5.35 a) $z = x + iy = r \cdot e^{i\theta} = r(\cos\theta + i\sin\theta)$

$$x = r\cos\theta \qquad y = r\cdot\sin\theta \quad \Rightarrow \quad r = \sqrt{x^2+y^2}\ , \quad \theta = \operatorname{atan}\frac{y}{x}$$

b) $z = x + iy$

$z^* = x - iy$

$$zz^* = x^2 + y^2 = |z|^2$$

c) $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$z^* = r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

d) $(z\cdot w)^* = \left(r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}\right)^* = \left(r_1 r_2 \cdot e^{i(\theta_1+\theta_2)}\right)^* = r_1 r_2 e^{-i(\theta_1+\theta_2)} =$

$$= r_1 e^{-i\theta_1} \cdot r_2 e^{-i\theta_2} = z^* \cdot w^*$$

$$\left(\frac{1}{z}\right)^* = \left(\frac{1}{r} e^{-i\theta_1}\right)^* = \frac{1}{r} e^{i\theta_1} = \frac{1}{re^{-i\theta_1}} = \frac{1}{z^*}$$

5.35 cont

e) $z = \dfrac{a}{b+ic}$, a, b, c - real

$$z = \frac{a}{b+ic} \cdot \frac{b-ic}{b-ic} = \frac{a(b-ic)}{b^2+c^2} = \frac{ab}{b^2+c^2} - i\,\frac{ac}{b^2+c^2}$$

$$|z|^2 = \frac{a^2b^2+a^2c^2}{(b^2+c^2)^2} = \frac{a^2(b^2+c^2)}{(b^2+c^2)^2} = \frac{a^2}{b^2+c^2}$$

5.41

$$\left.\begin{aligned} A(\omega) &= \frac{C}{\sqrt{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}} \\ A(\omega_0) &= \frac{C}{2\beta\omega_0} \end{aligned}\right\} \quad A^2(\omega) = \frac{1}{2}A^2(\omega_0)$$

$$\Downarrow$$

$$(\omega^2-\omega_0^2)^2+4\beta^2\omega^2 = 8\cdot\beta^2\omega_0^2$$

$\beta \ll \omega_0$

$\omega-\omega_0 = x$, $x \ll \omega_0$

$$\omega = \omega_0 + x$$

$$\omega^2 = \omega_0^2\left(1+\frac{x}{\omega_0}\right)^2 = \omega_0^2\left(1+2\frac{x}{\omega_0}\right) = \omega_0^2 + 2x\omega_0$$

$$\omega^2-\omega_0^2 = 2x\omega_0$$

$$4x^2\omega_0^2 + 4\beta^2\omega_0^2 + 8\beta^2 x\omega_0 = 8\beta^2\omega_0^2$$

$$x^2 + \beta^2 + \cancel{2x\cdot\omega_0\frac{\beta}{\omega_0}} = 2\beta^2$$

$$x^2 \approx \beta^2 \implies \boxed{\omega = \omega_0 \pm \beta}$$

5.42

$\tau = 8\,\mathrm{hrs}$

$L = 30\,\mathrm{m}$

$$Q = \pi \cdot \frac{\tau}{T} = \pi \cdot \frac{\tau \cdot \omega}{2\pi} = \frac{1}{2}\tau \cdot \sqrt{\frac{g}{L}} =$$

$$\omega = \sqrt{\frac{g}{L}} \quad , T = \frac{2\pi}{\omega}$$

$$= \frac{1}{2} \cdot 8 \cdot 3.6 \cdot 10^3 \cdot \sqrt{\frac{10}{30}} \approx$$

$$\approx 4 \cdot 3.6 \cdot 10^3 \cdot \frac{1}{1{,}7} \approx 8300$$

5.44

$$x = A\cdot\cos\omega_0 t \quad , \quad \omega_0 = \sqrt{\frac{k}{m}} \qquad \dot{x} = -A\omega_0 \sin\omega_0 t$$

a) $$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}A^2 m\omega_0^2 \sin^2\omega_0 t + \frac{1}{2}k A^2\cos^2\omega_0 t =$$

$$= \frac{1}{2}A^2 m\omega_0^2 \sin^2\omega_0 t + \frac{1}{2}m\omega_0^2 A^2\cos^2\omega_0 t = \frac{1}{2}m\omega_0^2 A^2$$

$$2\beta = \frac{b}{m}$$

b) $$dW = F_{damp}\cdot dx = -b\dot{x}\,dx$$

$$Q = \frac{\omega_0}{2\beta}$$

$$\frac{dW}{dt} = -b\dot{x}^2 = -2\beta m\cdot\dot{x}^2 \longleftarrow \text{dissipation rate}$$

$$\dot{x}^2 = A^2\omega_0^2\sin^2\omega_0 t \rightarrow \text{averaged over } T: \ \frac{1}{2}A^2\omega_0^2$$

$$\Delta E = T\cdot\frac{1}{2}A^2\omega_0^2(-2\beta m) = -\frac{2\pi}{\omega_0}\cdot A^2\omega_0^2\beta m = -2\pi A^2\omega_0\beta m$$

c) $$\frac{E}{\Delta E} = \frac{\frac{1}{2}\cancel{m}\omega_0^{\cancel{2}}\cancel{A^2}}{2\pi\cancel{A^2}\cancel{\omega_0}\beta\cancel{m}} = \frac{1}{4\pi}\frac{\omega_0}{\beta} = \frac{1}{2\pi}\frac{\omega_0}{2\beta} \Rightarrow \boxed{Q = 2\pi\frac{E}{\Delta E}}$$

5.47 $$\int_{-\tau/2}^{\tau/2} \cos(n\omega t)\cos(m\omega t)\,dt = \begin{cases} \tau/2 & , m=n \\ 0 & , m \neq n \end{cases}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\Rightarrow \cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$$

$$\cos(n\omega t)\cdot\cos(m\omega t) = \cos((n+m)\omega t) + \cos((n-m)\omega t)$$

$$\frac{1}{2}\int_{-\tau/2}^{\tau/2} \cos((n+m)\omega t)\,dt + \frac{1}{2}\int_{-\tau/2}^{\tau/2} \cos((n-m)\omega t)\,dt = \ldots$$

$$\int_{-\tau/2}^{\tau/2} \cos k\omega t\,dt = 0 \text{ if } k \neq 0 \text{ : integer number of periods!}$$

$$= \tau \text{ if } k=0 \quad \int_{-\tau/2}^{\tau/2} dt$$

$$\ldots = \begin{cases} \tau/2 & n=m \\ 0 & , n \neq m \end{cases} \quad \text{q.e.d.}$$

5.47 cont

$$\int_{-\tau/2}^{\tau/2} \sin(n\omega t)\,\sin(m\omega t)\,dt = \begin{cases} \tau/2 & , m=n \\ 0 & , m\neq n \end{cases}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\Rightarrow \sin\alpha\cdot\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\frac{1}{2}\int_{-\tau/2}^{\tau/2} \cos((n-m)\omega t)\,dt - \cancel{\frac{1}{2}\int_{-\tau/2}^{\tau/2} \cos((m+n)\omega t)\,dt} =$$

$$= \frac{1}{2}\times\begin{cases} \tau & , n=m \\ 0 & , n\neq m \end{cases} = \begin{cases} \tau/2 & , n=m \\ 0 & , n\neq m \end{cases}, \quad \text{q.e.d.}$$

5.47 cont.

$$\int_{-\tau/2}^{\tau/2} \sin(n\omega t)\cos(m\omega t)\,dt = 0$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\Rightarrow \sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$$\underbrace{\frac{1}{2}\int_{-\tau/2}^{\tau/2} \sin((n+m)\omega t)\,dt}_{n+m>0,\ \text{always zero}} + \underbrace{\frac{1}{2}\int_{-\tau/2}^{\tau/2} \sin((n-m)\omega t)\,dt}_{\substack{n\neq m \to \text{zero} \\ n=m \quad \sin((n-m)\omega t)=0,\ \text{zero}}} = 0, \quad \text{q.e.d}$$

5.48 $$f(t) = \sum_{n=0}^{\infty} a_n \cdot \cos n\omega t + b_n \cdot \sin \omega t$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \cos(n\omega t)\, dt$$

multiply both sides by $\cos(m\omega t)$ and integrate:

$$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \sin(n\omega t)\, dt$$

$$\int_{-\tau/2}^{\tau/2} f(t) \cdot \cos(m\omega t)\, dt = \sum_{n=0}^{\infty} \int_{-\frac{\tau}{2}}^{\tau/2} a_n \cdot \cos(n\omega t) \cos(m\omega t)\, dt +$$

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\, dt$$

$$+ \sum_{n=0}^{\infty} \int_{-\tau/2}^{\tau/2} b_n \sin(n\omega t) \cdot \cos(m\omega t)\, dt =$$

$$= a_m \cdot \frac{\tau}{2} \Rightarrow a_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \cos(m\omega t)\, dt$$

Similarly,

$$\int_{-\tau/2}^{\tau/2} f(t) \cdot \sin(m\omega t)\, dt = \sum_{n=0}^{\infty} \int_{-\tau/2}^{\tau/2} a_n \cos n\omega t \cdot \sin m\omega t\, dt + \int_{-\tau/2}^{\tau/2} b_n \sin n\omega t \sin m\omega t\, dt =$$

$$= b_m \cdot \frac{\tau}{2}, \quad b_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \sin m\omega t\, dt$$

5.48 cont

for a_0, just integrate:

$$\int_{-\tau/2}^{\tau/2} f(t)\,dt = \sum_{n=0}^{\infty} \int_{-\tau/2}^{\tau/2} a_n \cdot \cos n\omega t\,dt + \int_{-\tau/2}^{\tau/2} b_n \sin n\omega t\,dt =$$

$$= \int_{-\tau/2}^{\tau/2} a_0 \cdot dt = a_0 \cdot \tau$$

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\,dt$$