_0 = \int_{t_0}^{t_A} dt\sqrt{1-\frac{v^2}{c^2}} < t_A - t_0 \longrightarrow$$ TRAVELING TWIN is younger!

Proper time

$$\Delta t = \int_{t_1}^{t_2} dt = \int_0^A \frac{ds}{c}$$

line integral along the world line

Thus: $\int ds$ is largest along a straight line

(very different from Eucledian 3d: distance is shortest along a straight line)

Minkowski space: ct, x, y, z

Interval $s^2 = c^2t^2 - x^2 - y^2 - z^2$ ← analog of distance.

Coordinate transformation in different frames:

- shifts along $ct, x, y, z \rightarrow ct + c\Delta t, x+\Delta x, y+\Delta y, z+\Delta z$

  → interval is constant.

- rotations in xy, yz, or xz planes:

  i.e. in xy: $\begin{cases} ct = ct' \\ x = x'\cos\theta + y'\sin\theta \\ y = -x'\sin\theta + y'\cos\theta \\ z = z' \end{cases}$ → interval is constant.

- rotations in xt, yt, and zt

  → need to keep interval constant

  → linear combinations of coordinates

$$\cosh x = \frac{1}{2}\left[e^{x} + e^{-x}\right] \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\sinh x = \frac{1}{2}\left[e^{x} - e^{-x}\right]$$

Rotation in xt plane:

$$\begin{cases} ct = ct' \cosh \psi + x' \sinh \psi \\ x = ct' \sinh \psi + x' \cosh \psi \\ y = y' \\ z = z' \end{cases}$$

→ interval is constant

→ coordinate transformation between moving frames is a rotation in Minkowski space

What is the meaning of rotation angle ψ?

$\rightarrow$ track origin of frame K' in K:

$$x' = y' = z' = 0 \Rightarrow \begin{cases} ct = ct' \cosh\psi \\ x = ct' \sinh\psi \end{cases}$$

$$\Rightarrow \frac{\sinh\psi}{\cosh\psi} = \tanh\psi = \frac{x}{ct} = \frac{v}{c} = \beta$$

$$\cosh\psi = \frac{1}{\sqrt{1-\tanh^2\psi}} = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma$$

$$\sinh\psi = \beta\cdot\gamma$$

Thus:

$$\begin{cases} ct = \gamma\,(ct' + \beta x') \\ x = \gamma\,(\beta\cdot ct' + x') \\ y = y' \\ z = z' \end{cases}$$

$\leftarrow$ Lorentz transform

rotations do not commute!

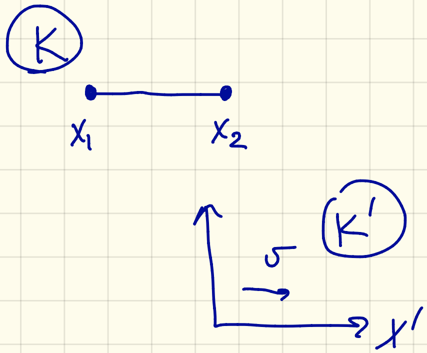
$$\left.\begin{aligned} x &= \gamma\,(x' + \beta \cdot ct') \\ y &= y' \\ z &= z' \\ ct &= \gamma\,(\beta x' + ct') \end{aligned}\right\} \quad \overset{\text{v.s.}}{\longleftrightarrow} \quad \left\{\begin{aligned} x &= \frac{x' + v t'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - v^2/c^2}} \end{aligned}\right.$$

Our units of time are wrong / obsolete!

"Natural" units will have $c \equiv 1$

$(ct \equiv t\ ,\ \beta \equiv v)$

Proper length of an object is its length in a system where it is at rest



in K: $\ell_0 = x_2 - x_1$

in K':

$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} \; ; \; x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

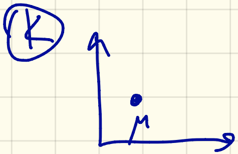
$$x_2 - x_1 = \ell_0 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} = \frac{\ell}{\sqrt{1 - v^2/c^2}}$$

$$\ell = \ell_0 \cdot \sqrt{1 - v^2/c^2}$$

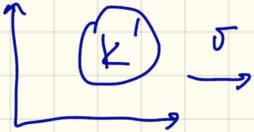
→ length is largest in the rest system

Proper time (again!)

two events (i.e. particle decay)



$$t_2 - t_1 = \tau_0$$



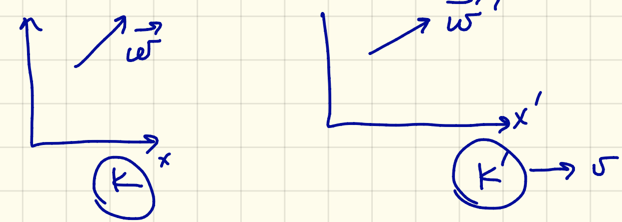
$$t_2 = \frac{t_2' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}} \; ; \qquad t_1 = \frac{t_1' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}}$$

$$t_2 - t_1 = \tau_0 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}} = \frac{\tau}{\sqrt{1 - v^2/c^2}}$$

$$\tau = \sqrt{1 - v^2/c^2}\, \tau_0$$ ← time dilation

moving clocks run slower

Speed transformation:



$$dx = \frac{dx' + v\,dt'}{\sqrt{1 - v^2/c^2}} \qquad dy = dy'$$

$$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1 - v^2/c^2}} \qquad dz = dz'$$

$$w_x = \frac{dx}{dt} \qquad w_x' = \frac{dx'}{dt'}$$

$$w_y = \frac{dy}{dt} \qquad w_y' = \frac{dy'}{dt'}$$

$$w_z = \frac{dz}{dt} \qquad w_z' = \frac{dz'}{dt'}$$

$$\vec{w} = \frac{d\vec{r}}{dt}$$

$$w_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'} = \frac{w_x' + v}{1 + \frac{v w_x'}{c^2}}$$

$$w_y = \frac{dy}{dt} = \frac{dy'\sqrt{1 - v^2/c^2}}{dt' + \frac{v}{c^2}dx'} = \frac{w_y' \cdot \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v w_x'}{c^2}}$$

$$w_z = \frac{w_z'\sqrt{1 - v^2/c^2}}{1 + \frac{v w_x'}{c^2}}$$

$\rightarrow$ if $c \to \infty$:
$$w_x = w_x' + v$$
$$w_y = w_y'$$
$$w_z = w_z'$$

if $v/c \ll 1$

$$\frac{1}{1+\frac{v w_x'}{c^2}} = 1 - \frac{v w_x'}{c^2} + \ldots$$

$$w_x = (w_x' + v)\left(1 - \frac{v w_x'}{c^2}\right) = w_x' + v - \frac{v w_x'^2}{c^2} - \cancel{\frac{v^2 w_x'}{c^2}}$$

$$w_y = w_y'\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\left(1 - \frac{v w_x'}{c^2}\right) = w_y' - \frac{v w_x' w_y'}{c^2}$$

$$w_z = w_z' - \frac{v}{c^2} w_x' w_z'$$

Or, in a coordinate-free form

$$\boxed{\vec{w} = \vec{w}' - \frac{1}{c^2}\vec{w}'(\vec{v}\cdot\vec{w}')}$$

$\leftarrow$ note asymmetry in $\vec{v}$ and $\vec{w}'$

Lorentz transformations are not commutative!

4-vectors

$x^0 = ct$

$x^1 = x$

$x^2 = y$

$x^3 = z$

A^i vector, if it transforms according to Lorentz transformations.

$A^i : (A^0, A^1, A^2, A^3) \to$ contravariant

$A_i = (A^0, -A^1, -A^2, -A^3) \to$ covariant

can indicate 4-vector by underscore: $\underline{A}$

Scalar product:

$$A_i A^i = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$

omit the sum over co/contra indices $A^i A_i \equiv \sum_{i=0}^{3} A^i A_i$

$$A^i = (A_0, \vec{A}) \quad , \quad A_i = (A^0, -\vec{A})$$

3-d vector

Another way to write Lorentz transformation: rotation matrix

$$\begin{array}{c} ct \\ x \\ y \\ z \end{array} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\cdot\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} , \quad \text{or} \quad \underline{x} = \Lambda \cdot \underline{x}'$$

Performing sequential boosts: multiply corresponding Λ's

Boost from K to K' and back (should get unit matrix)

$$\begin{bmatrix} \gamma & \gamma\cdot\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma^2-\gamma^2\beta^2 & \cancel{-\gamma^2\beta+\gamma^2\beta} & 0 & 0 \\ \cancel{\gamma^2\beta-\gamma^2\beta} & -\gamma^2\beta^2+\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\gamma^2(1-\beta^2) = \frac{1}{1-\beta^2}(1-\beta^2) = 1$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4-tensor — 16 values A^{ik} that transform as products of components of two 4-vectors: $A^{ik} = B^i \cdot C^k$

$A^{ik} \to$ contravariant A_{ik} - covariant $A^i{}_k$, $A_i{}^k \to$ mixed.

$A^{00} = A_{00}$ $A_{01} = -A^{01}$ $A_{11} = A^{11}$ $A_1{}^1 = -A^{11}$

position of indices is important

$A^i{}_k = A^{ik}$ $\leftarrow$ even if true in one system, will not be true in another.

Unit tensor:

$\delta_i{}^k A^i = A^k$ for all vectors

$\hookrightarrow \left.\begin{array}{l} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{array}\right\}$ $\boxed{\to \text{invariant in all systems!}}$

raise / lower indicies in δ_i^k

$$g_{ik} = g^{ik} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$ → metric tensor,

same in all coordinate systems.

$$A_i B^i = g_{ik} A^i B^k$$ ← scalar product.

→ another way to express index lowering/raising rules.

→ vector · vector = scalar $\quad A^i B_i = C$

→ scalar × vector = vector $\quad A \cdot B^i = C^i$

→ $A^i B^j = C^{ij}$ → tensor

→ pseudovector = $(ct, \vec{a})$
pseudoscalar, etc... } same, as in 3-D.

Quotient rule:
if C scalar
A^i vector
and
$A^i K_i = C$
Then K_i is also a vector

Example of speed addition & quotient rule use:

$$w = \frac{w' + v}{1 + \frac{v w'}{c^2}} \qquad (\text{if } w' \parallel v)$$

if $w' = c$: $$w = \frac{c + v}{1 + \frac{v c}{c^2}} = c \, \frac{1 + v/c}{1 + v/c} = c$$

$\rightarrow$ so, what, nothing happens to light?

$$E = E_0 \cdot \cos(\vec{k} \cdot \vec{r} - \omega t)$$

$\vec{k}$ is wave vector

ω is frequency

$\frac{\omega}{|k|} = c$ - speed of (wave) (light)

$$\omega t - \vec{k} \cdot \vec{r} = \left(\frac{\omega}{c}, \vec{k}\right) \cdot (ct, \vec{r})$$

since $(ct, \vec{r})$ is a 4-vector

$\left(\frac{\omega}{c}, \vec{k}\right)$ is also 4-vector

So, assuming

$$\underline{K}' = \left(\frac{\omega'}{c}, \vec{k}'\right) \qquad \frac{\omega'}{k'} = c \Rightarrow k' = \frac{\omega'}{c}$$

Lorentz transform along $\hat{x}$

$$k^0 = \gamma\left(k'^0 + \beta k'^1\right) = \gamma \frac{\omega'}{c} + \gamma \cdot \beta \frac{\omega'}{c} \cos\theta$$

$$\frac{\omega}{c} = \frac{\omega'}{c} \gamma \cdot \left(1 + \frac{v}{c} \cos\theta\right)$$

$$\omega = \frac{\omega_0 \left(1 + \frac{v}{c} \cos\theta\right)}{\sqrt{1 - v^2/c^2}}$$

red shift

if $v \parallel -x$, $\cos\theta = -1$

$$\omega = \omega_0 \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \boxed{\omega_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}}$$

if $v \parallel x$, $\cos\theta = +1$

blue shift

$$\omega = \omega_0 \frac{1 + \frac{v}{c}}{\sqrt{\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right)}} = \boxed{\omega_0 \sqrt{\frac{1 + v/c}{1 - v/c}}}$$

Kinematics

t_0 is proper time

$$ds = c\,dt_0 = c\,dt\sqrt{1-v^2/c^2}$$

4-position: $\underline{x} = (ct, \vec{r})$

4-speed: $$\underline{u} = \frac{d}{dt_0}\underline{x} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}(ct,\vec{r}) = \gamma\cdot(c,\vec{v})$$

$$u^i u_i = \gamma^2(c^2 - v^2) = c^2 \leftarrow$$ 4-speed has constant "length"

4-acceleration: $$\underline{\alpha} = \frac{d}{dt_0}\underline{u}$$

usefull derivatives:

$$\frac{d}{dt}(\gamma) = \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = -\frac{1}{2}\gamma^3\left(-2\frac{v}{c^2}\right)\frac{dv}{dt} = \boxed{\gamma^3\frac{v}{c^2}\frac{dv}{dt} = \frac{d\gamma}{dt}}$$

$$\frac{d}{dt}(\beta\gamma) = \frac{1}{c}\frac{d}{dt}\frac{v}{\sqrt{1-v^2/c^2}} = \frac{1}{c}\left[\gamma\frac{dv}{dt} + \gamma^3\frac{v^2}{c^2}\frac{dv}{dt}\right] = \frac{1}{c}\gamma^3\frac{dv}{dt}\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right]$$

$$\boxed{\frac{d}{dt}(\beta\gamma) = \frac{1}{c}\gamma^3\frac{dv}{dt}}$$

$$u^i u_i = c^2 \Rightarrow \frac{d}{dt_0}(u^i u_i) = 0 \Rightarrow 2u^i \alpha_i = 0$$

4-acceleration is $\perp$ to 4-speed

motion along 1-d (x):

$$\boxed{\underline{u} = \gamma(c, v, 0, 0)}$$

$$\alpha^2 = \alpha^3 = 0$$

$$\alpha^0 = \frac{1}{\sqrt{1-v^2/c^2}} \frac{d}{dt} \frac{c}{\sqrt{1-v^2/c^2}} = \gamma c \frac{d}{dt}(\gamma) = \gamma c \cdot \gamma^3 \frac{v}{c^2} \frac{dv}{dt} = \gamma^4 \beta \frac{dv}{dt}$$

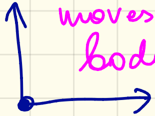
$$\alpha^1 = \frac{1}{\sqrt{1-v^2/c^2}} \frac{d}{dt} \frac{v}{\sqrt{1-v^2/c^2}} = \gamma c \frac{d}{dt}(\beta\gamma) = \gamma c \cdot \frac{1}{c}\gamma^3 \frac{dv}{dt} = \gamma^4 \frac{dv}{dt}$$

$$\boxed{\underline{\alpha} = \gamma^4 \frac{dv}{dt}(\beta, 1, 0, 0)}$$

Constant acceleration in 1-d:

$a = \text{const}$ in co-moving system : inertial system in which speed of the object is momentarily zero.

K



K′

moves with the body, $v(t)$

$$\underline{u} = \gamma\,(c, v_{(t)}, 0, 0) \qquad \underline{u} = (c, 0, 0, 0)$$

$$\underline{\alpha} = \gamma^4 \frac{dv}{dt}\left(\frac{v}{c}, 1, 0, 0\right) \qquad \underline{\alpha} = (0, a, 0, 0)$$

$\alpha_i \alpha^i$ is invariant !

(same in all inertial systems)

$$-\gamma^8 \left(\frac{dv}{dt}\right)^2 \underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{= \gamma^{-2}} = -a^2$$

$$\gamma^3 \frac{dv}{dt} = a \to \text{equation on } v(t)$$

$$\frac{dv}{\left(1+ v^2/c^2\right)^{3/2}} = a\,dt \qquad v\big|_{t=0} = 0$$

$$\int \frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{v}{\sqrt{1-v^2/c^2}} + \text{const}$$

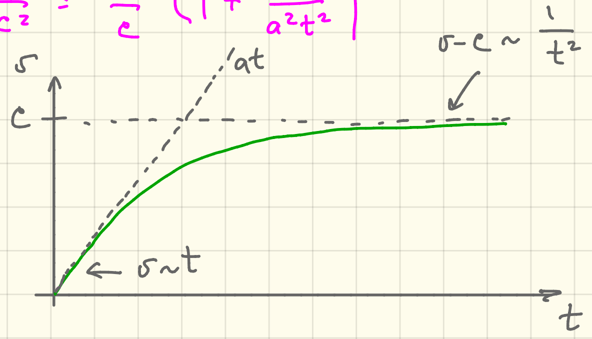
$$\frac{v}{\sqrt{1-v^2/c^2}} = at\,, \quad v^2 = a^2t^2 - \frac{a^2t^2}{c^2}\cdot v^2\,, \quad v^2\left(1+\frac{a^2t^2}{c^2}\right) = a^2t^2$$

$$\boxed{v = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}}$$

for $t \to 0$, $v \ll c$: $v = at$ → classical

for $t \to \infty$ $\sqrt{1+\frac{a^2t^2}{c^2}} = \frac{at}{c}\left(1+\frac{c^2}{a^2t^2}\right)$

$$v = c\left(1-\frac{c^2}{a^2t^2}\right)$$



Now: the dynamics

classical:

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt \quad \leftarrow \text{minimize } S$$

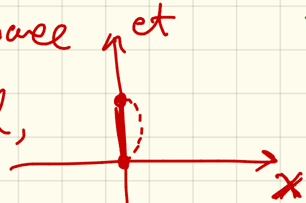
relativistic

$\alpha > 0$ scalar

$$S = -\alpha \int_a^b ds \quad \leftarrow$$ The only Lorentz-invariant way to write action of free body.

because of metrics of space

$\frac{1}{c}\int_a^b ds =$ time interval,



longest for "shortest" path

$$S = -\alpha \int_{t_1}^{t_2} c\sqrt{1-\frac{v^2}{c^2}}\, dt$$

$$\rightarrow \quad \mathcal{L} = -\alpha c \sqrt{1-\frac{v^2}{c^2}}$$

for $v \ll c$, $\mathcal{L} = -\alpha c \left(1 - \frac{v^2}{2c^2}\right) = -\alpha c + \frac{\alpha}{c} \cdot \frac{v^2}{2} \longleftrightarrow \frac{mv^2}{2}$

Therefore, $\alpha = mc$, so $\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2}$

Momentum:

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}}$$

$$p_x = \frac{\partial \mathcal{L}}{\partial v_x} = -mc^2\left(\frac{1}{2}\frac{-1/c^2}{\sqrt{1-v^2/c^2}}\cdot\frac{d}{dv_x}\left(v_x^2+v_y^2+v_z^2\right)\right) = \frac{m v_x}{\sqrt{1-v^2/c^2}}$$

Similarly,

$$p_y = \frac{\partial \mathcal{L}}{\partial v_y} = \frac{m v_y}{\sqrt{1-v^2/c^2}}, \quad p_z = \frac{\partial \mathcal{L}}{\partial v_z} = \frac{m v_z}{\sqrt{1-v^2/c^2}}$$

So:

$$\underline{P} = m\cdot\underline{u} = m\gamma\,(c, \vec{v})$$ ← 4-vector.

What is $p^0 = m\gamma c$?

↳ i.e. what is the physics meaning of it?

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\cdot\vec{v} + mc^2\sqrt{1-\frac{v^2}{c^2}} = mc^2\,\frac{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}{\sqrt{1-v^2/c^2}} =$$

Energy!

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}} = \mathcal{E} = c\cdot m\cdot u_0$$

$$= m\gamma c^2 = c\cdot p^0$$

$$\boxed{\underline{P} = m\,\underline{u} = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)}$$

Energy-momentum 4-vector

For small speeds

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-v^2/c^2}} = mc^2\left(1+\frac{v^2}{2c^2}\right) =$$

$$= mc^2 + \frac{1}{2}mv^2$$

in classical, constant does not matter

classical kinetic energy!

Force: $\vec{F} = \dfrac{d\vec{p}}{dt}$

if $\vec{F} \parallel \vec{v}$ $(\parallel \hat{x})$ $\quad \dfrac{dp}{dt} = m \dfrac{d}{dt} \dfrac{v}{\sqrt{1-v^2/c^2}} = m\gamma^3 \dfrac{dv}{dt} = \underline{\underline{ma}}$

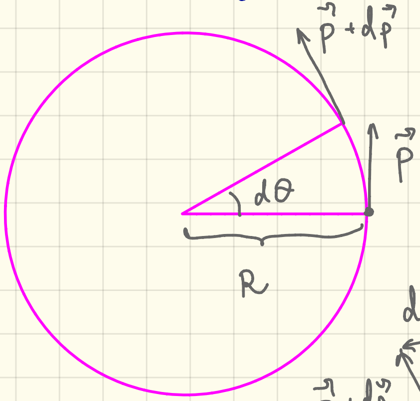
acceleration in co-moving system!

if $\vec{F} \perp \vec{v} \Rightarrow v = \text{const}$

$$\frac{d\vec{p}}{dt} = m \frac{d}{dt} \frac{\vec{v}}{\sqrt{1-v^2/c^2}} = \frac{m}{\sqrt{1-v^2/c^2}} \cdot \frac{d\vec{v}}{dt} = \vec{F}$$

$$\left.\begin{aligned} & d\vec{v} \cdot \vec{v} = 0 \\ & \quad " \\ & d(\vec{v}^2) = 0 \\ & v = \text{const} \end{aligned}\right\}$$

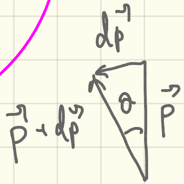
Relativistic cyclotron:



$$F = \frac{dp}{dt} = \frac{p \cdot d\theta}{dt} = p\omega = p \cdot \frac{v}{R}$$

if $p = mv$

$$F = m\frac{v^2}{R}$$



$$F_{mag} = q \cdot v \cdot B$$

$$q \cdot \cancel{v} \cdot B = p\frac{\cancel{v}}{R}, \qquad R = \frac{p}{q \cdot B}$$

Period of rotation:

$$T = \frac{2\pi R}{v}, \quad p = \gamma m v \Rightarrow T = 2\pi \frac{\gamma m \cancel{v}}{q \cdot B} \frac{1}{\cancel{v}}$$

$$T = 2\pi \frac{m}{q \cdot B} \cdot \gamma$$

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} \longrightarrow \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

$$\mathcal{H} = \vec{p}\cdot\vec{v} - \mathcal{L} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

$$\left.\right\} \underline{P} = m\underline{u} = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$$

$m \rightarrow$ is a <u>scalar</u>

Force becomes not a straightforward thing.

if uniform acceleration, $a = \gamma^3 \frac{dv}{dt}$

$$F_{\parallel} = \frac{dp}{dt} = m\gamma^3\frac{dv}{dt} = ma$$

but if $\vec{F} \perp \vec{v}$ $\quad \vec{F}_{\perp} = m\gamma\frac{d\vec{v}}{dt}$

$|v| = \text{const}$

$F_{\parallel}$ and $F_{\perp}$ look very different!

$$\underline{P} = m \cdot \underline{u} = m\gamma\,(c, \vec{v})$$

$$\underline{P}^2 = m^2 \underline{u}^2 = m^2 c^2 \rightarrow \text{scalar.}$$

in the rest frame, $\underline{P} = (mc, \vec{0}) = \left(\frac{\mathcal{E}_{rest}}{c}, \vec{0}\right)$

$$\underline{\mathcal{E}_{rest} = mc^2}$$

For small speeds

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = mc^2\left(1 + \frac{v^2}{2c^2}\right) = \underbrace{mc^2}_{} + \frac{1}{2}mv^2$$

in relativity, this constant has meaning!

That's what keeps $\left(\frac{\mathcal{E}}{c}, \vec{P}\right)$ 4-vector.

In relativity it's possible to have $m=0$!

$$\frac{m}{\sqrt{1-v^2/c^2}} \qquad \begin{array}{l} m \to 0 \\ v \to c \end{array}$$

$$\underline{P} = \left(\frac{\varepsilon}{c}, \vec{p}\right) \qquad \underline{P}^2 = 0 = \frac{\varepsilon^2}{c^2} - p^2 \Rightarrow \varepsilon = p \cdot c$$

recall 4-d wave vector:

$$\underline{K} = \left(\frac{\omega}{c}, \vec{k}\right) \qquad |\vec{k}| = \frac{\omega}{c}$$

$$\underline{K}^2 = \frac{\omega^2}{c^2} - k^2 = 0$$

in fact, $\underline{\mathbf{P}} = \hbar \underline{K}$, $\vec{p} = \hbar \vec{k}$, $\varepsilon = \hbar\omega$