Rutgers Physics 381 Mechanics I (Fall'18/Gershtein)

Final Exam – December 17, 2018

This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration - 3hr
You may find the following formulae useful:

$$\vec{F}_{centrifugal} = m\left[\left[\vec{\Omega}\times\vec{r}\right]\times\vec{\Omega}\right] \qquad \vec{F}_{coriolis} = 2m\left[\dot{\vec{r}}\times\vec{\Omega}\right] \qquad e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

1. Compute the path integral of the force $\vec{F} = x\hat{x} + y\hat{y}$ from (0,0,0) to (1,1,0) along two paths:

- From (0,0,0) to (1,0,0) along the x axis and then to (1,1,0) along the y axis
- Along the path $y = x^2, \quad z = 0$

Show that the force is conservative

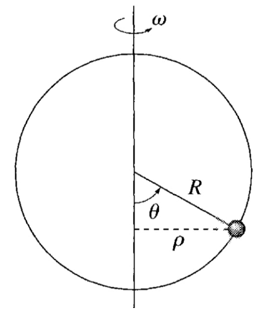
2. A particle of mass **m** on a frictionless horizontal table is attached to a massless string, whose other end passes through a hole in the table, where you are holding it. The particle is moving in a circle of radius $\mathbf{r_0}$ with angular velocity $\boldsymbol{\omega_0}$.

a) What is its kinetic energy?

You now slowly pull the string down through the hole until a length $\mathbf{r_0/2}$ remains between the hole and the particle.

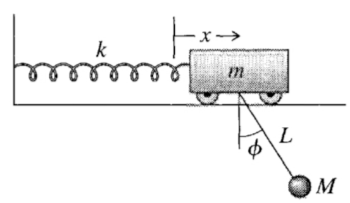
b) What is the angular velocity and kinetic energy now?

c) Verify that the extra kinetic energy equals the work done pulling the string



3. A bead is free to move around the frictionless wire hoop of radius R, which is spinning at a fixed rate ω about its vertical axis. The constant bead position is specified by the angle θ. Find all the equilibrium positions at which the bead can remain with θ constant, **using Newton laws in a coordinate system that rotates with the hoop** (we solved this problem in class using Lagrange formalism)

4. A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in the sketch. Write the Lagrangian in terms of the two generalized coordinates x and φ, where x is extension of the spring from its equilibrium length. Find the two Lagrange equations (but do not solve them!)



5. A particle moves in a potential $U(x) = U_0\cosh(x) = U_0\dfrac{e^x + e^{-x}}{2}$.

- Is there a stable equilibrium position?
- What is the frequency of small oscillations around it?

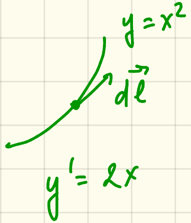
① $\vec{F} = x\hat{x} + y\hat{y}$

a) $\underset{A}{(0,0,0)} \to \underset{B}{(1,0,0)} \to \underset{C}{(1,1,0)}$

$$\left.\begin{aligned} \int_A^B \vec{F}\, d\vec{l} &= \int_0^1 \vec{F}(x,0)\,\hat{x}\, dx = \int_0^1 x\, dx = \frac{1}{2} \\ \int_B^C \vec{F}\, dl &= \int_0^1 \vec{F}(1,y)\,\hat{y}\, dy = \int_0^1 y\, dy = \frac{1}{2} \end{aligned}\right\} \Rightarrow \int_A^C \vec{F}\, dl = 1$$

b) $\int_A^C \vec{F}\, d\vec{l} = \int \vec{F}(x,x^2)\, d\vec{l} =$

dl is tangential to curve



$$\vec{dl} = \hat{x}\, dx + \hat{y}\, 2x\, dx$$

$$= \int_0^1 (x\hat{x} + y\hat{y})(\hat{x} + 2x\hat{y})\, dx =$$

$$= \int_0^1 (x\hat{x} + y\hat{y})(\hat{x} + 2x\hat{y})\,dx = \int_0^1 (x + 2x\cdot y)\,dx = \int_0^1 (x + 2x^3)\,dx =$$

$$= \int_0^1 x\,dx + 2\int_0^1 x^3\,dx = \frac{1}{2} + 2\cdot\frac{1}{4} = 1.$$

Force is conservative: calculate curl:

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \hat{x}\,\frac{\partial y}{\partial z} + \hat{y}\,\frac{\partial x}{\partial z} + \hat{z}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

Since $\vec{\nabla}\times\vec{F}$ is 0, and there is no time dependence, $\vec{F}$ is <u>conservative</u>!

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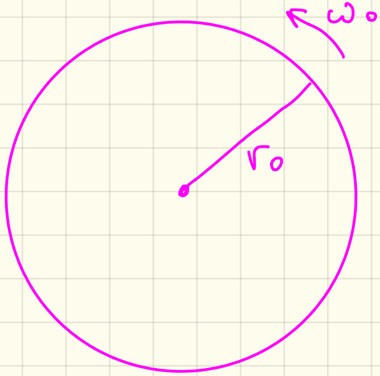


a) $K_0 = \frac{1}{2} m v_0^2 = \frac{1}{2} m \omega_0^2 r_0^2$

b) when radius changes, l_z conserves (no torques on the particle)

$$r v = r_0 v_0 \Rightarrow v = v_0 \cdot \frac{r_0}{r} = 2 v_0$$

$$K = \frac{1}{2} m v^2 = 2 m \omega_0^2 r_0^2 = 4 K_0$$

Work done pulling:

Tension of the string is

$T = m \frac{v^2}{r}$. For a given r, $v = v_0 \cdot \frac{r_0}{r} = \omega_0 r_0^2 \frac{1}{r}$

$$\text{Work} = -\int_{r_0}^{r_0/2} m \frac{v^2}{r} dr = -\int_{r_0}^{r_0/2} m \omega_0^2 r_0^4 \frac{1}{r^3} dr = -m \omega_0^2 r_0^4 \int_{r_0}^{r_0/2} \frac{dr}{r^3}$$

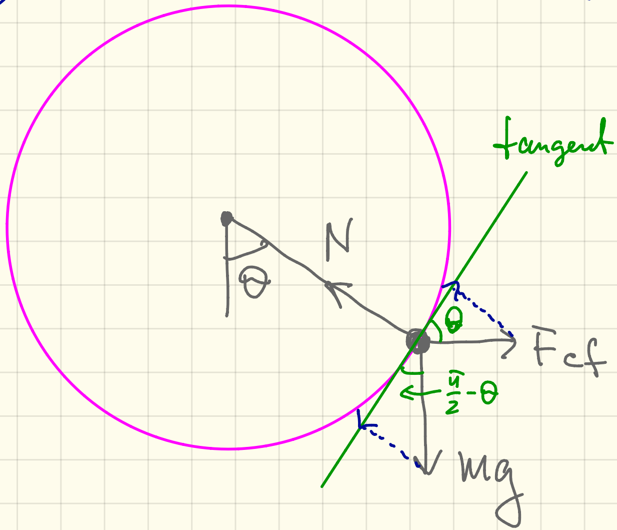
$$-m\omega_0^2 r_0^4 \int_{r_0}^{r_0/2} \frac{dr}{r^3} = -m\omega_0^2 r_0^4 \left(-\frac{1}{2}\frac{1}{r^2}\right)\Bigg|_{r_0}^{r_0/2} =$$

$$= -m\omega_0^2 r_0^4 \left(-\frac{1}{2}\frac{4}{r_0^2} + \frac{1}{2}\frac{1}{r_0^2}\right) = m\omega_0^2 r_0^2 \frac{1}{2}(4-1) =$$

$$W = \frac{3}{2} m\omega_0^2 r_0^2 \underline{= K - K_0}$$

indeed!

③



θ corresponds to an equilibrium if there are no tangential forces on the bead.

$$F_{\parallel} = F_{cf} \cdot \cos\theta - mg\cos\left(\frac{\pi}{2} - \theta\right) =$$

$$= m\omega^2 R\sin\theta \cdot \cos\theta - mg\sin\theta =$$

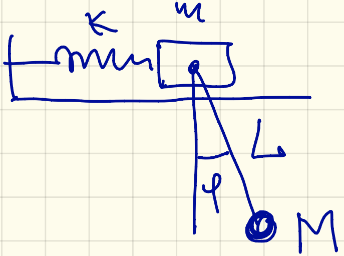
$$= m\omega^2 R\sin\theta\left(\cos\theta - \frac{g}{R\omega^2}\right) = 0$$

Turns to zero if

$\sin\theta = 0$: $\boxed{\theta_1 = 0, \quad \theta_2 = \pi}$

$\cos\theta = \frac{g}{R\omega^2} \Rightarrow$ if $\frac{g}{R\omega^2} < 1$ then $\boxed{\theta_{3,4} = \pm \operatorname{acos}\frac{g}{R\omega^2}}$

(4)



Coordinates: x, φ

Speed of the pendulum:

$$v_x = \dot{x} + L\dot{\varphi}\cos\varphi$$

$$v_y = L\dot{\varphi}\sin\varphi$$

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left(\dot{x}^2 + L^2\dot{\varphi}^2 + 2L\dot{x}\dot{\varphi}\cos\varphi\right)$$

$$U = -MgL\cos\varphi + \frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\varphi}^2 + ML\dot{x}\dot{\varphi}\cos\varphi + MgL\cos\varphi - \frac{1}{2}kx^2$$

$$\frac{\partial\mathcal{L}}{\partial x} = -kx \qquad \frac{\partial\mathcal{L}}{\partial\varphi} = -\sin\varphi\left(ML\dot{x}\dot{\varphi} + MgL\right)$$

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\varphi}^2 + ML\dot{x}\dot{\varphi}\cos\varphi + MgL\cos\varphi - \frac{1}{2}kx^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (m+M)\dot{x} + ML\dot{\varphi}\cos\varphi \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = (m+M)\ddot{x} + ML\ddot{\varphi}\cos\varphi - ML\dot{\varphi}^2\sin\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = ML^2\dot{\varphi} + ML\dot{x}\cos\varphi \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = ML^2\ddot{\varphi} + ML\ddot{x}\cos\varphi - ML\dot{x}\dot{\varphi}\sin\varphi$$

Euler-Lagrange equations:

$$\begin{cases} (m+M)\ddot{x} + ML\ddot{\varphi}\cos\varphi = ML\dot{\varphi}^2\sin\varphi - kx \\ ML^2\ddot{\varphi} + ML\ddot{x}\cos\varphi = \cancel{ML\dot{x}\dot{\varphi}\sin\varphi} - \cancel{\sin\varphi\, ML\dot{x}\dot{\varphi}} - MgL\sin\varphi \end{cases}$$

5 $$U = U_0 \frac{e^x + e^{-x}}{2}$$

$$U' = U_0 \frac{1}{2}\left(e^x - e^{-x}\right) = 0$$: $x = 0$ is equilibrium

$$U'' = U_0 \frac{1}{2}\left(e^x + e^{-x}\right) > 0$$: stable equilibrium.

Taylor expand around $x = 0$:

$$U = \frac{1}{2} U_0 \left(1 + x + \frac{1}{2}x^2 + 1 - x + \frac{1}{2}x^2\right) = \frac{1}{2} U_0 \left(2 + x^2\right) =$$

$$U = U_0 + \frac{1}{2} U_0 x^2$$

$$\omega = \sqrt{\frac{U_0}{m}}$$