From experiment: speed of light is the same in every inertial frame.

follows: events simultaneous in one frame may not be simultaneous in others.

Science is study of cause and effect.

Is this the end of science?
Event: \((ct, x, y, z)\) — depends on a choice of frame

Interval between two events:
\[\Delta S^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2\]

Suppose Event 1: light emitted at \((ct_1, x_1, y_1, z_1)\)
Event 2: light arrived at \((ct_2, x_2, y_2, z_2)\)
\[\Delta S^2 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = c^2 \Delta t^2 - \Delta l^2 = c^2 \Delta t^2 - (c \Delta t)^2 = 0\]

Same events in another frame: \((ct_1', x_1', y_1', z_1')\), \((ct_2', x_2', y_2', z_2')\)

\[\Delta S'^2 = c^2 \Delta t'^2 - \Delta l'^2 = c^2 \Delta t'^2 - (c \Delta t')^2 = 0\]

\[\text{if } \Delta S^2 = 0 \text{ in one inertial frame, it is zero in every inertial frame}\]

\[\text{may be different from } \Delta t, \Delta l\]

\[\text{speed of light is universal}\]
what about non-zero intervals?

Take infinitesimal interval $ds^2$ in frame $K$

Suppose in frame $K'$ the same interval is $ds'^2$

How can they be related?

$$ds' = f \cdot ds \quad \rightarrow \quad as \quad ds \rightarrow 0 \quad so \quad must \quad ds'$$

$f$ is some function

$f$ can not depend on $x, y, z$

can not depend on $t$

can not depend on direction of $\overrightarrow{v}$

$f$ can only depend on $|\overrightarrow{v}|$

$\rightarrow$ momentum conserves!

$\rightarrow$ energy conserves!

$\rightarrow$ angular momentum conserves!
So, \( ds' = f(|\mathbf{v}|) \, ds \), \( f(0) = 1 \).

Now take 3 frames: \( K \), \( K' \), \( K'' \)

\( K' \) speed in \( K \) is \( \mathbf{v}_1 \)

\( K'' \) speed in \( K \) is \( \mathbf{v}_2 \)

\( K'' \) speed in \( K' \) is \( \mathbf{v}_{12} \)

For an interval \( ds \):

\[
\begin{align*}
\begin{cases}
  ds' = f(\mathbf{v}_1) \cdot ds \\
  ds'' = f(\mathbf{v}_2) \cdot ds \\
  ds''' = f(\mathbf{v}_{12}) ds'
\end{cases}
\end{align*}
\]

\[
\int f(\mathbf{v}_{12}) \cdot f(\mathbf{v}_1) \cdot ds = f(\mathbf{v}_2) \cdot ds
\]

\[
f(\mathbf{v}_{12}) = \frac{f(\mathbf{v}_2)}{f(\mathbf{v}_1)}
\]

But! \( \mathbf{v}_{12} \) depends on the angle between \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \)!

\( \rightarrow f \) has to be constant. \( \Rightarrow f \equiv 1 \).
Thus:

\[ ds^2 = (c\, dt)^2 - dx^2 - dy^2 - dz^2 \]

is the same in every frame!

Compare to 3-D space and Newton mechanics:
coordinates of both ends of a ruler may be different in different frames, but its length is the same.

Hypothesis: we live not in 3-D with absolute time and metrics \( d\ell^2 = dx^2 + dy^2 + dz^2 \), but in 4-D Minkowski space with metrics

\[ ds^2 = (c\, dt)^2 - dx^2 - dy^2 - dz^2 \]
Time and length may be relative, but the
interval is **absolute**!

Frames $K$ and $K'$

Event 1: $(c t_1, x_1, y_1, z_1)$  $(c t'_1, x'_1, y'_1, z'_1)$

Event 2: $(c t_2, x_2, y_2, z_2)$  $(c t'_2, x'_2, y'_2, z'_2)$

Interval $\Delta s^2 = c^2 \Delta t^2 - \Delta l^2 = c^2 (t_2 - t_1)^2 - (l_2 - l_1)^2 = \Delta s'^2$

Suppose $\Delta l = 0$ (two events happened at the same point)

then $\Delta s^2 > 0$ in every system.

Suppose $\Delta t = 0$ (two events happened at the same time)

then $\Delta s^2 < 0$ in every system.

if $\Delta s^2 < 0$ the events can not be causally related

if $\Delta s^2 > 0$, they can be. **Causality is safe.**
\( \Delta s^2 > 0 \): time-like interval \( \leftarrow \) more interesting, since cause-effect relationships are time-like.

\( \Delta s^2 < 0 \): space-like interval

Note: we defined interval as

\[ \Delta s^2 = c^2 \Delta t^2 - \Delta l^2 \]

Your textbook defined it as

\[ \Delta S_{\text{Book}}^2 = \Delta l^2 - c^2 \Delta t^2 = - \Delta S^2 \]

\( \rightarrow \) both are fine, but for the latter time-like intervals are imaginary numbers.

\( \rightarrow \) many concepts & derivations are much easier if a time-like interval is a real number.
Future and past are absolute

\[ x = -ct \]

\[ x = ct \]

\[ \Delta s^2 > 0: \text{time-like} \]

\[ \Delta s^2 > 0 \]

\[ \text{can find a frame where the events happen at the same point, separated by time } t = \frac{\Delta s}{c} \]

\[ \Delta s^2 < 0: \text{space-like} \]

\[ \Delta s^2 < 0 \]

\[ \text{can find a frame where the events happen at the same time, separated by distance } l = -i \Delta s, \text{ and therefore cannot be causally related.} \]
Proper Time:

- $K$ and $K'$ are inertial frames.
- Origin of $K'$ has speed $v$ in $K$.
- A clock is placed at the origin of $K'$.

In frame $K$, after time $dt$ the clock moves by $v dt = \sqrt{dx^2 + dy^2 + dz^2}$.

Interval between two positions:

$$ds^2 = (c dt)^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 \left( 1 - \frac{v^2}{c^2} \right)$$

Same interval in $K'$:

$$ds'^2 = ds^2 = c^2 dt'^2$$

Thus: $dt' = dt \sqrt{1 - \frac{v^2}{c^2}}$

- Moving clocks run slow.
- Time in rest frame of a body is called **proper time**.
World lines: trajectories in Minkowski space

Body at rest: uniform motion with $v = c$

Uniform motion, $v < c$

Non-uniform motion: body moves away from the origin and then back

World line can be looked at as infinite sum of infinitesimal uniform motions.
If the bodies traveling along green and magenta world lines are clocks:

\[ dt' = dt \sqrt{1 - \frac{\sigma^2}{c^2}} \]

\[ \sigma = \sigma(t) \]

Integrate along the world line to get time elapsed

\[ t_A' - t_0' = \int_{t_0}^{t_A} dt \sqrt{1 - \frac{\sigma^2}{c^2}} < t_A - t_0 \rightarrow \text{traveling twin is younger!} \]

Proper time

\[ \Delta t = \int_{t_1}^{t_2} dt = \int_0^A \frac{ds}{c} \]

Thus: \( \int ds \) is largest along a straight line

(very different from Euclidean 3d: distance is shortest along a straight line)
Minkowski space: $ct$, $x$, $y$, $z$

Interval $S^2 = c^2t^2 - x^2 - y^2 - z^2 \leftarrow$ analog of distance.

Coordinate transformation in different frames:

• shifts along $ct$, $x$, $y$, $z \rightarrow ct + c\Delta t$, $x + \Delta x$, $y + \Delta y$, $z + \Delta z$
  \rightarrow interval is constant.

• rotations in $xy$, $yz$, or $xz$ planes:
  i.e. in $xy$:
  \[
  \begin{align*}
  ct &= ct' \\
  x &= x'\cos \theta + y'\sin \theta \\
  y &= -x'\sin \theta + y'\cos \theta \\
  z &= z'
  \end{align*}
  \]
  \rightarrow interval is constant.

• rotations in $xt$, $yt$, and $zt$
  \rightarrow need to keep interval constant
  \rightarrow linear combinations of coordinates
\[
\cosh x = \frac{1}{2} \left[ e^x + e^{-x} \right] \\
\sinh x = \frac{1}{2} \left[ e^x - e^{-x} \right] \\
\cosh^2 x - \sinh^2 x = 1
\]

Rotation in \( xt \) plane:

\[
\begin{align*}
ct &= ct' \cosh \gamma + x' \sinh \gamma \\
x &= ct' \sinh \gamma + x' \cosh \gamma \\
y &= y' \\
z &= z'
\end{align*}
\]

\( \rightarrow \) interval is \( \text{constant} \)

\( \rightarrow \) coordinate transformation between moving frames is a rotation in Minkowski space
What is the meaning of rotation angle $\gamma$?

\[ x' = y' = z' = 0 \Rightarrow \begin{cases} ct = ct' \cosh \gamma \\ x = ct' \sinh \gamma \end{cases} \]

\[ \frac{\sinh \gamma}{\cosh \gamma} = \tanh \gamma = \frac{x}{ct} = \frac{v}{c} = \beta \]

\[ \cosh \gamma = \sqrt{1 - \tanh^2 \gamma} = \sqrt{1 - \frac{v^2}{c^2}} = \gamma \]

Thus:

\[ \begin{cases} ct = \delta \left( ct' + \beta x' \right) \\ x = x' \left( \beta \cdot ct' + x' \right) \\ y = y' \\ z = z' \end{cases} \]

\[ \Rightarrow \text{Lorentz transform rotations do not commute!} \]
Our units of time are wrong / obsolete!

"Natural" units will have \( c = 1 \)

(\( ct = t \), \( \beta = \sigma \))
Proper length of an object is its length in a system where it is at rest.

In \( K \):
\[ l_0 = x_2 - x_1 \]

In \( K' \):
\[ x_2 = \frac{x_2' + \nu t'}{\sqrt{1 - \frac{\nu^2 c^2}{c^2}}} \quad ; \quad x_1 = \frac{x_1' + \nu t'}{\sqrt{1 - \frac{\nu^2 c^2}{c^2}}} \]

\[ x_2 - x_1 = l_0 = \frac{x_2' - x_1'}{\sqrt{1 - \frac{\nu^2 c^2}{c^2}}} = \frac{l}{\sqrt{1 - \frac{\nu^2 c^2}{c^2}}} \]

\[ l = l_0 \cdot \sqrt{1 - \frac{\nu^2 c^2}{c^2}} \]

→ length is largest in the rest system.
Proper Time (again!)

Two events (i.e. particle decay)

\[ t_2 - t_1 = \tau_0 \]

\[
\begin{align*}
    t_2 &= \frac{t_2' + \frac{\vec{v}}{c^2} x'}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \quad ; \quad t_1 = \frac{t_1' + \frac{\vec{v}}{c^2} x'}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}
\end{align*}
\]

\[ t_2 - t_1 = \tau_0 = \frac{t_2' - t_1'}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \sqrt{1 - \frac{\vec{v}^2}{c^2}} \quad \tau_0 \leftarrow \text{time dilation, moving clocks run slower} \]
\[ dx = \frac{dx'}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ dy = dy' \]
\[ dz = dz' \]

\[ dt = \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ w_x = \frac{dx}{dt} \]
\[ w_y = \frac{dy}{dt} \]
\[ w_z = \frac{dz}{dt} \]

\[ w_x' = \frac{dx'}{dt'} \]
\[ w_y' = \frac{dy'}{dt'} \]
\[ w_z' = \frac{dz'}{dt'} \]

\[ \lim_{c \to \infty} : \quad w_x = w_x' + v \]
\[ w_y = w_y' \]
\[ w_z = w_z' \]
\[ \frac{1}{1 + \frac{\nu \omega_x'}{c^2}} = \left[ 1 - \frac{\nu \omega_x'}{c^2} \right] + \ldots \]

\[
\omega_x = (\omega_x' + \nu)(1 - \frac{\nu \omega_x'}{c^2}) = \omega_x' + \nu - \frac{\nu^2 \omega_x'^2}{c^2} - \frac{\nu \omega_x' \omega_{x'}}{c^2} \]

\[
\omega_y = (\omega_y' + \nu \omega_{x'} + \frac{\nu^2 \omega_x'}{c^2})(1 - \frac{\nu \omega_x'}{c^2}) = \omega_y' - \frac{\nu \omega_x' \omega_{y'}}{c^2} \]

\[
\omega_z = \omega_z' - \frac{\nu \omega_x' \omega_{z'}}{c^2} \]

Or, in a coordinate-free form:

\[
\dot{\mathbf{\omega}} = \mathbf{\omega}' - \frac{1}{c^2} \mathbf{\omega} \cdot (\mathbf{\nu} \times \mathbf{\omega}') \]

\[ \text{Note asymmetry in } \mathbf{v}^2 \text{ and } \mathbf{\omega}' \text{.} \]

\[ \text{Lorentz transformations are not commutative!} \]
$4$-vectors

\[ x^0 = ct \]
\[ x^1 = x \]
\[ x^2 = y \]
\[ x^3 = z \]

A $^4$ vector, if it transforms according to Lorentz transformations.

\[ A^i : (A^0, A^1, A^2, A^3) \rightarrow \text{contravariant} \]
\[ A_i = (A^0, -A^1, -A^2, -A^3) \rightarrow \text{covariant} \]

can indicate $4$-vector by underscore: \( \underline{A} \)

Scalar product:

\[ A_i A^i = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 \]

\(< \text{omit the sum over co/contrav indices} \>

\[ A^i = (A^0, \vec{A}) \quad \rightarrow \]
\[ A_i = (A^0, -\vec{A}) \]

$3$-d vector
Another way to write Lorentz transformation: rotation matrix

\[
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix} =
\begin{bmatrix}
    g & g' & 0 & 0 \\
    g' & g & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}, \text{ or } x' = \Lambda \cdot x
\]

Performing sequential boosts: multiply corresponding \(\Lambda\)’s

Boost from \(K\) to \(K'\) and back (should get unit matrix)

\[
\begin{bmatrix}
    g & g' & 0 & 0 \\
    g' & g & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} =
\begin{bmatrix}
    g^2 - g'^2 & -2g'^2 & 0 & 0 \\
    -2g^2 & g^2 - g'^2 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\gamma^2 \left(1 - \beta^2\right) = \frac{1}{1 - \beta^2} \left(1 - \beta^2\right) = 1\]

\[
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]
4-tensor: 16 values $A^{ik}$ that transform as products of components of two 4-vectors: $A^{ik} = B^i \cdot C^k$

$A^{ik} \rightarrow$ contravariant $A_{ik} \rightarrow$ covariant $A^i_k, A^k_i \rightarrow$ mixed.

$A^{00} = A_{00}, \quad A_{01} = -A^{01}, \quad A_{11} = A^{11}, \quad A^1_1 = -A^{11}$

Position of indices is important

$A^i_k = A^{ik} \leftarrow$ even if true in one system, will not be true in another.

Unit tensor:

$S^{ik} A^i = A^k$ for all vectors

$L \rightarrow 1$ if $i = k$

$0$ if $i \neq k$\]$

\rightarrow$ invariant in all systems!
\[ g_{ik} = g^{ik} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \rightarrow \text{metric tensor, same in all coordinate systems.} \]

\[ A^i B^i = g_{ik} A^i B^k \leftarrow \text{scalar product.} \]

\[ \rightarrow \text{another way to express index lowering/raising rules.} \]

\[ \text{vector \cdot vector} = \text{scalar} \quad A^i B_i = C \]

\[ \text{scalar \times vector} = \text{vector} \quad A^i B^i = C^i \]

\[ A^i B^j = C_{ij} \rightarrow \text{tensor} \]

\[ \text{pseudovector} = (ct, \vec{a}) \rightarrow \text{same, as in 3-D.} \]

\[ \text{pseudoscalar, etc.} \]

Quotient rule: if C scalar

\[ A^i \text{ vector} \quad A^i K_i = C \]

Then K_i is also a vector.
Example of speed addition & quotient rule use:

\[ w = \frac{w' + v}{1 + \frac{v w'}{c^2}} \] (if \( w' \ll c \))

if \( w' = c \):

\[ w = \frac{c + v}{1 + \frac{v c}{c^2}} = c \frac{1 + v/c}{1 + v/c} = c \]

→ so, what, nothing happens to light?

\[ E = E_0 \cos (k \cdot \vec{r} - \omega t) \]

\[ \omega t - k \cdot \vec{r} = (\frac{\omega}{c}, \vec{k}) \cdot (ct, \vec{r}) \]

since \((ct, \vec{r})\) is a 4-vector

\((\frac{\omega}{c}, \vec{k})\) is also 4-vector

\[ \frac{\omega}{|k|} = c \] - speed of light
So, assuming

\[ k'_i = \left( \frac{\omega'_i}{c}, \kappa'_i \right) \]

\[ \frac{\omega'_i}{k'_i} = c \Rightarrow k'_i = \frac{\omega'_i}{c} \]

Lorentz transform slant \( k \)

\[ k^0 = \kappa (k'^0 + \beta k'^1) = \kappa \frac{\omega'_i}{c} + \beta \kappa \frac{\omega'_i}{c} \cos \theta \]

\[ \frac{\omega}{c} = \frac{\omega'_i}{c} \kappa \left( 1 + \frac{\kappa^2}{c^2} \cos \theta \right) \]

\[ w = \kappa \left( 1 + \frac{\kappa^2}{c^2} \cos \theta \right) \sqrt{1 - \frac{\kappa^2}{c^2}} \]

If \( \kappa \gg 1 \), \( \cos \theta = \pm 1 \)

\[ w = \kappa \left( \frac{1 - \frac{\kappa^2}{c^2}}{1 + \frac{\kappa^2}{c^2}} \right) = \kappa \left( \frac{1 - \frac{\kappa^2}{c^2}}{1 + \frac{\kappa^2}{c^2}} \right) \]

\[ w = \kappa \left( \frac{1 - \frac{\kappa^2}{c^2}}{1 + \frac{\kappa^2}{c^2}} \right) = \kappa \left( \frac{1 - \frac{\kappa^2}{c^2}}{1 + \frac{\kappa^2}{c^2}} \right) \]

Red shift

Blue shift
Kinematics

4-position: \( \mathbf{x} = (ct, \mathbf{r}) \)

4-speed: \( \mathbf{u} = \frac{d}{dt_0} \mathbf{x} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d}{dt} (ct, \mathbf{r}) = \gamma (c, \mathbf{v}) \)

\[ \mathbf{u} \cdot \mathbf{u} = \gamma^2 (c^2 - v^2) = c^2 \]

4-speed has constant "length"

4-acceleration: \( \mathbf{a} = \frac{d}{dt_0} \mathbf{u} \)

useful derivatives:

\[
\frac{d}{dt} (\gamma) = \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = -\gamma^3 \left( -2 \frac{\gamma}{c^2} \right) \frac{dv}{dt} = \frac{\gamma^3 v}{c^2} \frac{ds}{dt} = \frac{d\gamma}{dt}
\]

\[
\frac{d}{dt} (\beta \gamma) = \frac{1}{c} \frac{d}{dt} \frac{\gamma}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{c} \left[ \gamma \frac{dv}{dt} + \gamma^3 \frac{v^2}{c^2} \frac{d\gamma}{dt} \right] = \frac{1}{c} \gamma^3 \frac{d\gamma}{dt} \left[ 1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right]
\]

\[
\frac{d}{dt} (\beta \gamma) = \frac{1}{c} \gamma^3 \frac{dv}{dt}
\]
\[ u^i u_i = c^2 \implies \frac{d}{dt} (u^i u_i) = 0 \implies 2 u^i \dot{u}_i = 0 \]

4-acceleration is \( \perp \) to 4-speed

\text{motion along } 1-d \ (x) :

\[ u = \gamma \left( c, 0, 0, 0 \right) \]

\[ L^2 = L^3 = 0 \]

\[ L^0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d}{dt} \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma^2 c \frac{d}{dt} (\beta) = \gamma^2 c \cdot \frac{\gamma^3 c}{c^2} \frac{dv}{dt} = \gamma^4 \beta \frac{dv}{dt} \]

\[ L^1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d}{dt} \frac{\gamma^3 c}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma^2 c \frac{d}{dt} (\gamma \beta) = \gamma^2 c \cdot \frac{\gamma^4}{c} \gamma^3 \frac{dv}{dt} = \gamma^4 \frac{dv}{dt} \]

\[ L = \gamma^4 \frac{dv}{dt} \left( \beta, 1, 0, 0 \right) \]
Constant acceleration in 1-d:

\[
a = \text{const in co-moving system: inertial system in which speed of the object is momentarily zero.}
\]

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}'(\mathbf{e}, \mathbf{v}_y, 0, 0) \\
\mathbf{u} &= (\mathbf{e}, 0, 0, 0) \\
\mathbf{x} &= \gamma^4 \frac{d\mathbf{x}'}{dt} (\frac{\mathbf{e}}{c}, 1, 0, 0) \\
\mathbf{x}' &= (0, a, 0, 0)
\end{align*}
\]

\[\mathbf{x}' \text{ is invariant!} \]

(same in all inertial systems)

\[\gamma^3 \frac{d\mathbf{u}}{dt} = a \rightarrow \text{equation on } \mathbf{u}(t)\]
\[
\frac{\text{d}v}{(1 - \frac{v^2}{c^2})^{3/2}} = a \, dt
\]

\[
v\bigg|_{t=0} = 0
\]

\[
\int \frac{\text{d}v}{(1 - \frac{v^2}{c^2})^{3/2}} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} + \text{const}
\]

\[
v = at, \quad v^2 = a^2 t^2 - \frac{a^2 t^2}{c^2} \cdot \sigma^2, \quad \sigma^2 \left(1 + \frac{a^2 t^2}{c^2}\right) = a^2 t^2
\]

\[
v = at
\]

\text{for } t \to 0, \quad \sigma \ll c: \quad v = at \rightarrow \text{classical}

\text{for } t \to \infty \quad \sqrt{1 + \frac{a^2 t^2}{c^2}} = \frac{at}{c} \left(1 + \frac{c^2}{a^2 t^2}\right)

\[
v = c \left(1 - \frac{c^2}{a^2 t^2}\right)
\]
Now: the dynamics

Classical:

\[ S = \int_{t_1}^{t_2} L \, dt \]

\[ \text{minimize } S \]

Relativistic:

\[ S = \text{constant} \]

\[ \text{because of metrics of space and time interval, } \frac{1}{c} \int ds = \text{time interval}, \]

\[ \text{longest for "shortest" path} \]

\[ S = -L \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} \, dt \]

\[ L = -L \left( 1 - \frac{v^2}{c^2} \right) \]

For \( v \ll c \),

\[ L = -2c \left( 1 - \frac{v^2}{2c^2} \right) = -2c + \frac{2}{c} \cdot \frac{v^2}{2} \]

\[ \rightarrow \frac{mc^2}{2} \]

Therefore, \( d = mc \), so

\[ L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \]
**Momentum:**

\[ \mathbf{p} = \frac{\partial \mathbf{L}}{\partial \mathbf{\dot{q}}} \]

Similar to the previous equation:

\[ p_x = \frac{\partial L}{\partial \dot{x}_1} = \frac{-e}{\sqrt{1 - v^2 c^2}} \cdot \frac{d}{dx} \left( \sqrt{x_1^2 + y_1^2 + z_1^2} \right) = \frac{m \sigma_x}{\sqrt{1 - v^2 c^2}} \]

\[ p_y = \frac{\partial L}{\partial \dot{y}_1} = \frac{m \sigma_y}{\sqrt{1 - v^2 c^2}} \]

\[ p_z = \frac{\partial L}{\partial \dot{z}_1} = \frac{m \sigma_z}{\sqrt{1 - v^2 c^2}} \]

So:

\[ \mathbf{p} = m \cdot \mathbf{u} = m \gamma (c, \mathbf{v}) \leftrightarrow 4\text{-vector}. \]

What is \( P^0 = m \gamma c \)?

\[ \text{i.e. what is the physics meaning of it?} \]
$\mathcal{L} = \sum p_i - U = \frac{m v_i^2}{\sqrt{1 - \frac{v_i^2}{c^2}}} + m c^2 \sqrt{1 - \frac{v^2}{c^2}} = m c^2 \frac{\sqrt{\frac{v^2}{c^2} + 1}}{\sqrt{1 - \frac{v^2}{c^2}}} = m c^2 \frac{\sqrt{\frac{v^2}{c^2} + 1}}{\sqrt{1 - \frac{v^2}{c^2}}} = E = c \cdot m \cdot u_o$

For small speeds:

$E = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = m c^2 \left(1 + \frac{v^2}{2c^2}\right) = m c^2 + \frac{1}{2} m v^2$

in classical, constant does not matter

classical kinetic energy!

$P = m u = \left(\frac{E}{c}, \vec{p}\right)$

Energy-momentum 4-vector
Force: \[ F = \frac{d^2}{dt^2} \]

if \( F \parallel \ddot{r} \) (\( F(\hat{x}) \)) \[ \frac{d}{dt} \frac{\ddot{r}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{mv^3}{c^2} \frac{d\dot{v}}{dt} = \frac{ma}{q} \]

acceleration in co-moving system!

if \( F \perp \ddot{r} \Rightarrow \dot{v} = \text{const} \)

\[ \frac{d}{dt} \frac{\ddot{r}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{ma}{c^2} \frac{d^2}{dt^2} \ddot{r} = F \]

\[ d\omega \cdot \vec{v} = 0 \]

\[ \lambda (\vec{\omega} \times \vec{v}) = 0 \]

\[ \dot{v} = \text{const} \]
Relativistic cyclotron:

\[ F = \frac{dp}{dt} = \frac{p \cdot d\theta}{dt} = p \omega = p \cdot \frac{v}{R} \]

if \( p = \frac{m \sigma}{R} \)

\[ F = m \frac{\sigma^2}{R} \]

\[ F_{mag} = q \cdot v \cdot B \]

\[ q \cdot v \cdot B = p \cdot \frac{v}{R}, \quad R = \frac{p}{q \cdot B} \]

Period of rotation:

\[ T = \frac{\Delta \sigma R}{v} \]

\( p = \delta m \sigma \Rightarrow T = 2\pi \frac{\delta m \sigma}{q \cdot B} \]

\[ T = 2\pi \frac{m}{q \cdot B} \cdot \frac{1}{\sigma} \]
\[ L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \rightarrow \vec{p} = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \mathbf{F} = \vec{p} \cdot \vec{v} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \vec{p} = m\vec{v} = \left( \frac{\varepsilon}{c}, \vec{p} \right) \]

\[ m \to \text{a scalar} \]

Force becomes not a straightforward thing.

If uniform acceleration, \( a = \varepsilon^3 \frac{dv}{dt} \)

\[ F_{\parallel} = \frac{d\vec{p}}{dt} = m\varepsilon^3 \frac{dv}{dt} = ma \]

\[ \text{but if } \vec{F} \perp \vec{v}, \quad \frac{d\vec{p}}{dt} = m\varepsilon^3 \frac{d\vec{v}}{dt} \]

\[ |\vec{F}| = \text{const} \quad F_{\perp} = m\varepsilon^3 \frac{d\vec{v}}{dt} \]

\[ F_{\parallel} \text{ and } F_{\perp} \text{ look very different!} \]
\[ P = m \cdot u = m \mathbf{v} (c, \mathbf{v}^2) \]

\[ p^2 = m^2 v^2 = mc^2 \rightarrow \text{scalar.} \]

in the rest frame, \( P = (mc, \mathbf{0}) = \left( \frac{E_{\text{rest}}}{c}, 0 \right) \)

\[ E_{\text{rest}} = mc^2 \]

For small speeds

\[ E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 \left( 1 + \frac{v^2}{2c^2} \right) = mc^2 + \frac{1}{2}mv^2 \]

In relativity, this constant has meaning!

That's what keeps \( \left( \frac{E}{c}, \mathbf{p} \right) \) 4-vector.
In relativity it's possible to have \( m = 0 \):

\[
\frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \quad m \to 0 \quad v \to c
\]

\[
P = \left( \frac{E}{c}, \mathbf{p} \right) \quad P^2 = 0 = \frac{E^2}{c^2} - p^2 \quad \Rightarrow \quad E = p \cdot c
\]

Recall 4-d wave vector:

\[
K = \left( \frac{\omega}{c}, \mathbf{k} \right) \quad |K|^2 = \frac{\omega}{c}
\]

\[
K^2 = \frac{\omega^2}{c^2} - k^2 = 0
\]

In fact, \( P = \hbar K \), \( \mathbf{p} = \hbar \mathbf{k} \), \( E = \hbar \omega \)