Problem 1. Two blocks are in contact on a frictionless horizontal table. A horizontal force is applied to one of the blocks, as shown in the drawing. Find the force of contact between the blocks. Express your answer using $m_1$, $m_2$, and $F$.

\[ F = (m_1 + m_2) a \quad a = \frac{F}{m_1 + m_2} \]

Newton's 2nd Law for the 2nd block is

\[ F_N = m_2 a \quad \text{Thus,} \quad F_N = \frac{m_2}{m_1 + m_2} \cdot F \]
Problem 2. A block of mass $m$ slides along a horizontal table with speed $v_0$, initially with no friction, see the sketch below. It hits a spring with spring constant $k$, and simultaneously begins to experience a constant friction force with the coefficient of friction $\mu$. Find the loss in mechanical energy when the block has first come momentarily to rest. (Recall that while the friction force is nonconservative, its work can be calculated easily, and work-kinetic energy theorem applies to all forces.)

Let $x$ be the spring's compression when the block comes to stop.

Work done by friction is

$$f_{\text{friction}} \cdot x = \text{lost mechanical energy} = W_{\text{lost}}$$

Since $f_{\text{friction}} = \mu F_n = \mu mg$, $W_{\text{lost}} = \mu mg x$

Work-kinetic energy theorem gives

$$\frac{mv_0^2}{2} = \frac{kx^2}{2} + \mu mg x$$

Thus,

$$x = \frac{\mu mg}{k} \left( \sqrt{1 + \frac{k v_0^2}{\mu^2 mg^2}} - 1 \right)$$

and

$$W_{\text{lost}} = \frac{(\mu mg)^2}{k} \left( \sqrt{1 + \frac{k v_0^2}{\mu^2 mg^2}} - 1 \right)$$
Problem 3. You are told that, at the known positions \( x_1 \) and \( x_2 \), an oscillating mass \( m \) has speeds \( v_1 \) and \( v_2 \). What are the amplitude \( A \) and the angular frequency \( \omega \) of the oscillations?

\[
\begin{align*}
\dot{x} &= A \cos (\omega t + \delta) \\
\ddot{x} &= -A \sin (\omega t + \delta) \cdot \omega \\
\end{align*}
\]

At \( t_1 \):

\[
\begin{align*}
\dot{x}_1 &= A \cos (\omega t_1 + \delta) \\
\ddot{x}_1 &= -A \omega \sin (\omega t_1 + \delta) = \pm A \sqrt{1 - \cos^2 (\omega t_1 + \delta)} = \pm A \sqrt{1 - \frac{x_1^2}{A^2}} \cdot \omega
\end{align*}
\]

At \( t_2 \), therefore,

\[
\begin{align*}
x_2 &= A \cos (\omega t_2 + \delta) \\
\ddot{x}_2 &= \pm A \sqrt{1 - \frac{x_2^2}{A^2}} \cdot \omega
\end{align*}
\]

Thus,

\[
\frac{\dot{x}_1}{\dot{x}_2} = \sqrt{\frac{1 - \frac{x_1^2}{A^2}}{1 - \frac{x_2^2}{A^2}}} \quad \text{and we obtain}
\]

\[
A = \left( \frac{x_2^2 \ddot{x}_1^2 - \ddot{x}_2^2 x_1^2}{\ddot{x}_1^2 - \ddot{x}_2^2} \right)^{\frac{1}{2}}
\]

Then,

\[
\omega^2 = \frac{\ddot{x}_2^2}{\frac{A^2}{A^2} (1 - \frac{x_2^2}{A^2})}
\]

\[
\omega = \left( \frac{\ddot{x}_1^2 - \ddot{x}_2^2}{x_1^2 - x_1^2} \right)^{\frac{1}{2}}
\]
Problem 4. Prove that for circular orbits around a stationary gravitational force center (such as the sun) the speed of the orbiting body is inversely proportional to the square root of the orbital radius $R$.
Hint: a simple Newtonian approach may suffice.

For a circular orbit,

$$ m \frac{v^2}{R} = F = \frac{mMG}{R^2} $$

Thus, $v = \sqrt{\frac{MG}{R}}$
Problem 5. A small projectile is fired from the surface of the earth at an angle $\alpha=60^\circ$ from the vertical. The initial speed $v_0=\left(\frac{GM_e}{R_e}\right)^{0.5}$, where $R_e$ is the earth’s radius, and $M_e$ is the earth’s mass. How high does the projectile rise? (Find $r_{\text{max}}$ shown in the sketch below).

Hints. (1) It is easier to apply the conservation laws directly instead of using the orbit equations. Think about the two relevant conservation laws. (2) Once you write down the equations, substitute the given expression for $v_0$ there. This will simplify your calculations.

Angular momentum conservation (origin O) gives

$$\ell_1 = \ell_2$$

$$m v_0 R_e \sin \alpha = m u R_{\text{max}}$$

$$u = \frac{v_0 R_e}{R_{\text{max}}} \sin \alpha$$

Energy conservation law $E_1 = E_2$ gives

$$\frac{m v_0^2}{2} - \frac{GM_e m}{R_e} = \frac{m}{2} \frac{v_0^2 R_e^2}{R_{\text{max}}^2} \sin^2 \alpha - \frac{GM_e m}{R_{\text{max}}}$$

Substituting $v_0 = \sqrt{\frac{GM_e}{R_e}}$, $\sin^2 \alpha = \frac{3}{4}$, and introducing $x = \frac{R_{\text{max}}}{R_e}$, the equation becomes

$$x^2 - 2x + 0.75 = 0$$

$$x = \frac{3}{2} \quad R_{\text{max}} = \frac{3}{2} R_e$$
**Problem 6.** Consider a pendulum of length \( l \) and mass \( m \) whose plane of oscillations is rotating about the vertical axis with a constant angular velocity \( \Omega \), as shown below.

(a) Find the speed of the mass in the plane of the drawing using the generalized coordinate \( \theta \) and the corresponding generalized velocity. Then find the speed of the mass normal to the plane of the paper. Using these results, find the total velocity-squared \((v^2)\) of the mass.

(b) Write the Lagrangian.

(c) Write the Lagrange's equation. You do not need to solve it.

(d) Write the Lagrange's equation using the small angle approximation. Is the equilibrium position \( \theta = 0 \) stable or unstable? (Your answer will depend on the parameters of the problem.) Find the frequency of the oscillations about \( \theta = 0 \) for the case of stable equilibrium.

\[
\begin{align*}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \quad \text{gives} \quad \ddot{\theta} = -2l \sin \theta \cos \theta - \frac{g}{l} \sin \theta \\
\end{align*}
\]

\( (a) \quad U = l \dot{\theta}^2 + l \dot{\theta} \sin \theta \quad S = \frac{1}{2} l^2 \dot{\theta}^2 + \frac{1}{2} l^2 \dot{\theta}^2 \sin^2 \theta \)

\( (b) \quad T = \frac{1}{2} m \dot{\theta}^2 \quad U = -mg l \cos \theta + \cos \theta \)

\( L = T - U = \frac{1}{2} ml^2 (\dot{\theta}^2 + l^2 \dot{\theta}^2 \sin^2 \theta) + mg l \cos \theta \)

\( (c) \quad \frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \quad \text{gives} \quad \ddot{\theta} = -2l \sin \theta \cos \theta - \frac{g}{l} \sin \theta \\
\)

\( (d) \quad \text{For small } \theta, \quad \sin \theta \approx \theta \quad \cos \theta \approx 1 \quad \text{and we get} \quad \ddot{\theta} = -\theta \left( \frac{g}{l} - l^2 \right) \)

For \( \frac{g}{l} - l^2 > 0 \), this is simple harmonic motion (stable equilibrium) with frequency \( \omega = \sqrt{\frac{g}{l} - l^2} \).

For \( \frac{g}{l} - l^2 < 0 \), this is an exponentially diverging solution and the equilibrium is unstable.
**Problem 7.** Consider a double pendulum with given masses $m_1$ and $m_2$, and lengths $l_1$ and $l_2$ oscillating in the $xy$ plane, as shown below. The generalized coordinates describing the double pendulum are $\varphi_1$ and $\varphi_2$.

(a) Express the Cartesian coordinates $(x_1, y_1)$, and $(x_2, y_2)$ of the two masses using the generalized coordinates. Use the coordinate system shown in the sketch.

(b) Take the time derivative of the results of (a), and obtain the Cartesian velocity vectors of the masses expressed through the generalized coordinates and velocities.

(c) Using the results of (a) and (b), write down the Lagrangian of the system. You do not have to simplify the obtained expression, and you do not need to write the Lagrange's equations.

(d) Set $m_2=0$. Write down the Lagrange's equation for $\varphi_1$. You should get the equation of motion of a simple pendulum, of course.

\[
\begin{align*}
(a) \quad x_1 &= l_1 \sin \varphi_1, \quad y_1 = l_1 \cos \varphi_1, \\
&\quad x_2 = l_1 \sin \varphi_1 + l_2 \sin \varphi_2, \\
&\quad y_2 = l_1 \cos \varphi_1 + l_2 \cos \varphi_2
\end{align*}
\]

\[
\begin{align*}
(b) \quad \dot{x}_1 &= l_1 \cos \varphi_1 \dot{\varphi}_1, \quad \ddot{y}_1 = -l_1 \sin \varphi_1 \dot{\varphi}_1, \\
&\quad \dot{x}_2 = l_1 \cos \varphi_1 \dot{\varphi}_1 + l_2 \cos \varphi_2 \dot{\varphi}_2, \\
&\quad \ddot{y}_2 = -l_1 \sin \varphi_1 \dot{\varphi}_1 - l_2 \sin \varphi_2 \dot{\varphi}_2
\end{align*}
\]

\[
\begin{align*}
(c) \quad T &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \\
&\quad + \frac{1}{2} m_2 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \right]
\end{align*}
\]

\[
U = -m_1 g y_1 - m_2 g y_2 = -m_1 g l_1 \cos \varphi_1 - m_2 g (l_1 \cos \varphi_1 + l_2 \cos \varphi_2)
\]

\[
L = T - U
\]

\[
(d) \quad L = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + m_1 g l_1 \cos \varphi_1, \quad \frac{\partial}{\partial \dot{\varphi}_1} \frac{\partial L}{\partial \ddot{\varphi}_1} = \ddot{\varphi}_1 \quad \text{gives} \quad \ddot{\varphi}_1 = \frac{g}{l_1} \sin \varphi_1, \quad \text{as expected.}
\]