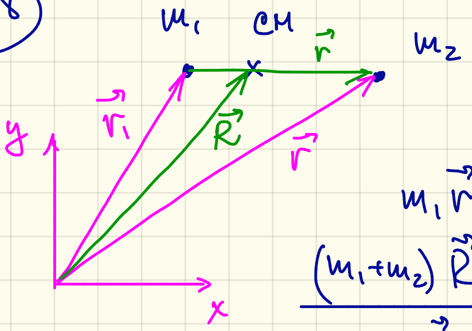
8.8



$$\vec r = \vec r_2 - \vec r_1$$

$$\vec R = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1+m_2}$$

$$m_1 \vec r = m_1 \vec r_2 - m_1 \vec r_1$$

$$(m_1+m_2)\vec R = m_1 \vec r_1 + m_2 \vec r_2$$

$$(m_1+m_2)\vec R + m_1 \vec r = (m_1+m_2)\vec r_2$$

$$\boxed{\begin{aligned} \vec r_2 &= \vec R + \frac{m_1}{m_1+m_2}\vec r \\ \vec r_1 &= \vec R - \frac{m_2}{m_1+m_2}\vec r \end{aligned}}$$

$$T = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} = \frac{1}{2} m_1 \left(\dot{\vec R}^2 + \frac{m_2^2}{(m_1+m_2)^2}\dot{\vec r}^2 + \frac{2m_2}{m_1+m_2}\dot{\vec R}\dot{\vec r} \right) +$$

$$+ \frac{1}{2} m_2 \left(\dot{\vec R}^2 + \frac{m_1^2}{(m_1+m_2)^2}\dot{\vec r}^2 - \frac{2m_1}{m_1+m_2}\dot{\vec R}\dot{\vec r} \right) =$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec R}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec r}^2 + \dot{\vec R}\dot{\vec r}\left(\cancel{\frac{m_1 m_2}{m_1+m_2}} - \cancel{\frac{m_1 m_2}{m_1+m_2}} \right)$$

$$\vec R = (X, Y)$$

$$\vec r = (x, y)$$

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\boxed{\mathcal{L} = \frac{1}{2} M (\dot X^2 + \dot Y^2) + \frac{1}{2}\mu(\dot x^2 + \dot y^2) - \frac{1}{2}k(x^2+y^2)}$$

$$\mathcal{L} = \frac{1}{2} M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2+y^2)$$

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial \mathcal{L}}{\partial Y} = 0 \rightarrow$$ total momentum conserves

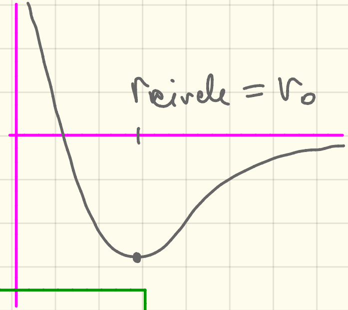
$$\dot{X} = \text{const}, \quad \dot{Y} = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \mu \dot{x} \qquad \Rightarrow \boxed{\begin{aligned} \ddot{x} &= -\frac{k}{\mu} \cdot x \\ \ddot{y} &= -\frac{k}{\mu} y \end{aligned}}$$

x and y independently oscillate with frequency $\omega = \sqrt{\frac{k}{\mu}}$

8.12

a)
$$U_{eff} = -G\frac{m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$$



$$\frac{dU_{eff}}{dr} = \frac{G m_1 m_2}{r^2} - \frac{\ell^2}{\mu r^3} = 0$$

$$\ell^2 = G m_1 m_2 \mu r \qquad \boxed{r_0 = \frac{\ell^2}{G m_1 m_2 \mu}}$$

b) "stable" means that $\frac{d^2U_{eff}}{dr^2}(r_0) > 0$

$$\frac{d^2U_{eff}}{dr^2} = -\frac{2Gm_1m_2}{r^3} + 3\frac{\ell^2}{\mu r^4} = \frac{1}{\mu r^4}\left(3\ell^2 - 2Gm_1m_2\mu r\right)\Big|_{r=r_0} =$$

$$= \frac{1}{\mu r^4}\left(3\ell^2 - 2Gm_1m_2\mu \cdot \frac{\ell^2}{Gm_1m_2\mu}\right) = \boxed{\frac{\ell^2}{\mu r^4} > 0}$$

Oscillations for small deviations from r_{circle}:

$$r = r_0 + \rho, \quad \rho \ll r_0$$

8.12 cont

$$\mu \ddot{r} = -\frac{dU_{eff}}{dr}, \quad r_0 = \frac{\ell^2}{\mu \gamma}$$

$$-\frac{dU_{eff}}{dr} = -\frac{\gamma}{(r_0+\rho)^2} + \frac{\ell^2}{\mu (r_0+\rho)^3} = -\frac{\gamma}{r_0^2}\left(1 - \frac{2\rho}{r_0}\right) + \frac{\ell^2}{\mu r_0^3}\left(1 - \frac{3\rho}{r_0}\right) =$$

$$= \frac{2\gamma}{r_0^3}\rho - \frac{3\ell^2}{\mu r_0^4}\rho = \rho \frac{1}{\mu r_0^4}\left(2\mu\gamma r_0 - 3\ell^2\right) =$$

$$= -\rho \frac{\ell^2}{\mu r_0^4} = -\rho \frac{\ell^2}{\mu \gamma}\frac{\gamma}{r_0^4} = -\frac{\rho\gamma}{r_0^3}$$

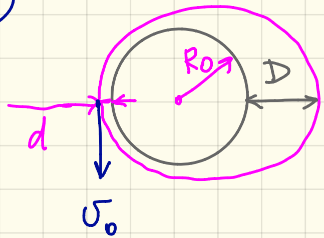
$$\mu \ddot{\rho} = -\frac{\gamma}{r_0^3}\rho \Rightarrow \text{oscillation with } \omega = \sqrt{\frac{\gamma}{\mu r_0^3}}$$

Planet's orbital period:

$$\underbrace{\mu \omega_0^2 r_0}_{\text{centripital acceleration}} = \underbrace{\frac{\gamma}{r_0^2}}_{\text{force}} \Rightarrow \omega_0^2 = \frac{\gamma}{\mu r_0^3}$$

!

8.18



$d = 250\ \text{km} = 2.5 \cdot 10^5\ \text{m}$

$R_0 = 6.4 \cdot 10^6\ \text{m}$

$v_0 = 8.5 \cdot 10^3\ \text{m/s}$

$\mu \approx m$

$\gamma = GmM$

$\ell = (R_0 + d)\mu \cdot v_0$

$$r(\varphi) = \frac{\ell^2}{\gamma\mu}\frac{1}{1+\varepsilon\cos\varphi}$$

$$\frac{\gamma}{\mu R_0^2} = g$$

$$r(0) = (R_0+d) = \frac{\ell^2}{\gamma\mu}\frac{1}{1+\varepsilon}$$

$$1+\varepsilon = \frac{(R_0+d)^2\mu^2 v_0^2}{\gamma\mu}\frac{1}{R_0+d} = \frac{(R_0+d)\mu v_0^2}{\gamma} = \frac{(R_0+d)v_0^2}{gR_0^2} =$$

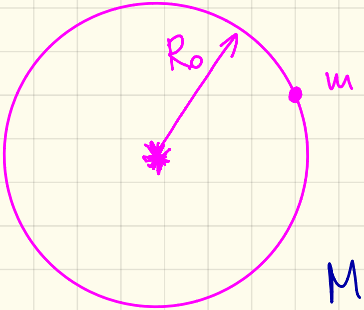
$$= \frac{6.65\cdot 10^6 \cdot (8.5\cdot 10^3)^2}{10\cdot(6.4\cdot 10^6)^2} = \frac{6.65\cdot 8.5^2}{6.4^2}\frac{1}{10} \approx 1.17$$

$$\boxed{\varepsilon = 0.17}$$

$$R_0 + D = \frac{\ell^2}{\gamma\mu}\frac{1}{1-\varepsilon} \Rightarrow \frac{R_0+D}{R_0+d} = \frac{1+\varepsilon}{1-\varepsilon} \approx 1.41$$

$$D = 1.41\cdot d + 0.41\cdot R_0 \approx \boxed{3\cdot 10^6\ \text{m}}$$

8.29



$$T = \frac{1}{2} m v^2$$

$$U = -G \frac{mM}{R_0}$$

$$\frac{m v^2}{R_0} = G \frac{mM}{R_0^2}$$

$$T = -\frac{1}{2} U$$

$$M \to \frac{1}{2} M$$

$$T \to T = \frac{1}{2} m v^2 \quad \leftarrow \text{same}$$

$$U \to -\frac{1}{2} G \frac{mM}{R_0} \quad \leftarrow \text{halved}$$

$$E_{tot} = T + U = \frac{U}{2} \to T + \frac{1}{2} U = 0$$

if half of Sun's mass disappear, total energy will become zero:

- unbound orbit
- parabola

8.30

$$r(\varphi) = \frac{c}{1+\varepsilon\cos\varphi}\,, \qquad c = \frac{\ell^2}{\gamma\mu}$$

$\varepsilon > 1$:

$$r + r\varepsilon\cos\varphi = c$$

$$r = c - \varepsilon x$$

$$x^2 + y^2 = c^2 + \varepsilon x^2 - 2\varepsilon c x$$

$$x^2(\varepsilon^2-1) - 2\varepsilon c x = y^2 - c^2$$

$$x^2 - 2\frac{\varepsilon}{\varepsilon^2-1}cx = \frac{y^2}{\varepsilon^2-1} - \frac{c^2}{\varepsilon^2-1}$$

$$\frac{\varepsilon c}{\varepsilon^2-1} = x_0$$

$$a = \frac{c}{\varepsilon^2-1} \qquad b = \frac{c}{\sqrt{\varepsilon^2-1}}$$

$$\left(x - \frac{\varepsilon c}{\varepsilon^2-1}\right)^2 - \frac{y^2}{\varepsilon^2-1} = \frac{\varepsilon^2 c^2}{(\varepsilon^2-1)^2} - \frac{c^2(\varepsilon^2-1)}{(\varepsilon^2-1)^2} = \frac{c^2}{(\varepsilon^2-1)^2}$$

$$\frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \longleftarrow \quad \underline{\text{hyperbola!}}$$

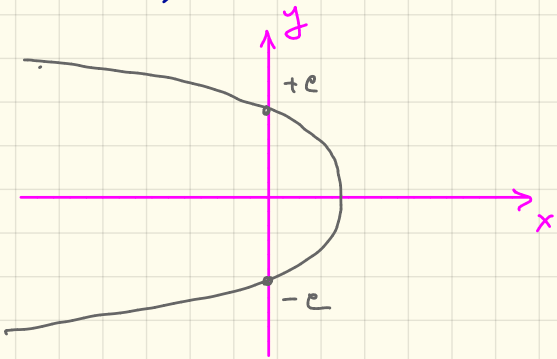
8.30 cont

$\varepsilon = 1$: $\quad r + r\cos\varphi = c \quad$ (rotate so $\delta = 0$)

$$r^2 = (c-x)^2 = x^2 + y^2$$

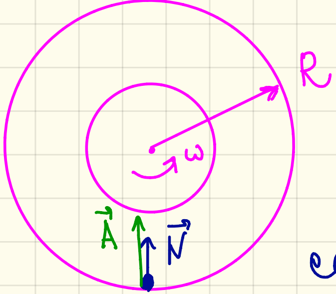
$$x^2 + c^2 - 2cx = x^2 + y^2$$

$$x = \frac{1}{2c}\left(c^2 - y^2\right)$$



9.2

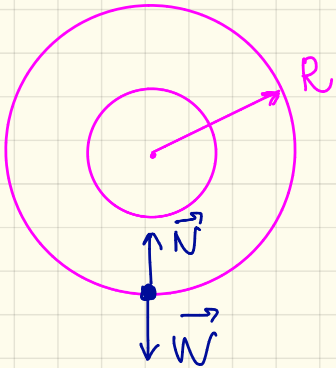
a)



In inertial system:

Normal force $\vec{N}$ provides centripital acceleration $\vec{A} = \omega^2 R$

b)



In astronaut frame:

weight $\vec{W}$ (inertial force) is compensated by the normal force $\vec{N} = -\vec{W}$

$$\vec{W} = m\vec{A} \Rightarrow A = \omega^2 R = g$$

$$\omega = \sqrt{\frac{g}{R}} = \sqrt{\frac{10}{40}} = \frac{1}{2}$$

c)

apparent gravity $g_a = \omega^2 R$

if R changes $\frac{2}{40} = 5\%$, g_a changes by 5% as well.