2.55 ★★★ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** pointing in the y direction and **B** in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{xo}$ (positive or negative). **(a)** Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$. **(b)** Prove that there is a unique value of v_{xo}, called the drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam with many different speeds.) **(c)** Solve the equations of motion to give the particle's velocity as a function of t, for arbitrary values of v_{xo}. [*Hint:* The equations for (v_x, v_y) should look very like Equations (2.68) except for an offset of v_x by a constant. If you make a change of variables of the form $u_x = v_x - v_{dr}$ and $u_y = v_y$, the equations for (u_x, u_y) will have exactly the form (2.68), whose general solution you know.] **(d)** Integrate the velocity to find the position as a function of t and sketch the trajectory for various values of v_{xo}.

and sketch z and z^* in the complex plane:

$$\textbf{(a)}\ z = 1 + i \qquad \textbf{(b)}\ z = 1 - i\sqrt{3}$$
$$\textbf{(c)}\ z = \sqrt{2}e^{-i\pi/4} \qquad \textbf{(d)}\ z = 5e^{i\omega t}.$$

In part (d), ω is a constant and t is the time.

2.47 ★ For each of the following two pairs of numbers compute $z + w$, $z - w$, zw, and z/w.

$$\textbf{(a)}\ z = 6 + 8i \ \text{ and } \ w = 3 - 4i \qquad \textbf{(b)}\ z = 8e^{i\pi/3} \ \text{ and } \ w = 4e^{i\pi/6}.$$

Notice that for adding and subtracting complex numbers, the form $x + iy$ is more convenient, but for multiplying and especially dividing, the form $re^{i\theta}$ is more convenient. In part (a), a clever trick for finding z/w without converting to the form $re^{i\theta}$ is to multiply top and bottom by w^*; try this one both ways.

2.48 ★ Prove that $|z| = \sqrt{z^*z}$ for any complex number z.

2.49 ★ Consider the complex number $z = e^{i\theta} = \cos\theta + i \sin\theta$. **(a)** By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. **(b)** Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

2.50 ★ Use the series definition (2.72) of e^z to prove that[12] $de^z/dz = e^z$.

2.51 ★★ Use the series definition (2.72) of e^z to prove that $e^z e^w = e^{z+w}$. [*Hint:* If you write down the left side as a product of two series, you will have a huge sum of terms like $z^n w^m$. If you group together all the terms for which $n + m$ is the same (call it p) and use the binomial theorem, you will find you have the series for the right side.]

SECTION 2.7 Solution for the Charge in a B Field

2.52 ★ The transverse velocity of the particle in Sections 2.5 and 2.7 is contained in (2.77), since $\eta = v_x + iv_y$. By taking the real and imaginary parts, find expressions for v_x and v_y separately. Based on these expressions describe the time dependence of the transverse velocity.

2.53 ★ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** and **B**, both pointing in the z direction. The net force on the particle is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Write down the equation of motion for the particle and resolve it into its three components. Solve the equations and describe the particle's motion.

2.54 ★★ In Section 2.5 we solved the equations of motion (2.68) for the transverse velocity of a charge in a magnetic field by the trick of using the complex number $\eta = v_x + iv_y$. As you might imagine, the equations can certainly be solved without this trick. Here is one way: **(a)** Differentiate the first of equations (2.68) with respect to t and use the second to give you a second-order differential equation for v_x. This is an equation you should recognize [if not, look at Equation (1.55)] and you can write down its general solution. Once you know v_x, (2.68) tells you v_y. **(b)** Show that the general solution you get here is the same as the general solution contained in (2.77), as disentangled in Problem 2.52.

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[12] If you are the type who worries about mathematical niceties, you may be wondering if it is permissible to differentiate an infinite series. Fortunately, in the case of a power series (such as this), there is a theorem that guarantees the series can be differentiated for any z inside the "radius of convergence." Since the radius of convergence of the series for e^z is infinite, we can differentiate it for *any* value of z.

(2.86) to write $\dot{v}$ as $v\,dv/dy$, and then solve the equation of motion by separating variables (put all terms involving v on one side and all terms involving y on the other). Integrate both sides to give y in terms of v, and hence v as a function of y. Show that the baseball's maximum height is

$$y_{max} = \frac{v_{ter}^2}{2g} \ln\left(\frac{v_{ter}^2 + v_o^2}{v_{ter}^2}\right). \tag{2.89}$$

If $v_o = 20$ m/s (about 45 mph) and the baseball has the parameters given in Example 2.5 (page 61), what is y_{max}? Compare with the value in a vacuum.

2.42 ⋆⋆ Consider again the baseball of Problem 2.41 and write down the equation of motion for the downward journey. (Notice that with a quadratic drag the downward equation is different from the upward one, and has to be treated separately.) Find v as a function of y and, given that the downward journey starts at y_{max} as given in (2.89), show that the speed when the ball returns to the ground is $v_{ter}v_o/\sqrt{v_{ter}^2 + v_o^2}$. Discuss this result for the cases of very much and very little air resistance. What is the numerical value of this speed for the baseball of Problem 2.41? Compare with the value in a vacuum.

2.43 ⋆⋆⋆ [Computer] The basketball of Problem 2.31 is thrown from a height of 2 m with initial velocity $\mathbf{v}_o = 15$ m/s at 45° above the horizontal. **(a)** Use appropriate software to solve the equations of motion (2.61) for the ball's position (x, y) and plot the trajectory. Show the corresponding trajectory in the absence of air resistance. **(b)** Use your plot to find how far the ball travels in the horizontal direction before it hits the floor. Compare with the corresponding range in a vacuum.

2.44 ⋆⋆⋆ [Computer] To get an accurate trajectory for a projectile one must often take account of several complications. For example, if a projectile goes very high then we have to allow for the reduction in air resistance as atmospheric density decreases. To illustrate this, consider an iron cannonball (diameter 15 cm, density 7.8 g/cm^3) that is fired with initial velocity 300 m/s at 50 degrees above the horizontal. The drag force is approximately quadratic, but since the drag is proportional to the atmospheric density and the density falls off exponentially with height, the drag force is $f = c(y)v^2$ where $c(y) = \gamma D^2 \exp(-y/\lambda)$ with γ given by (2.6) and $\lambda \approx 10,000$ m. **(a)** Write down the equations of motion for the cannonball and use appropriate software to solve numerically for $x(t)$ and $y(t)$ for $0 \le t \le 3.5$ s. Plot the ball's trajectory and find its horizontal range. **(b)** Do the same calculation ignoring the variation of atmospheric density [that is, setting $c(y) = c(0)$], and yet again ignoring air resistance entirely. Plot all three trajectories for $0 \le t \le 3.5$ s on the same graph. You will find that in this case air resistance makes a huge difference and that the variation of air resistance makes a small, but not negligible, difference.

SECTION 2.6 Complex Exponentials

2.45 ⋆ **(a)** Using Euler's relation (2.76), prove that any complex number $z = x + iy$ can be written in the form $z = re^{i\theta}$, where r and θ are real. Describe the significance of r and θ with reference to the complex plane. **(b)** Write $z = 3 + 4i$ in the form $z = re^{i\theta}$. **(c)** Write $z = 2e^{-i\pi/3}$ in the form $x + iy$.

2.46 ⋆ For any complex number $z = x + iy$, the **real** and **imaginary parts** are defined as the real numbers $\text{Re}(z) = x$ and $\text{Im}(z) = y$. The **modulus** or **absolute value** is $|z| = \sqrt{x^2 + y^2}$ and the **phase** or **angle** is the value of θ when z is expressed as $z = re^{i\theta}$. The **complex conjugate** is $z^* = x - iy$. (This last is the notation used by most physicists; most mathematicians use $\bar{z}$.) For each of the following complex numbers, find the real and imaginary parts, the modulus and phase, and the complex conjugate, gives you a simple approximation when x is large.] **(d)** Show that for t small, Equation (2.58) for the position gives $y \approx \frac{1}{2}gt^2$. [Use the Taylor series for $\cosh x$ and for $\ln(1+\delta)$.]

2.36 ★★ Consider the following quote from Galileo's *Dialogues Concerning Two New Sciences:*

> Aristotle says that "an iron ball of 100 pounds falling from a height of one hundred cubits reaches the ground before a one-pound ball has fallen a single cubit." I say that they arrive at the same time. You find, on making the experiment, that the larger outstrips the smaller by two finger-breadths, that is, when the larger has reached the ground, the other is short of it by two finger-breadths.

We know that the statement attributed to Aristotle is totally wrong, but just how close is Galileo's claim that the difference is just "two finger breadths"? **(a)** Given that the density of iron is about 8 g/cm^3, find the terminal speeds of the two iron balls. **(b)** Given that a cubit is about 2 feet, use Equation (2.58) to find the time for the heavier ball to land and then the position of the lighter ball at that time. How far apart are they?

2.37 ★★ The result (2.57) for the velocity of a falling object was found by integrating Equation (2.55) and the quickest way to do this is to use the integral $\int du/(1-u^2) = \operatorname{arctanh} u$. Here is another way to do it: Integrate (2.55) using the method of "partial fractions," writing

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right),$$

which lets you do the integral in terms of natural logs. Solve the resulting equation to give v as a function of t and show that your answer agrees with (2.57).

2.38 ★★ A projectile that is subject to quadratic air resistance is thrown vertically *up* with initial speed v_o. **(a)** Write down the equation of motion for the upward motion and solve it to give v as a function of t. **(b)** Show that the time to reach the top of the trajectory is

$$t_{top} = (v_{ter}/g)\arctan(v_o/v_{ter}).$$

(c) For the baseball of Example 2.5 (with $v_{ter} = 35$ m/s), find t_{top} for the cases that $v_o = 1$, 10, 20, 30, and 40 m/s, and compare with the corresponding values in a vacuum.

2.39 ★★ When a cyclist coasts to a stop, he is actually subject to two forces, the quadratic force of air resistance, $f = -cv^2$ (with c as given in Problem 2.26), and a constant frictional force f_{fr} of about 3 N. The former is dominant at high and medium speeds, the latter at low speed. (The frictional force is a combination of ordinary friction in the bearings and rolling friction of the tires on the road.) **(a)** Write down the equation of motion while the cyclist is coasting to a stop. Solve it by separating variables to give t as a function of v. **(b)** Using the numbers of Problem 2.26 (and the value $f_{fr} = 3$ N given above) find how long it takes the cyclist to slow from his initial 20 m/s to 15 m/s. How long to slow to 10 and 5 m/s? How long to come to a full stop? If you did Problem 2.26, compare with the answers you got there ignoring friction entirely.

2.40 ★★ Consider an object that is coasting horizontally (positive x direction) subject to a drag force $f = -bv - cv^2$. Write down Newton's second law for this object and solve for v by separating variables. Sketch the behavior of v as a function of t. Explain the time dependence for t large. (Which force term is dominant when t is large?)

2.41 ★★ A baseball is thrown vertically up with speed v_o and is subject to a quadratic drag with magnitude $f(v) = cv^2$. Write down the equation of motion for the upward journey (measuring y vertically *up*) and show that it can be rewritten as $\dot{v} = -g[1 + (v/v_{ter})^2]$. Use the "$v\,dv/dx$ rule"

2.27 ★ I kick a puck of mass m up an incline (angle of slope = θ) with initial speed v_o. There is no friction between the puck and the incline, but there is air resistance with magnitude $f(v) = cv^2$. Write down and solve Newton's second law for the puck's velocity as a function of t on the upward journey. How long does the upward journey last?

2.28 ★ A mass m has speed v_o at the origin and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the "$v\,dv/dx$ rule" (2.86) in Problem 2.12 to write the equation of motion in the separated form $m\,v\,dv/F(v) = dx$, and then integrate both sides to give x in terms of v (or vice versa). Show that it will eventually travel a distance $2m\sqrt{v_o}/c$.

2.29 ★ The terminal speed of a 70-kg skydiver in spread-eagle position is around 50 m/s (about 115 mi/h). Find his speed at times $t = 1, 5, 10, 20, 30$ seconds after he jumps from a stationary balloon. Compare with the corresponding speeds if there were no air resistance.

2.30 ★ Suppose we wish to approximate the skydiver of Problem 2.29 as a sphere (not a very promising approximation, but nevertheless the kind of approximation physicists sometimes like to make). Given the mass and terminal speed, what should we use for the diameter of the sphere? Does your answer seem reasonable?

2.31 ★★ A basketball has mass $m = 600$ g and diameter $D = 24$ cm. **(a)** What is its terminal speed? **(b)** If it is dropped from a 30-m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

2.32 ★★ Consider the following statement: If at all times during a projectile's flight its speed is much less than the terminal speed, the effects of air resistance are usually very small. **(a)** Without reference to the explicit equations for the magnitude of v_{ter}, explain clearly why this is so. **(b)** By examining the explicit formulas (2.26) and (2.53) explain why the statement above is even more useful for the case of quadratic drag than for the linear case. [*Hint:* Express the ratio f/mg of the drag to the weight in terms of the ratio v/v_{ter}.]

2.33 ★★ The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as follows:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

for any z, real or complex. **(a)** Sketch the behavior of both functions over a suitable range of real values of z. **(b)** Show that $\cosh z = \cos(iz)$. What is the corresponding relation for $\sinh z$? **(c)** What are the derivatives of $\cosh z$ and $\sinh z$? What about their integrals? **(d)** Show that $\cosh^2 z - \sinh^2 z = 1$. **(e)** Show that $\int dx/\sqrt{1+x^2} = \text{arcsinh}\,x$. [*Hint:* One way to do this is to make the substitution $x = \sinh z$.]

2.34 ★★ The hyperbolic function $\tanh z$ is defined as $\tanh z = \sinh z/\cosh z$, with $\cosh z$ and $\sinh z$ defined as in Problem 2.33. **(a)** Prove that $\tanh z = -i\tan(iz)$. **(b)** What is the derivative of $\tanh z$? **(c)** Show that $\int dz \tanh z = \ln\cosh z$. **(d)** Prove that $1 - \tanh^2 z = \text{sech}^2 z$, where $\text{sech}\, z = 1/\cosh z$. **(e)** Show that $\int dx/(1 - x^2) = \text{arctanh}\,x$.

2.35 ★★ **(a)** Fill in the details of the arguments leading from the equation of motion (2.52) to Equations (2.57) and (2.58) for the velocity and position of a dropped object subject to quadratic air resistance. Be sure to do the two integrals involved. (The results of Problem 2.34 will help.) **(b)** Tidy the two equations by introducing the parameter $\tau = v_{ter}/g$. Show that when $t = \tau$, v has reached 76% of its terminal value. What are the corresponding percentages when $t = 2\tau$ and 3τ? **(c)** Show that when $t \gg \tau$, the position is approximately $y \approx v_{ter}t + \text{const}$. [*Hint:* The definition of $\cosh x$ (Problem 2.33) the gun can hit any object inside the surface

$$z = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

Describe this surface and comment on its dimensions.

2.22 ★★★ [Computer] The equation (2.39) for the range of a projectile in a linear medium cannot be solved analytically in terms of elementary functions. If you put in numbers for the several parameters, then it *can* be solved numerically using any of several software packages such as Mathematica, Maple, and MatLab. To practice this, do the following: Consider a projectile launched at angle θ above the horizontal ground with initial speed v_o in a linear medium. Choose units such that $v_o = 1$ and $g = 1$. Suppose also that the terminal speed $v_{ter} = 1$. (With $v_o = v_{ter}$, air resistance should be fairly important.) We know that in a vacuum, the maximum range occurs at $\theta = \pi/4 \approx 0.75$. **(a)** What is the maximum range in a vacuum? **(b)** Now solve (2.39) for the range in the given medium at the same angle $\theta = 0.75$. **(c)** Once you have your calculation working, repeat it for some selection of values of θ within which the maximum range probably lies. (You could try $\theta = 0.4, 0.5, \cdots, 0.8$.) **(d)** Based on these results, choose a smaller interval for θ where you're sure the maximum lies and repeat the process. Repeat it again if necessary until you know the maximum range and the corresponding angle to two significant figures. Compare with the vacuum values.

SECTION 2.4 Quadratic Air Resistance

2.23 ★ Find the terminal speeds in air of **(a)** a steel ball bearing of diameter 3 mm, **(b)** a 16-pound steel shot, and **(c)** a 200-pound parachutist in free fall in the fetal position. In all three cases, you can safely assume the drag force is purely quadratic. The density of steel is about 8 g/cm^3 and you can treat the parachutist as a sphere of density 1 g/cm^3.

2.24 ★ Consider a sphere (diameter D, density ϱ_{sph}) falling through air (density ϱ_{air}) and assume that the drag force is purely quadratic. **(a)** Use Equation (2.84) from Problem 2.4 (with $\kappa = 1/4$ for a sphere) to show that the terminal speed is

$$v_{ter} = \sqrt{\frac{8}{3} D g \frac{\varrho_{sph}}{\varrho_{air}}}. \qquad (2.88)$$

(b) Use this result to show that of two spheres of the same size, the denser one will eventually fall faster. **(c)** For two spheres of the same material, show that the larger will eventually fall faster.

2.25 ★ Consider the cyclist of Section 2.4, coasting to a halt under the influence of a quadratic drag force. Derive in detail the results (2.49) and (2.51) for her velocity and position, and verify that the constant $\tau = m/cv_o$ is indeed a time.

2.26 ★ A typical value for the coefficient of quadratic air resistance on a cyclist is around $c = 0.20$ N/(m/s)^2. Assuming that the total mass (cyclist plus cycle) is $m = 80$ kg and that at $t = 0$ the cyclist has an initial speed $v_o = 20$ m/s (about 45 mi/h) and starts to coast to a stop under the influence of air resistance, find the characteristic time $\tau = m/cv_o$. How long will it take him to slow to 15 m/s? What about 10 m/s? And 5 m/s? (Below about 5 m/s, it is certainly not reasonable to ignore friction, so there is no point pursuing this calculation to lower speeds.)

velocity is $v_o > 0$ at time $t = 0$. **(b)** At what time does it come instantaneously to rest? **(c)** By integrating $v(t)$, you can find $x(t)$. Do this and find how far the mass travels before coming instantaneously to rest.

SECTION 2.3 Trajectory and Range in a Linear Medium

2.15 ⋆ Consider a projectile launched with velocity (v_{xo}, v_{yo}) from horizontal ground (with x measured horizontally and y vertically up). Assuming no air resistance, find how long the projectile is in the air and show that the distance it travels before landing (the horizontal range) is $2v_{xo}v_{yo}/g$.

2.16 ⋆ A golfer hits his ball with speed v_o at an angle θ above the horizontal ground. Assuming that the angle θ is fixed and that air resistance can be neglected, what is the minimum speed $v_o(\text{min})$ for which the ball will clear a wall of height h, a distance d away? Your solution should get into trouble if the angle θ is such that $\tan\theta < h/d$. Explain. What is $v_o(\text{min})$ if $\theta = 25°$, $d = 50$ m, and $h = 2$ m?

2.17 ⋆ The two equations (2.36) give a projectile's position (x, y) as a function of t. Eliminate t to give y as a function of x. Verify Equation (2.37).

2.18 ⋆ Taylor's theorem states that, for any reasonable function $f(x)$, the value of f at a point $(x + \delta)$ can be expressed as an infinite series involving f and its derivatives at the point x:

$$f(x+\delta) = f(x) + f'(x)\delta + \frac{1}{2!}f''(x)\delta^2 + \frac{1}{3!}f'''(x)\delta^3 + \cdots \tag{2.87}$$

where the primes denote successive derivatives of $f(x)$. (Depending on the function this series may converge for *any* increment δ or only for values of δ less than some nonzero "radius of convergence.") This theorem is enormously useful, especially for small values of δ, when the first one or two terms of the series are often an excellent approximation.[11] **(a)** Find the Taylor series for $\ln(1+\delta)$. **(b)** Do the same for $\cos\delta$. **(c)** Likewise $\sin\delta$. **(d)** And e^δ.

2.19 ⋆ Consider the projectile of Section 2.3. **(a)** Assuming there is no air resistance, write down the position (x, y) as a function of t, and eliminate t to give the trajectory y as a function of x. **(b)** The correct trajectory, including a linear drag force, is given by (2.37). Show that this reduces to your answer for part (a) when air resistance is switched off (τ and $v_{ter} = g\tau$ both approach infinity). [*Hint:* Remember the Taylor series (2.40) for $\ln(1-\epsilon)$.]

2.20 ⋆⋆ [Computer] Use suitable graph-plotting software to plot graphs of the trajectory (2.36) of a projectile thrown at 45°above the horizontal and subject to linear air resistance for four different values of the drag coefficient, ranging from a significant amount of drag down to no drag at all. Put all four trajectories on the same plot. [*Hint:* In the absence of any given numbers, you may as well choose convenient values. For example, why not take $v_{xo} = v_{yo} = 1$ and $g = 1$. (This amounts to choosing your units of length and time so that these parameters have the value 1.) With these choices, the strength of the drag is given by the one parameter $v_{ter} = \tau$, and you might choose to plot the trajectories for $v_{ter} = 0.3, 1, 3,$ and ∞ (that is, no drag at all), and for times from $t = 0$ to 3. For the case that $v_{ter} = \infty$, you'll probably want to write out the trajectory separately.]

2.21 ⋆⋆⋆ A gun can fire shells in any direction with the same speed v_o. Ignoring air resistance and using cylindrical polar coordinates with the gun at the origin and z measured vertically up, show that

[11] For more details on Taylor's series see, for example, Mary Boas, *Mathematical Methods in the Physical Sciences* (Wiley, 1983), p. 22 or Donald McQuarrie, *Mathematical Methods for Scientists and Engineers* (University Science Books, 2003), p. 94.

2.9 ★ We solved the differential equation (2.29), $m\dot{v}_y = -b(v_y - v_{\text{ter}})$, for the velocity of an object falling through air, by inspection — a most respectable way of solving differential equations. Nevertheless, one would sometimes like a more systematic method, and here is one. Rewrite the equation in the "separated" form

$$\frac{m\,dv_y}{v_y - v_{\text{ter}}} = -b\,dt$$

and integrate both sides from time 0 to t to find v_y as a function of t. Compare with (2.30).

2.10 ★★ For a steel ball bearing (diameter 2 mm and density 7.8 g/cm^3) dropped in glycerin (density 1.3 g/cm^3 and viscosity 12 N·s/m^2 at STP), the dominant drag force is the linear drag given by (2.82) of Problem 2.2. **(a)** Find the characteristic time τ and the terminal speed v_{ter}. [In finding the latter, you should include the buoyant force of Archimedes. This just adds a third force on the right side of Equation (2.25).] How long after it is dropped from rest will the ball bearing have reached 95% of its terminal speed? **(b)** Use (2.82) and (2.84) (with $\kappa = 1/4$ since the ball bearing is a sphere) to compute the ratio $f_{\text{quad}}/f_{\text{lin}}$ at the terminal speed. Was it a good approximation to neglect f_{quad}?

2.11 ★★ Consider an object that is thrown vertically up with initial speed v_o in a linear medium. **(a)** Measuring y *upward* from the point of release, write expressions for the object's velocity $v_y(t)$ and position $y(t)$. **(b)** Find the time for the object to reach its highest point and its position y_{max} at that point. **(c)** Show that as the drag coefficient approaches zero, your last answer reduces to the well-known result $y_{\text{max}} = \frac{1}{2}v_o^2/g$ for an object in the vacuum. [*Hint:* If the drag force is very small, the terminal speed is very big, so v_o/v_{ter} is very small. Use the Taylor series for the log function to approximate $\ln(1+\delta)$ by $\delta - \frac{1}{2}\delta^2$. (For a little more on Taylor series see Problem 2.18.)]

2.12 ★★ Problem 2.7 is about a class of one-dimensional problems that can always be reduced to doing an integral. Here is another. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx'. \tag{2.85}$$

[*Hint:* Use the chain rule to prove the following handy relation, which we could call the "$v\,dv/dx$ rule": If you regard v as a function of x, then

$$\dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}. \tag{2.86}$$

Use this to rewrite Newton's second law in the separated form $m\,d(v^2) = 2F(x)\,dx$ and then integrate from x_o to x.] Comment on your result for the case that $F(x)$ is actually a constant. (You may recognise your solution as a statement about kinetic energy and work, both of which we shall be discussing in Chapter 4.)

2.13 ★★ Consider a mass m constrained to move on the x axis and subject to a net force $F = -kx$ where k is a positive constant. The mass is released from rest at $x = x_o$ at time $t = 0$. Use the result (2.85) in Problem 2.12 to find the mass's speed as a function of x; that is, $dx/dt = g(x)$ for some function $g(x)$. Separate this as $dx/g(x) = dt$ and integrate from time 0 to t to find x as a function of t. (You may recognize this as one way — not the easiest — to solve the simple harmonic oscillator.)

2.14 ★★★ Use the method of Problem 2.7 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_o e^{v/V}$, where F_o and V are constants. **(a)** Find $v(t)$ if the initial very large, the quadratic drag is dominant and the linear can be neglected; vice versa when R is very small. **(b)** Find the Reynolds number for a steel ball bearing (diameter 2 mm) moving at 5 cm/s through glycerin (density 1.3 g/cm^3 and viscosity 12 N·s/m^2 at STP).

2.4 ★★ The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. **(a)** Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ϱ, show that the rate at which the projectile encounters fluid (mass/time) is ϱAv. **(b)** Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is ϱAv^2. It is certainly not true that all the fluid that the projectile encounters is accelerated to the full speed v, but one might guess that the actual force would have the form

$$f_{\text{quad}} = \kappa \varrho A v^2 \tag{2.84}$$

where κ is a number less than 1, which would depend on the shape of the projectile, with κ small for a streamlined body, and larger for a body with a flat front end. This proves to be true, and for a sphere the factor κ is found to be $\kappa = 1/4$. **(c)** Show that (2.84) reproduces the form (2.3) for f_{quad}, with c given by (2.4) as $c = \gamma D^2$. Given that the density of air at STP is $\varrho = 1.29$ kg/m^3 and that $\kappa = 1/4$ for a sphere, verify the value of γ given in (2.6).

SECTION 2.2 Linear Air Resistance

2.5 ★ Suppose that a projectile which is subject to a linear resistive force is thrown vertically down with a speed v_{yo} which is *greater* than the terminal speed v_{ter}. Describe and explain how the velocity varies with time, and make a plot of v_y against t for the case that $v_{yo} = 2v_{\text{ter}}$.

2.6 ★ **(a)** Equation (2.33) gives the velocity of an object dropped from rest. At first, when v_y is small, air resistance should be unimportant and (2.33) should agree with the elementary result $v_y = gt$ for free fall in a vacuum. Prove that this is the case. [*Hint:* Remember the Taylor series for $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, for which the first two or three terms are certainly a good approximation when x is small.] **(b)** The position of the dropped object is given by (2.35) with $v_{yo} = 0$. Show similarly that this reduces to the familiar $y = \frac{1}{2}gt^2$ when t is small.

2.7 ★ There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in this chapter) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. Write down Newton's second law and separate the variables by rewriting it as $m\,dv/F(v) = dt$. Now integrate both sides of this equation and show that

$$t = m\int_{v_o}^{v} \frac{dv'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. Use this method to solve the special case that $F(v) = F_o$, a constant, and comment on your result. This method of *separation of variables* is used again in Problems 2.8 and 2.9.

2.8 ★ A mass m has velocity v_o at time $t = 0$ and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the method of Problem 2.7 to find v in terms of the time t and the other given parameters. At what time (if any) will it come to rest?

Here D denotes the linear size of the object. For a sphere, D is the diameter and, for a sphere in air at STP, $\beta = 1.6 \times 10^{-4}$ N·s/m^2 and $\gamma = 0.25$ N·s^2/m^4.

The Lorentz Force on a Charged Particle

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad \text{[Eq. (2.62) \& Problem 2.53]}$$

Problems for Chapter 2

Stars indicate the approximate level of difficulty, from easiest (★) to most difficult (★★★).

SECTION 2.1 Air Resistance

2.1 ★ When a baseball flies through the air, the ratio $f_{\text{quad}}/f_{\text{lin}}$ of the quadratic to the linear drag force is given by (2.7). Given that a baseball has diameter 7 cm, find the approximate speed v at which the two drag forces are equally important. For what approximate range of speeds is it safe to treat the drag force as purely quadratic? Under normal conditions is it a good approximation to ignore the linear term? Answer the same questions for a beach ball of diameter 70 cm.

2.2 ★ The origin of the linear drag force on a sphere in a fluid is the viscosity of the fluid. According to Stokes's law, the viscous drag on a sphere is

$$f_{\text{lin}} = 3\pi\eta D v \tag{2.82}$$

where η is the viscosity[8] of the fluid, D the sphere's diameter, and v its speed. Show that this expression reproduces the form (2.3) for f_{lin}, with b given by (2.4) as $b = \beta D$. Given that the viscosity of air at STP is $\eta = 1.7 \times 10^{-5}$ N·s/m^2, verify the value of β given in (2.5).

2.3 ★ **(a)** The quadratic and linear drag forces on a moving sphere in a fluid are given by (2.84) and (2.82) (Problems 2.4 and 2.2). Show that the ratio of these two kinds of drag force can be written as $f_{\text{quad}}/f_{\text{lin}} = R/48$,[9] where the dimensionless **Reynolds number** R is

$$R = \frac{Dv\varrho}{\eta} \tag{2.83}$$

where D is the sphere's diameter, v its speed, and ϱ and η are the fluid's density and viscosity. Clearly the Reynolds number is a measure of the relative importance of the two kinds of drag.[10] When R is

[8] For the record, the viscosity η of a fluid is defined as follows: Imagine a wide channel along which fluid is flowing (x direction) such that the velocity v is zero at the bottom ($y = 0$) and increases toward the top ($y = h$), so that successive layers of fluid slide across one another with a velocity gradient dv/dy. The force F with which an area A of any one layer drags the fluid above it is proportional to A and to dv/dy, and η is defined as the constant of proportionality; that is, $F = \eta A\, dv/dy$.

[9] The numerical factor 48 is for a sphere. A similar result holds for other bodies, but the numerical factor is different for different shapes.

[10] The Reynolds number is usually defined by (2.83) for flow involving any object, with D defined as a typical linear dimension. One sometimes hears the claim that R is the ratio $f_{\text{quad}}/f_{\text{lin}}$. Since $f_{\text{quad}}/f_{\text{lin}} = R/48$ for a sphere, this claim would be better phrased as "R is roughly of the order of $f_{\text{quad}}/f_{\text{lin}}$."

1.48 ★★ Find expressions for the unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ of cylindrical polar coordinates (Problem 1.47) in terms of the Cartesian $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$. Differentiate these expressions with respect to time to find $d\hat{\boldsymbol{\rho}}/dt$, $d\hat{\boldsymbol{\phi}}/dt$, and $d\hat{\mathbf{z}}/dt$.

1.49 ★★ Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

1.50 ★★★ [Computer] The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. **(a)** If you have access to software, such as Mathematica, Maple, or Matlab, that can solve differential equations numerically, solve the differential equation for the case that the board is released from $\phi_o = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of ϕ against time for two or three periods. **(b)** On the same picture, plot the approximate solution (1.57) with the same $\phi_o = 20°$. Comment on your two graphs. Note: If you haven't used the numerical solver before, you will need to learn the necessary syntax. For example, in Mathematica you will need to learn the syntax for "NDSolve" and how to plot the solution that it provides. This takes a bit of time, but is something that is very well worth learning.

1.51 ★★★ [Computer] Repeat all of Problem 1.50 but using the initial value $\phi_o = \pi/2$.

SECTION 1.7 Two-Dimensional Polar Coordinates

1.41 ⋆ An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law (1.48) in polar coordinates and find the tension in the string.

1.42 ⋆ Prove that the transformations from rectangular to polar coordinates and vice versa are given by the four equations (1.37). Explain why the equation for ϕ is not quite complete and give a complete version.

1.43 ⋆ **(a)** Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coordinates is equal to

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$. **(b)** Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

1.44 ⋆ Verify by direct substitution that the function $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$ of (1.56) is a solution of the second-order differential equation (1.55), $\ddot{\phi} = -\omega^2\phi$. (Since this solution involves two arbitrary constants — the coefficients of the sine and cosine functions — it is in fact the general solution.)

1.45 ⋆⋆ Prove that if $\mathbf{v}(t)$ is any vector that depends on time (for example the velocity of a moving particle) but which has *constant magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\hat{\mathbf{r}}/dt$ has to be in the direction of $\hat{\boldsymbol{\phi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

1.46 ⋆⋆ Consider the experiment of Problem 1.27, in which a frictionless puck is slid straight across a rotating turntable through the center O. **(a)** Write down the polar coordinates r, ϕ of the puck as functions of time, as measured in the inertial frame $\mathcal{S}$ of an observer on the ground. (Assume that the puck was launched along the axis $\phi = 0$ at $t = 0$.) **(b)** Now write down the polar coordinates r', ϕ' of the puck as measured by an observer (frame $\mathcal{S}'$) at rest on the turntable. (Choose these coordinates so that ϕ and ϕ' coincide at $t = 0$.) Describe and sketch the path seen by this second observer. Is the frame $\mathcal{S}'$ inertial?

1.47 ⋆⋆ Let the position of a point P in three dimensions be given by the vector $\mathbf{r} = (x, y, z)$ in rectangular (or Cartesian) coordinates. The same position can be specified by **cylindrical polar coordinates**, ρ, ϕ, z, which are defined as follows: Let P' denote the projection of P onto the xy plane; that is, P' has Cartesian coordinates $(x, y, 0)$. Then ρ and ϕ are defined as the two-dimensional polar coordinates of P' in the xy plane, while z is the third Cartesian coordinate, unchanged. **(a)** Make a sketch to illustrate the three cylindrical coordinates. Give expressions for ρ, ϕ, z in terms of the Cartesian coordinates x, y, z. Explain in words what ρ is ("ρ is the distance of P from ______"). There are many variants in notation. For instance, some people use r instead of ρ. Explain why this use of r is unfortunate. **(b)** Describe the three unit vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r}$ in terms of these unit vectors. **(c)** Differentiate your last answer twice to find the cylindrical components of the acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle. To do this, you will need to know the time derivatives of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. You could get these from the corresponding two-dimensional results (1.42) and (1.46), or you could derive them directly as in Problem 1.48.

1.34 ★★★ Prove that in the absence of external forces, the total *angular* momentum (defined as $\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$) of an N-particle system is conserved. [*Hints*: You need to mimic the argument from (1.25) to (1.29). In this case you need more than Newton's third law: In addition you need to assume that the interparticle forces are *central*; that is, $\mathbf{F}_{\alpha\beta}$ acts along the line joining particles α and β. A full discussion of angular momentum is given in Chapter 3.]

SECTION 1.6 Newton's Second Law in Cartesian Coordinates

1.35 ★ A golf ball is hit from ground level with speed v_o in a direction that is due east and at an angle θ above the horizontal. Neglecting air resistance, use Newton's second law (1.35) to find the position as a function of time, using coordinates with x measured east, y north, and z vertically up. Find the time for the golf ball to return to the ground and how far it travels in that time.

1.36 ★ A plane, which is flying horizontally at a constant speed v_o and at a height h above the sea, must drop a bundle of supplies to a castaway on a small raft. **(a)** Write down Newton's second law for the bundle as it falls from the plane, assuming you can neglect air resistance. Solve your equations to give the bundle's position in flight as a function of time t. **(b)** How far before the raft (measured horizontally) must the pilot drop the bundle if it is to hit the raft? What is this distance if $v_o = 50$ m/s, $h = 100$ m, and $g \approx 10$ m/s^2? **(c)** Within what interval of time ($\pm\Delta t$) must the pilot drop the bundle if it is to land within ± 10 m of the raft?

1.37 ★ A student kicks a frictionless puck with initial speed v_o, so that it slides straight up a plane that is inclined at an angle θ above the horizontal. **(a)** Write down Newton's second law for the puck and solve to give its position as a function of time. **(b)** How long will the puck take to return to its starting point?

1.38 ★ You lay a rectangular board on the horizontal floor and then tilt the board about one edge until it slopes at angle θ with the horizontal. Choose your origin at one of the two corners that touch the floor, the x axis pointing along the bottom edge of the board, the y axis pointing up the slope, and the z axis normal to the board. You now kick a frictionless puck that is resting at O so that it slides across the board with initial velocity $(v_{ox}, v_{oy}, 0)$. Write down Newton's second law using the given coordinates and then find how long the puck takes to return to the floor level and how far it is from O when it does so.

1.39 ★★ A ball is thrown with initial speed v_o up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. Write down Newton's second law using these axes and find the ball's position as a function of time. Show that the ball lands a distance $R = 2v_o^2 \sin\theta \cos(\theta + \phi)/(g \cos^2\phi)$ from its launch point. Show that for given v_o and ϕ, the maximum possible range up the inclined plane is $R_{max} = v_o^2/[g(1 + \sin\phi)]$.

1.40 ★★★ A cannon shoots a ball at an angle θ above the horizontal ground. **(a)** Neglecting air resistance, use Newton's second law to find the ball's position as a function of time. (Use axes with x measured horizontally and y vertically.) **(b)** Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight? [*Hint:* Using your solution to part (a) you can write down r^2 as $x^2 + y^2$, and then find the condition that r^2 is always increasing.]

standing on the ground (which we shall take to be an inertial frame) beside a perfectly flat horizontal turntable, rotating with constant angular velocity ω. I lean over and shove a frictionless puck so that it slides across the turntable, straight through the center. The puck is subject to zero net force and, as seen from my inertial frame, travels in a straight line. Describe the puck's path as observed by someone sitting at rest on the turntable. This requires careful thought, but you should be able to get a qualitative picture. For a quantitative picture, it helps to use polar coordinates; see Problem 1.46.

SECTION 1.5 The Third Law and Conservation of Momentum

1.28 ★ Go over the steps from Equation (1.25) to (1.29) in the proof of conservation of momentum, but treat the case that $N = 3$ and write out all the summations explicitly to be sure you understand the various manipulations.

1.29 ★ Do the same tasks as in Problem 1.28 but for the case of four particles ($N = 4$).

1.30 ★ Conservation laws, such as conservation of momentum, often give a surprising amount of information about the possible outcome of an experiment. Here is perhaps the simplest example: Two objects of masses m_1 and m_2 are subject to no external forces. Object 1 is traveling with velocity $\mathbf{v}$ when it collides with the stationary object 2. The two objects stick together and move off with common velocity $\mathbf{v}'$. Use conservation of momentum to find $\mathbf{v}'$ in terms of $\mathbf{v}$, m_1, and m_2.

1.31 ★ In Section 1.5 we proved that Newton's third law implies the conservation of momentum. Prove the converse, that if the law of conservation of momentum applies to every possible group of particles, then the interparticle forces must obey the third law. [*Hint:* However many particles your system contains, you can focus your attention on just two of them. (Call them 1 and 2.) The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.]

1.32 ★★ If you have some experience in electromagnetism, you could do the following problem concerning the curious situation illustrated in Figure 1.8. The electric and magnetic fields at a point $\mathbf{r}_1$ due to a charge q_2 at $\mathbf{r}_2$ moving with constant velocity $\mathbf{v}_2$ (with $v_2 \ll c$) are[15]

$$\mathbf{E}(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_o}\frac{q_2}{s^2}\hat{\mathbf{s}} \quad \text{and} \quad \mathbf{B}(\mathbf{r}_1) = \frac{\mu_o}{4\pi}\frac{q_2}{s^2}\mathbf{v}_2 \times \hat{\mathbf{s}}$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector pointing from $\mathbf{r}_2$ to $\mathbf{r}_1$. (The first of these you should recognize as Coulomb's law.) If $\mathbf{F}_{12}^{\text{el}}$ and $\mathbf{F}_{12}^{\text{mag}}$ denote the electric and magnetic forces on a charge q_1 at $\mathbf{r}_1$ with velocity $\mathbf{v}_1$, show that $F_{12}^{\text{mag}} \leq (v_1 v_2/c^2) F_{12}^{\text{el}}$. This shows that in the non-relativistic domain it is legitimate to ignore the magnetic force between two moving charges.

1.33 ★★★ If you have some experience in electromagnetism and with vector calculus, prove that the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's third law. [*Hints:* Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are short segments of the loops, then according to the Biot–Savart law, the force on $d\mathbf{r}_1$ due to $d\mathbf{r}_2$ is

$$\frac{\mu_o}{4\pi}\frac{I_1 I_2}{s^2} d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$. The force $\mathbf{F}_{12}$ is found by integrating this around both loops. You will need to use the "$BAC - CAB$" rule to simplify the triple product.]

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[15] See, for example, David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, (1999), p. 440.

1.19 ★★ If **r**, **v**, **a** denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}).$$

1.20 ★★ The three vectors **A**, **B**, **C** point from the origin O to the three corners of a triangle. Use the result of Problem 1.18 to show that the area of the triangle is given by

$$(\text{area of triangle}) = \tfrac{1}{2}|(\mathbf{B}\times\mathbf{C}) + (\mathbf{C}\times\mathbf{A}) + (\mathbf{A}\times\mathbf{B})|.$$

1.21 ★★ A parallelepiped (a six-faced solid with opposite faces parallel) has one corner at the origin O and the three edges that emanate from O defined by vectors **a**, **b**, **c**. Show that the volume of the parallelepiped is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

1.22 ★★ The two vectors **a** and **b** lie in the xy plane and make angles α and β with the x axis. **(a)** By evaluating $\mathbf{a}\cdot\mathbf{b}$ in two ways [namely using (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a}\times\mathbf{b}$ prove that

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

1.23 ★★ The unknown vector **v** satisfies $\mathbf{b}\cdot\mathbf{v} = \lambda$ and $\mathbf{b}\times\mathbf{v} = \mathbf{c}$, where λ, **b**, and **c** are fixed and known. Find **v** in terms of λ, **b**, and **c**.

SECTION 1.4 Newton's First and Second Laws; Inertial Frames

1.24 ★ In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple excercise to get you started: Find the general solution of the first-order equation $df/dt = f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f = dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

1.25 ★ Answer the same questions as in Problem 1.24, but for the differential equation $df/dt = -3f$.

1.26 ★★ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following: I am standing on a level floor at the origin of an inertial frame $\mathcal{S}$ and kick a frictionless puck due north across the floor. **(a)** Write down the x and y coordinates of the puck as functions of time as seen from my inertial frame. (Use x and y axes pointing east and north respectively.) Now consider two more observers, the first at rest in a frame $\mathcal{S}'$ that travels with constant velocity v due east relative to $\mathcal{S}$, the second at rest in a frame $\mathcal{S}''$ that travels with constant *acceleration* due east relative to $\mathcal{S}$. (All three frames coincide at the moment when I kick the puck, and $\mathcal{S}''$ is at rest relative to $\mathcal{S}$ at that same moment.) **(b)** Find the coordinates x', y' of the puck and describe the puck's path as seen from $\mathcal{S}'$. **(c)** Do the same for $\mathcal{S}''$. Which of the frames is inertial?

1.27 ★★ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following experiment: I am

1.14 ⋆ Prove that for any two vectors **a** and **b**,

$$|\mathbf{a} + \mathbf{b}| \leq (a + b).$$

[*Hint:* Work out $|\mathbf{a} + \mathbf{b}|^2$ and compare it with $(a + b)^2$.] Explain why this is called the triangle inequality.

1.15 ⋆ Show that the definition (1.9) of the cross product is equivalent to the elementary definition that $\mathbf{r} \times \mathbf{s}$ is perpendicular to both **r** and **s**, with magnitude $rs \sin\theta$ and direction given by the right-hand rule. [*Hint:* It is a fact (though quite hard to prove) that the definition (1.9) is independent of your choice of axes. Therefore you can choose axes so that **r** points along the x axis and **s** lies in the xy plane.]

1.16 ⋆⋆ **(a)** Defining the scalar product $\mathbf{r} \cdot \mathbf{s}$ by Equation (1.7), $\mathbf{r} \cdot \mathbf{s} = \sum r_i s_i$, show that Pythagoras's theorem implies that the magnitude of any vector **r** is $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$. **(b)** It is clear that the length of a vector does not depend on our choice of coordinate axes. Thus the result of part (a) guarantees that the scalar product $\mathbf{r} \cdot \mathbf{r}$, as defined by (1.7), is the same for any choice of orthogonal axes. Use this to prove that $\mathbf{r} \cdot \mathbf{s}$, as defined by (1.7), is the same for any choice of orthogonal axes. [*Hint:* Consider the length of the vector $\mathbf{r} + \mathbf{s}$.]

1.17 ⋆⋆ **(a)** Prove that the vector product $\mathbf{r} \times \mathbf{s}$ as defined by (1.9) is distributive; that is, that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) + (\mathbf{r} \times \mathbf{v})$. **(b)** Prove the product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.$$

Be careful with the order of the factors.

1.18 ⋆⋆ The three vectors **a**, **b**, **c** are the three sides of the triangle ABC with angles α, β, γ as shown in Figure 1.15. **(a)** Prove that the area of the triangle is given by any one of these three expressions:

$$\text{area} = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| = \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| = \tfrac{1}{2}|\mathbf{c} \times \mathbf{a}|.$$

(b) Use the equality of these three expressions to prove the so-called law of sines, that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

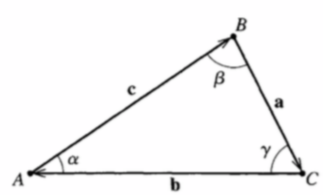

Figure 1.15 Triangle for Problem 1.18.

1.4 ★ One of the many uses of the scalar product is to find the angle between two given vectors. Find the angle between the vectors $\mathbf{b} = (1, 2, 4)$ and $\mathbf{c} = (4, 2, 1)$ by evaluating their scalar product.

1.5 ★ Find the angle between a body diagonal of a cube and any one of its face diagonals. [*Hint:* Choose a cube with side 1 and with one corner at O and the opposite corner at the point $(1, 1, 1)$. Write down the vector that represents a body diagonal and another that represents a face diagonal, and then find the angle between them as in Problem 1.4.]

1.6 ★ By evaluating their dot product, find the values of the scalar s for which the two vectors $\mathbf{b} = \hat{\mathbf{x}} + s\hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} - s\hat{\mathbf{y}}$ are orthogonal. (Remember that two vectors are orthogonal if and only if their dot product is zero.) Explain your answers with a sketch.

1.7 ★ Prove that the two definitions of the scalar product $\mathbf{r} \cdot \mathbf{s}$ as $rs\cos\theta$ (1.6) and $\sum r_i s_i$ (1.7) are equal. One way to do this is to choose your x axis along the direction of $\mathbf{r}$. [Strictly speaking you should first make sure that the definition (1.7) is independent of the choice of axes. If you like to worry about such niceties, see Problem 1.16.]

1.8 ★ **(a)** Use the definition (1.7) to prove that the scalar product is distributive, that is, $\mathbf{r} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{v}$. **(b)** If $\mathbf{r}$ and $\mathbf{s}$ are vectors that depend on time, prove that the product rule for differentiating products applies to $\mathbf{r} \cdot \mathbf{s}$, that is, that

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{s}) = \mathbf{r} \cdot \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{s}.$$

1.9 ★ In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, and c, that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$.

1.10 ★ A particle moves in a circle (center O and radius R) with constant angular velocity ω counterclockwise. The circle lies in the xy plane and the particle is on the x axis at time $t = 0$. Show that the particle's position is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}}R\cos(\omega t) + \hat{\mathbf{y}}R\sin(\omega t).$$

Find the particle's velocity and acceleration. What are the magnitude and direction of the acceleration? Relate your results to well-known properties of uniform circular motion.

1.11 ★ The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}}b\cos(\omega t) + \hat{\mathbf{y}}c\sin(\omega t),$$

where b, c, and ω are constants. Describe the particle's orbit.

1.12 ★ The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}}b\cos(\omega t) + \hat{\mathbf{y}}c\sin(\omega t) + \hat{\mathbf{z}}v_o t$$

where b, c, v_o and ω are constants. Describe the particle's orbit.

1.13 ★ Let $\mathbf{u}$ be an arbitrary fixed unit vector and show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2.$$

Explain this result in words, with the help of a picture.

Inertial Frames

An inertial frame is any reference frame in which Newton's first law holds, that is, a nonaccelerating, nonrotating frame.

Unit Vectors of a Coordinate System

If (ξ, η, ζ) are an orthogonal system of coordinates, then

$$\hat{\boldsymbol{\xi}} = \text{ unit vector in direction of increasing } \xi \text{ with } \eta \text{ and } \zeta \text{ fixed}$$

and so on, and any vector $\mathbf{s}$ can be expanded as $\mathbf{s} = s_\xi \hat{\boldsymbol{\xi}} + s_\eta \hat{\boldsymbol{\eta}} + s_\zeta \hat{\boldsymbol{\zeta}}$.

Newton's Second Law in Various Coordinate Systems

Vector Form	Cartesian (x, y, z)	2D Polar (r, ϕ)	Cylindrical Polar (ρ, ϕ, z)
$\mathbf{F} = m\ddot{\mathbf{r}}$	$\begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$	$\begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$	$\begin{cases} F_r = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$
	Eq. (1.35)	Eq. (1.48)	Problem 1.47 or 1.48

Problems for Chapter 1

The problems for each chapter are arranged according to section number. A problem listed for a given section requires an understanding of that section and earlier sections, but not of later sections. Within each section problems are listed in approximate order of difficulty. A single star (⋆) indicates straightforward problems involving just one main concept. Two stars (⋆⋆) identify problems that are slightly more challenging and usually involve more than one concept. Three stars (⋆⋆⋆) indicate problems that are distinctly more challenging, either because they are intrinsically difficult or involve lengthy calculations. Needless to say, these distinctions are hard to draw and are only approximate.

Problems that need the use of a computer are flagged thus: [Computer]. These are mostly classified as ⋆⋆⋆ on the grounds that it usually takes a long time to set up the necessary code — especially if you're just learning the language.

SECTION 1.2 Space and Time

1.1 ⋆ Given the two vectors $\mathbf{b} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$ find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} + 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

1.2 ⋆ Two vectors are given as $\mathbf{b} = (1, 2, 3)$ and $\mathbf{c} = (3, 2, 1)$. (Remember that these statements are just a compact way of giving you the components of the vectors.) Find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} - 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

1.3 ⋆ By applying Pythagoras's theorem (the usual two-dimensional version) twice over, prove that the length r of a three-dimensional vector $\mathbf{r} = (x, y, z)$ satisfies $r^2 = x^2 + y^2 + z^2$.