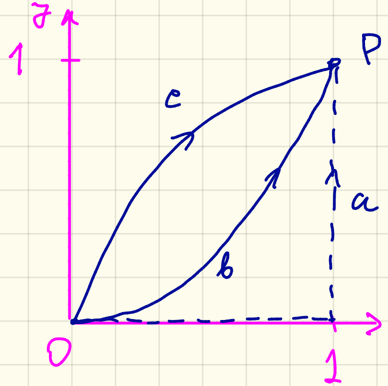
4.2 $\quad \vec{F} = (x^2, 2xy)$



$$W = \int \vec{F}\, d\vec{l} = \int F_x\, dx + \int F_y\, dy$$

a) $$W = \int_0^1 x^2 dx + \int_0^1 2y\, dy = \frac{1}{3} + 2\cdot\frac{1}{2} = \frac{4}{3}$$

b) $y = x^2 \qquad dy = 2x\, dx$

$$W = \int_0^1 x^2 dx + \int_0^1 2x^3\cdot 2x\, dx = \frac{1}{3} + 4\cdot\frac{1}{5} = \frac{5+12}{15} = \frac{17}{15}$$

$$\frac{4}{3} + 1 = \frac{7}{3}$$

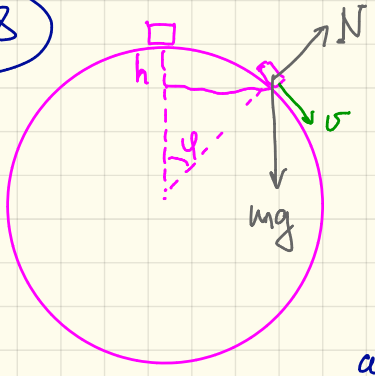
c) $x = t^3,\ y = t^2 \rightarrow dx = 3t^2 dt,\ y = 2t\, dt$

$$W = \int_0^1 t^6\cdot 3t^2 dt + \int_0^1 2t^3 t^2\cdot 2t\, dt = 3\cdot\frac{1}{9} + 4\cdot\frac{1}{7} = \frac{1}{3} + \frac{4}{7} = \frac{19}{21}$$

$t = x^{1/3},\ y = x^{2/3} \qquad dy = \frac{2}{3} x^{-1/3} dx$

$$W = \int_0^1 x^2 dx + \int_0^1 2x\cdot x^{2/3}\cdot\frac{2}{3} x^{-1/3} dx = \frac{1}{3} + \frac{4}{3}\int_0^1 x^{4/3} dx = \frac{1}{3} + \frac{4}{3}\cdot\frac{3}{7} = \frac{1}{3} + \frac{4}{7}$$

4.8



$v(h)$: $mgh = \frac{1}{2}mv^2 \Rightarrow v^2 = 2gh$

$\varphi(h)$: $h = R(1-\cos\varphi) \Rightarrow 1-\cos\varphi = \frac{h}{R}$

$$\frac{mv^2}{R} = mg\cos\varphi - N \qquad \cos\varphi = 1 - \frac{h}{R}$$

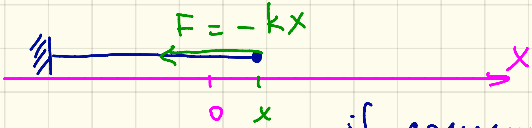
at lift-off, $N = 0$.

$$\frac{2\cancel{g}h}{R} = \cancel{g}\cdot\cos\varphi = \cancel{g}\left(1-\frac{h}{R}\right)$$

$$2h = R\left(1-\frac{h}{R}\right) = R - h$$

$$3h = R, \quad \boxed{h = \frac{1}{3}R}$$

4.9

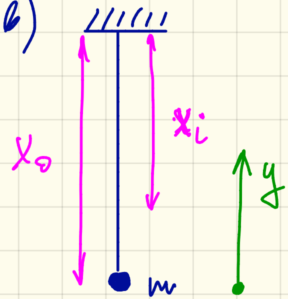


a) if conservative, path does not matter

work of an external force to stretch from equilibrium to x is

$$W = -\int_0^x F\,dx = -\int_0^x -kx\,dx = \frac{1}{2}kx^2 \Rightarrow U = \frac{1}{2}kx^2$$

b)



$$(x_0 - x_i)k = mg$$

$$W = -\int_0^y \left(-mg + k\left((x_0 - x_i) - y\right)\right) dy =$$

$$= mgy - k(x_0 - x_i)\,y + \frac{1}{2}ky^2 = mgy - mgy + \frac{1}{2}ky^2$$

$$U = \frac{1}{2}ky^2$$

4.12 $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right)$

a) $f = x^2 + z^3$: $\quad \vec{\nabla} f = (2x, 0, 3z^2)$

b) $f = ky$: $\quad \vec{\nabla} f = (0, k, 0)$

c) $f = \sqrt{x^2+y^2+z^2} \equiv r$: $\quad \frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2} \cdot 2x = \frac{x}{r}$

$$\vec{\nabla} f = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \hat{r}$$

d) $f = \frac{1}{r}$: $\quad \frac{\partial f}{\partial x} = -\frac{1}{2} \frac{1}{(x^2+y^2+z^2)^{3/2}} \cdot 2x = -\frac{x}{r^3}$

$$\vec{\nabla} f = -\left(\frac{x}{r^3}; \frac{y}{r^3}, \frac{z}{r^3}\right) = -\frac{\hat{r}}{r^2}$$

4.15

$f(r) = x^2 + 2y^2 + 3z^2$

$\vec{\nabla} f = (2x, 4y, 6z)$

$\vec{\nabla} f(1,1,1) = (2, 4, 6)$

$df = \vec{\nabla} f \cdot d\vec{r} = 0.02 + 0.12 + 0.30 =$

$= 0.44$

$f(1,1,1) = 1 + 2 + 3 = 6$

$f(1.01, 1.03, 1.05) =$

$= 1.01^2 + 2 \cdot 1.03^2 + 3 \cdot 1.05^2 -$

$= 1.0201 + 2.1218 + 3.3075 =$

$= 6.4494$

$df = 0.4494$

4.16

$$U = k(x^2 + y^2 + z^2)$$

$$\vec{F} = -\vec{\nabla} U = -k\,(2x,\ 2y,\ 2z) = -2k\,\vec{r}$$