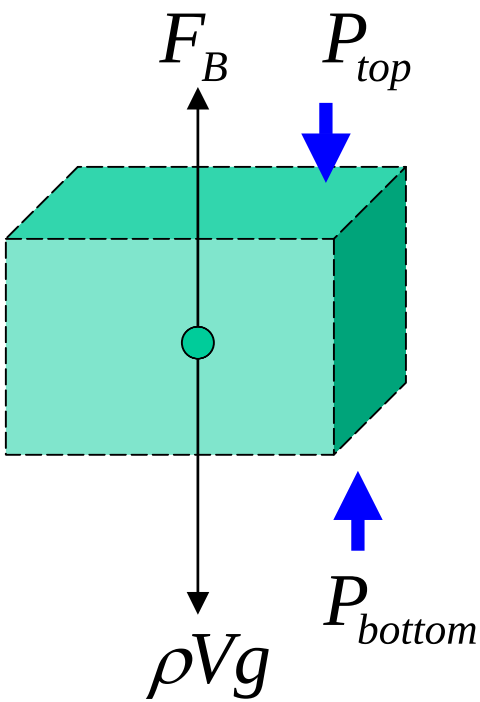
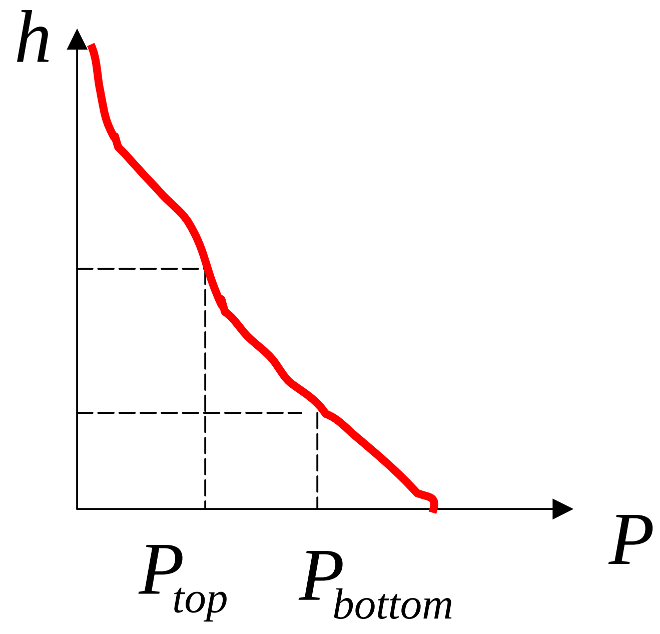


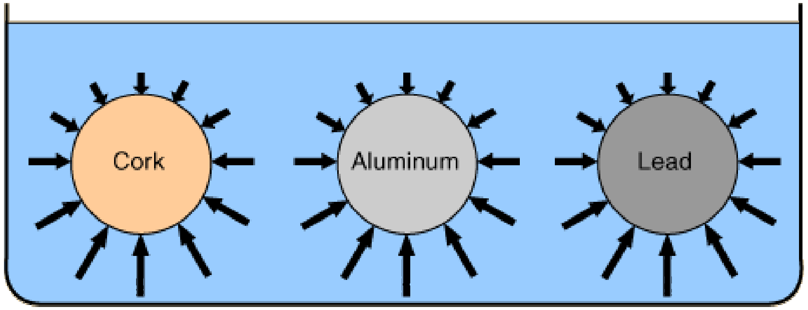
Buoyant force

The volume of gas (liquid) is at rest with respect to its surrounding gas (liquid): the force of gravity is balanced by the buoyant force

$$F_B = (P_{bottom} - P_{top})A$$



Note that the buoyant force does not care what's ***inside*** this volume (a brick, a gas, or vacuum): it depends only on the volume and the density of the ***outside*** gas (liquid).



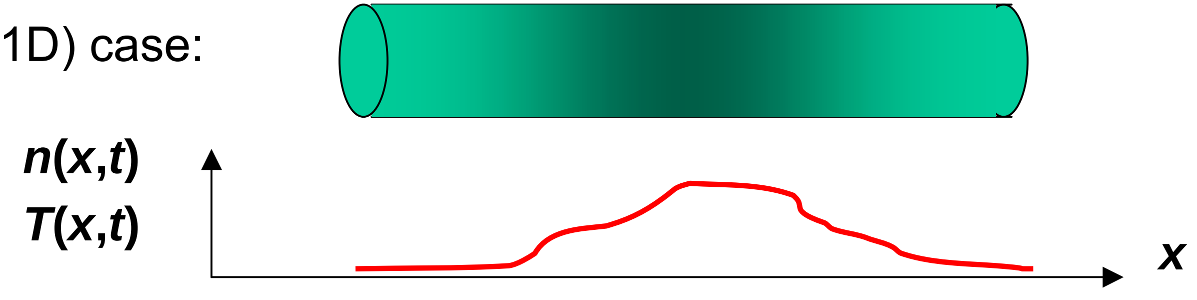
Lecture 3. Transport Phenomena (Ch.1)

Lecture 2 – various processes in **macro systems near the state of equilibrium** can be described by a handful of macro parameters. ***Quasi-static processes*** – sufficiently slow processes, at any moment the system is ***almost*** in equilibrium.

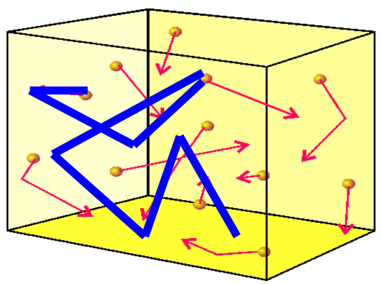
It is important to know how much time it takes for a system to approach an equilibrium state. A system is not in equilibrium when the macroscopic parameters (***T***, ***P***, etc.) are not constant throughout the system. To approach equilibrium, these non-uniformities have to be ironed out through **the transport of energy, momentum, and mass** from one part of the system to another. The mechanism of transport is molecular collisions. Our goal - to estimate the characteristic rates of approaching equilibrium, and, thus, to impose limitations on the rates of "quasi-static" processes.

1. **Transfers of Q ("Heat" Conduction)**
2. **Transfers of Mass (Diffusion)**

One-dimensional (1D) case:



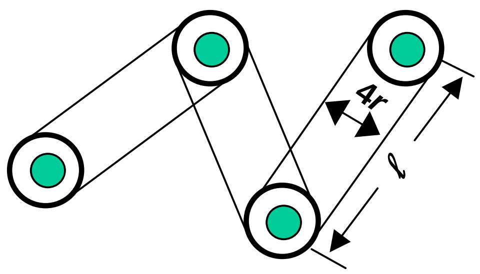
The Mean Free Path of Molecules



Transports energy, momentum, mass – due to random thermal motion of molecules in gases and liquids.

The mean free path *l* - the average distance traveled by a molecule btw two successive collisions.

An estimate: one molecule is moving with a constant speed ***v***, the other molecules are fixed. Model of hard spheres, the radius of molecule ***r*** ~ $1{\cdot}10^{-10}$ m.



The av. distance traversed by a molecule until the 1st collision is the distance in which the av. # of molecules in this cylinder is 1.

$$\pi (2r)^2 l \times \frac{N}{V} = 1 \quad \Rightarrow \quad l = \frac{1}{4\pi r^2}\frac{V}{N} = \frac{1}{\sigma n}$$

n = N/V – the density of molecules

$\sigma = 4\pi r^2$ – the cross section

Maxwell:

$$l = \frac{1}{\sqrt{2}\,\sigma n}$$

The average time interval between successive collisions - **the collision time**:

$$\tau = \frac{l}{\bar{v}}$$

$\bar{v}$ - the most probable speed of a molecule

Some Numbers:

$$l \propto \frac{1}{\sigma\, n}$$ $\Rightarrow$ for an ideal gas: $PV = Nk_B T$ $\quad P = nk_B T$ $\quad \Rightarrow \quad l \propto \frac{1}{n} \propto \frac{T}{P}$

air at norm. conditions: $$\frac{1}{n} = \frac{V}{N} = \frac{k_B T}{P} = \frac{1.38 \cdot 10^{-23}\,\text{J/K} \times 300\text{K}}{10^5\,\text{Pa}} \approx 4 \cdot 10^{-26}\,\text{m}^3$$

the intermol. distance $$d = \sqrt[3]{\frac{V}{N}} \sim 3 \cdot 10^{-9}\ \text{m}$$

P = 10^5 Pa: ***l*** ~ 10^{-7} m - 30 times greater than *d*

P = 10^{-2} Pa (10^{-4}mbar): ***l*** ~ 1 m (size of a typical vacuum chamber)

- at this ***P***, there are still ~$2.5 \cdot 10^{12}$ molecule/cm^3 **(!)**

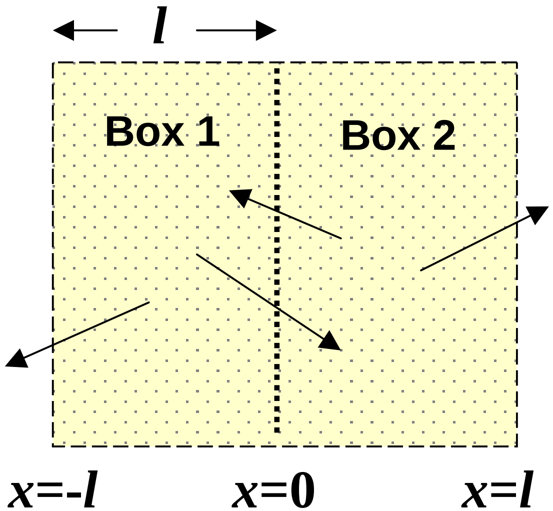
$$\frac{l}{d} \propto n^{-2/3} \propto P^{-2/3}$$

The collision time at norm. conditions: $\tau \sim 10^{-7}$m / 500m/s = $2 \cdot 10^{-10}$ s

For $\boldsymbol{H_2}$ gas in interstellar space, where the density is ~ 1 molecule/ cm^3,

l ~ 10^{13} m - ~ 100 times greater than the Sun-Earth distance ($1.5 \cdot 10^{11}$ m)

Transport in Gases (Liquids)



Simplified approach: consider the "ballistic" molecule exchange between two "boxes" within the gas (thickness of each box should be comparable to the mean free path of molecules, ***l***). During the average time between molecular collisions, τ, roughly half the molecules in Box 1 will move to the right in Box 2, while roughly half the molecules in Box 2 will move to the left in Box 1.

Each molecule "carries" some quantity φ (mass, kin. energy, etc.), within each box - $\Phi = N\varphi = A\, l\, n\, \varphi$. E.g., the flux of the number of molecules across the border per unit area of the border, J_x:

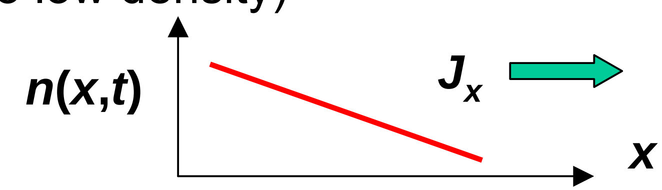
$$J_n \equiv \frac{\Delta N}{A\Delta t} = \frac{1}{6}\bar{v}\left[n(x=-l) - n(x=l)\right] = \frac{1}{6}\bar{v}\left(-2l\frac{\partial n}{\partial x}\right) = -\frac{1}{3}\bar{v}\, l\frac{\partial n}{\partial x} = -D\frac{\partial n}{\partial x}$$

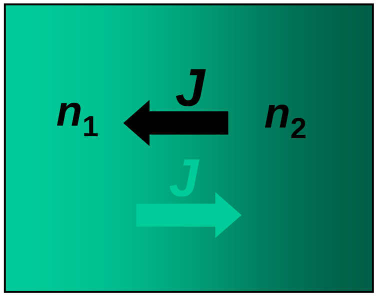
in a 3D case, on average 1/6 of the molecules have a velocity along +*x* or –*x*

the diffusion constant

"-" - if ***∂n/∂x*** is negative, the flux is in the positive ***x*** direction (the current flows from high density to low density)

In a 3D case, $\vec{J}_n = -D\nabla n \qquad \vec{J}_U = -K_{th}\nabla T$





Diffusion

Diffusion – the flow of randomly moving particles caused by variations of the *concentration* of particles. Example: a mixture of two gases, the total concentration $\boldsymbol{n = n_1 + n_2}$ **=const** over the volume ($\boldsymbol{P}$ **= const**).

Fick's Law:

$$J_x = -\frac{1}{3} l \bar{v} \frac{\partial n}{\partial x} = -D \frac{\partial n}{\partial x}$$

$$D = \frac{1}{3} l \bar{v}$$

- ***the diffusion coefficient***

(numerical pre-factor depends on the dimensionality: 3D – 1/3; 2D – 1/2)

$$D = \frac{1}{3} l \bar{v}$$

its dimensions $[L]^2 [t]^{-1}$, its units $m^2 s^{-1}$

Typically, at normal conditions, $\boldsymbol{l} \sim 10^{-7}$ m, $\boldsymbol{v} \sim 300$ m/s $\Rightarrow$ $\boldsymbol{D} \sim 10^{-5}\ m^2 s^{-1}$
(in liquids, $\boldsymbol{D}$ is much smaller, $\sim 10^{-10}\ m^2 s^{-1}$)

For electrons in well-ordered semiconductor heterostructures at low *T*:
$\boldsymbol{l} \sim 10^{-5}$ m, $\boldsymbol{v} \sim 10^5$ m/s $\Rightarrow$ $\boldsymbol{D} \sim 1\ m^2 s^{-1}$

Diffusion Coefficient of an Ideal Gas (*Pr. 1.70*)

for an ideal gas: $l \propto \frac{1}{n} \propto \frac{k_B T}{P}$

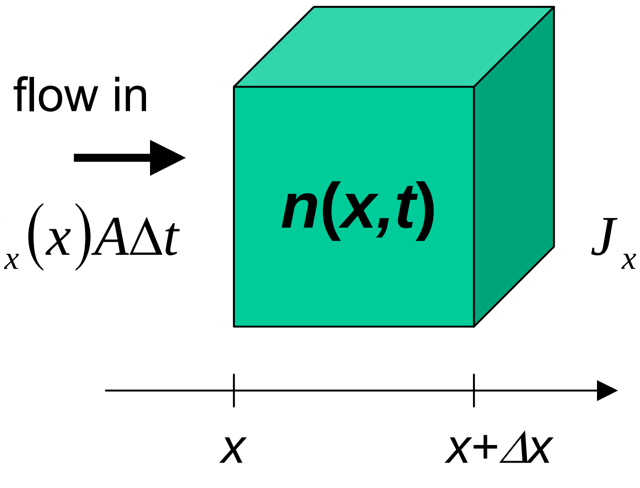
from the equipartition theorem: $\bar{v} \propto T^{1/2}$

$$D \propto T^{1/2} \frac{T}{P} = \frac{T^{3/2}}{P}$$

therefore, at a const. temperature: $D \propto \frac{1}{P}$

and at a const. pressure: $D \propto T^{3/2}$

The Diffusion Equation



$J_x(x)A\Delta t$ $\quad$ $J_x(x+\Delta x)A\Delta t$

change of ***n*** inside:

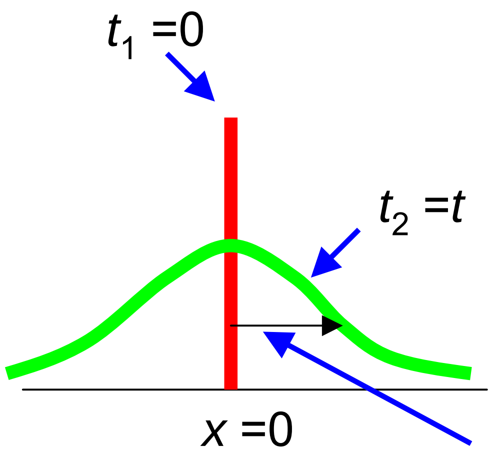
$$\frac{\partial n}{\partial t} = -\frac{\partial J_x}{\partial x}$$

combining with $J_x = -D\frac{\partial n}{\partial x}$

we'll get the equation that describes one-dimensional diffusion:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$$

the diffusion equation



the solution which corresponds to an initial condition that all particles are at x =0 at t =0:

$$n(x,t) = \frac{C}{\sqrt{Dt}}\exp\left(-\frac{x^2}{4Dt}\right)$$

C is a normalization factor

the ***rms*** displacement of particles: $\sqrt{\langle x^2 \rangle} \approx \sqrt{Dt}$

Brownian Motion (self-diffusion)

Historical background:

The experiment by the botanist R. Brown concerning the drifting of tiny (~ 1μm) specks in liquids and gases, had been known since 1827.

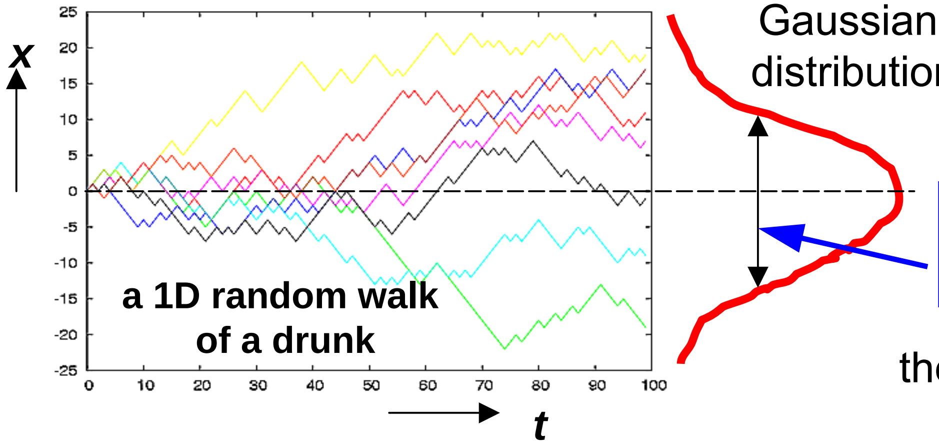
Brownian motion was in focus of the struggle for and against the atomic structure of matter, which went on during the second half of the 19th century and involved many prominent physicists.

Ernst Mach: "If the belief in the existence of atoms is so crucial in your eyes, I hereby withdraw from the physicist's way of thought..."

Albert Einstein explained the phenomenon on the basis of the kinetic theory (1905), connected in a quantitative manner the Brownian motion and such macroscopic quantities as the coefficients of mobility and viscosity – and brought the debate to a conclusion in a short time.

Observing the Brownian motion under a microscope, Jean Perrin measured the Boltzman constant and Avogadro number in 1908 (Nobel 1926).

Brownian Motion (cont.)



Gaussian distribution $n(x,t) = \frac{C}{\sqrt{Dt}} \exp\left(-\frac{x^2}{4Dt}\right)$

$$\sqrt{\left\langle x^2 \right\rangle_{\Delta t}} = \sqrt{D\Delta t}$$

the *rms* displacement

A body that participates in a random walk, or a subject of random collisions with the gas molecules. Its average displacement is zero. However, ***the average square distance grows linearly with time***:

after *N* steps, the position is $\vec{R}_{N+1}$ $\quad \vec{R}_{N+1} = \vec{R}_{N+1} + l\,\vec{n}$

$\vec{n}$ - a randomly oriented unit vector

$$\vec{R}_{N+1}^2 = \left(\vec{R}_N + l\,\vec{n}\right)^2 = \vec{R}_N^2 + l^2 + 2l\,\vec{n}\cdot\vec{R}_N$$

after averaging ($\left\langle \vec{R}_{N+1} \right\rangle = 0$): $\quad \left\langle \vec{R}_{N+1}^2 \right\rangle = \left\langle \vec{R}_N^2 \right\rangle + l^2 \quad \Rightarrow \quad \left\langle \vec{R}_N^2 \right\rangle = N\,l^2 \propto t$

For air at normal conditions ($l \approx 10^{-7}\,\mathrm{m} \quad \bar{v} \approx 500\,\mathrm{m/s} \quad D \approx 1.7\cdot 10^{-5}\,\mathrm{m^2/s}$), it takes

$\Delta t = \frac{L^2}{D} \sim 10^5\,\mathrm{s}$ for a molecule to “diffuse” over 1m: odor spreads by convection

For electrons in metals at 300K ($l \approx 10^{-7}\,\mathrm{m} \quad \bar{v} \approx 10^6\,\mathrm{m/s} \quad D \approx 3\cdot 10^{-2}\,\mathrm{m^2/s}$), it takes

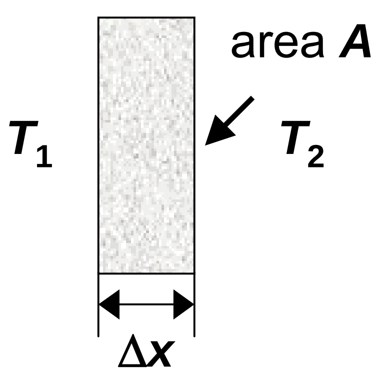
$\Delta t = \frac{L^2}{D} \sim 30\,\mathrm{s}$ to “diffuse” over 1m. For the electron gas in metals, convection can be ignored: the electron velocities are randomized by impurity/phonon scattering.

Static Energy Flow by "Heat" Conduction

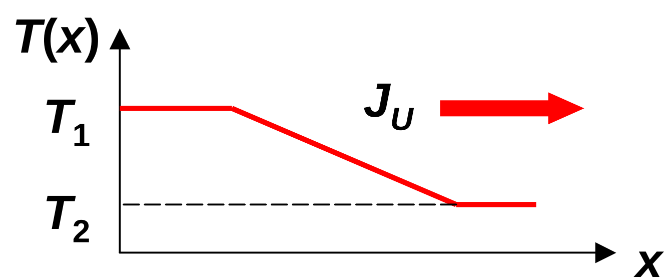
In general, the energy transport due to molecular motion is described by the equation of heat conduction:

$$\frac{\partial T}{\partial t} = -\frac{1}{C}\frac{\partial J_U}{\partial x} = \frac{K_{th}}{C}\frac{\partial^2 T}{\partial x^2} \qquad J_U = -K_{th}\frac{\partial T}{\partial x}$$

Thus, in principle, if you know the initial conditions, e.g. $T(x,t=t_0)$, you can describe the process by solving the equation. Often, you are asked to consider a different situation: a ***static*** flow of energy from a "hot" object to a "cold" one. (At what rate the internal energy is transferred between two systems with $T_1 \neq T_2$ or between parts of a non-equilibrium system (if one can introduce T_i) ?) The temperature distribution in this formulation is time-independent, and we need to calculate the flux of thermal energy J_U due to the ***heat conduction*** (diffusion/intermixing of particles with different energies, interactions between the particles that vibrate but do not move "translationally").



Heat conduction (static heat flow, ΔT = const)



Fourier Heat Conduction Law

$$\delta Q \propto \frac{\Delta T \Delta t}{\Delta x} A \quad \Rightarrow \quad \boxed{J_U \equiv \frac{\delta Q}{\Delta t} = -K_{th} A \frac{\Delta T}{\Delta x}}$$

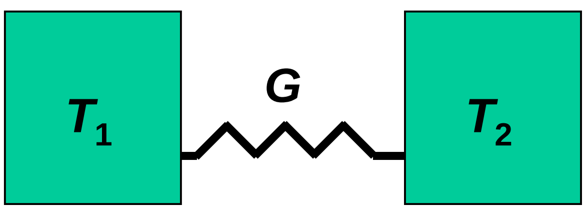
"-" - if ***T*** increases from left to right, energy flows from right to left

K_{th} [*W/K·m*] – the thermal conductivity (material-specific)

Pr. 1.56 For a window glass (K_{th} =0.8W/m·K, 3 mm thick, A=1m²) and Δ***T*** = 20K:

$$\frac{\delta Q}{\Delta t} = (0.8\ \mathrm{W/m \cdot K})(1\ \mathrm{m}^2)\frac{20\ \mathrm{K}}{0.003\ \mathrm{m}} \approx 5300\ \mathrm{W}$$

~ 10 times greater than in reality, a thin layer of still air must contribute to thermal insulation.



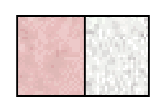
$$J_U(\mathrm{power}) = G\Delta T, \qquad G = K_{th}\frac{A}{\Delta x}$$

G – the thermal conductivity [W/K]

R* =1/*G – the thermal resistivity

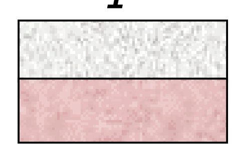
	Electricity	Thermal Physics
What "flows"	Charge *Q*	Th. Energy, δQ
Flux	Currant *dQ/dt*	Power δ*Q/dt*
Driving "force"	El.-stat. pot. difference	Temperature difference
"Resistance"	El. resistance *R*	Th. resistance *R*

***Connection in series* (*Pr. 1.57*):**

T_1  T_2 $\quad R_{tot} = R_1 + R_2$

Connection in parallel:

$$R_{tot}^{-1} = R_1^{-1} + R_2^{-1}$$

T_1  T_2

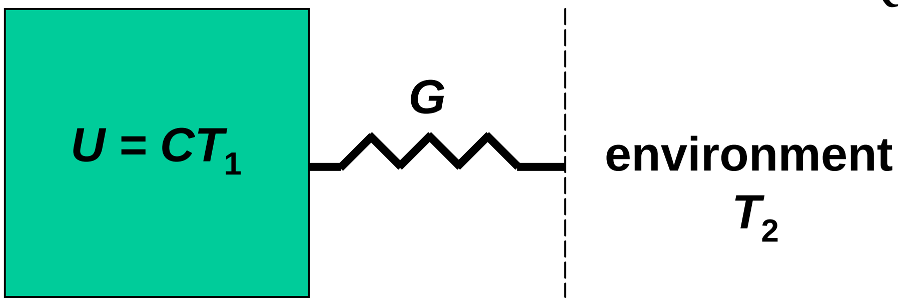
Relaxation Time due to Thermal Conductivity

(a rough estimate)

the heat capacity (specific heat)

$$\tau \sim \frac{U}{\delta Q / dt} = \frac{C\,T}{G\,\Delta T} \approx \left[T_1 - T_2 \approx T_1 \right] \approx \frac{C}{G}$$

the thermal conductivity



the thermal conductivity $G = K_{th} \frac{A}{\Delta x}$

Problem 1.60: A frying pan is quickly heated on the stovetop to 200⁰C. It has an iron handle that is 20 cm long. Estimate how much time should pass before the end of the handle is too hot to grab (the density of iron ρ = 7.9 g/cm³, its specific heat *c* = 0.45 J/g·K, the thermal conductivity K_{th}=80 W/m·K).

$$\tau \approx \frac{C}{G} = \frac{\rho\, c\, A L}{K_{th} \frac{A}{L}} = \frac{\rho\, c\, L^2}{K_{th}} = \frac{7900\, kg \cdot m^{-3} \times 450\, J \cdot kg^{-1} \cdot K^{-1} \times (0.1 m)^2}{80\, J \cdot s^{-1} \cdot m^{-1} \cdot K^{-1}} \sim 400 s$$

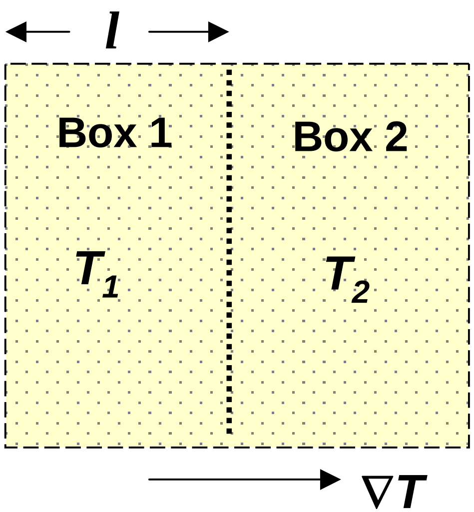
Thermal Conductivity of an Ideal Gas

Energy "flow", $\Delta t \sim \tau$:

the time between two consecutive collisions $\tau = \frac{l}{\bar{v}}$

$$\frac{\delta Q}{\tau} = \frac{1}{2}\frac{(U_1 - U_2)}{\tau} = \frac{1}{2}\frac{C_V (T_1 - T_2)}{\tau} = \frac{1}{2}\frac{C_V\, l}{\tau}\frac{dT}{dx} = \frac{1}{2} C_V \bar{v} \frac{dT}{dx}$$

$$\Updownarrow$$



$$\frac{\delta Q}{\tau} = -K_{th} A \frac{\Delta T}{\Delta x} \quad \Rightarrow \quad K_{th} = \frac{1}{2}\frac{C_V \bar{v}}{A} = \frac{1}{2}\frac{C_V\, l\, \bar{v}}{A l} = \frac{1}{2}\frac{C_V}{V} l\, \bar{v}$$

the specific heat capacity

$$c_V \equiv \frac{C_V}{V} = \frac{\frac{f}{2} N k_B}{V} = \frac{f}{2} n\, k_B$$

$$\boxed{K_{th} = \frac{f}{4} n\, k_B\, l\, \bar{v}}$$

The thermal conductivity of air at norm. conditions:

$$K_{th} = \frac{f}{4} n\, k_B\, l\, \bar{v} = \frac{5}{4} \times 2.4 \cdot 10^{25}\,\mathrm{m}^{-3} \times 1.38 \cdot 10^{-23}\,\mathrm{J/K} \times 10^{-7}\,\mathrm{m} \times 500\,\mathrm{m/s} \approx 0.02 \frac{\mathrm{W}}{\mathrm{m \cdot K}}$$

(exp. value – 0.026 W/m·K)

Thermal Conductivity of Gases (cont.)

$$K_{ht} = \frac{f}{4} n\, k_B\, l\, \bar{v} \quad \Rightarrow \quad l = (\sigma\, n)^{-1}, \bar{v} \propto \sqrt{\frac{T}{m}} \quad \Rightarrow \quad K_{th} \propto \sqrt{\frac{T}{m}}$$

1. $K_{th} \propto 1/\sqrt{m}$

- an argon-filled window helps to reduce Q

2. Thermal conductivity of an ideal gas is independent of the gas density!

(at higher densities, more molecules participate in the energy transfer, but they carry the energy over a shorter distance)

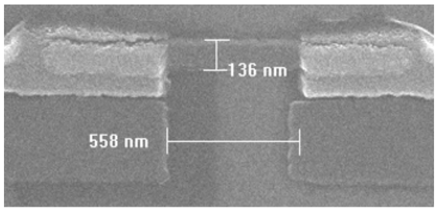
	Thermal Conductivity at 300 K (W/mK)
Air	0.026
Ar	0.018
CO	0.025
CO_2	0.017
H	0.182
He	0.151
N2	0.026
Ne	0.049
O_2	0.027

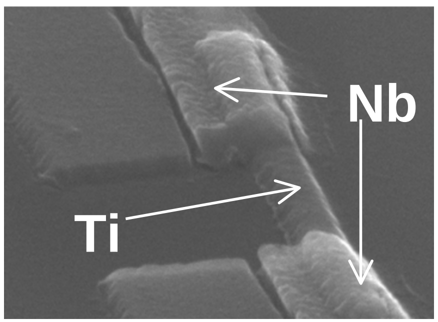
This conclusion holds only if ***L*** **>>** ***l*** .

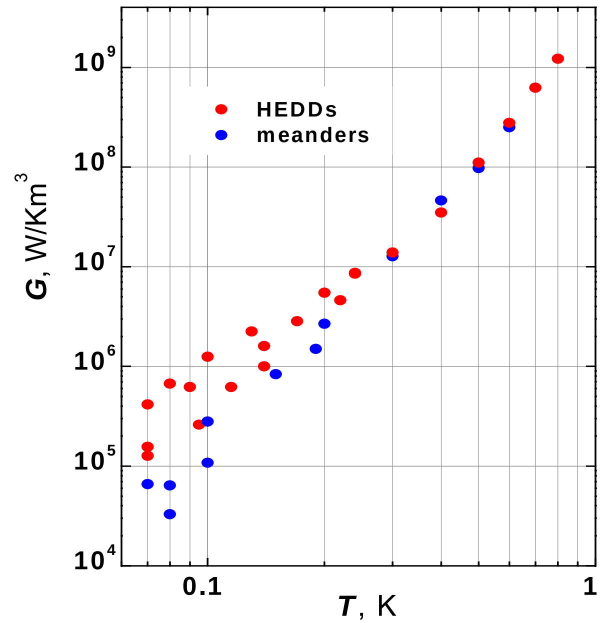
For ***L*** **<** ***l*** , $K_{th} \propto n$

Sate-of-the-art Bolometers (direct detectors of e.-m. radiation)

$$\frac{\delta Q}{dt}(\text{power}) = RI^2 = G_{e-ph}(T)\Delta T = G_{e-ph}(T)(T_e - T_{ph})$$







specific heat of electrons

$$\tau = \gamma V T_c / G_{e-ph}$$

$\hbar\omega$

T_e **electrons**

T_{ph} **phonons**

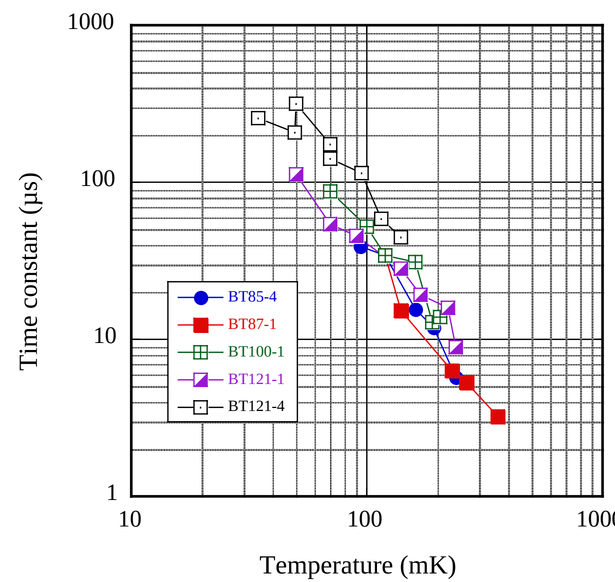
T **heat sink**

$$G_{e-ph} = C_e/\tau_{e-ph}$$

$$G = C_{ph}/\tau_{es}$$

$$G_{e-ph} \ll G$$

$$\tau \sim \tau_{e-ph}$$



Momentum Transfer, Viscosity

Drag – transfer of the momentum in the direction perpendicular to velocity.

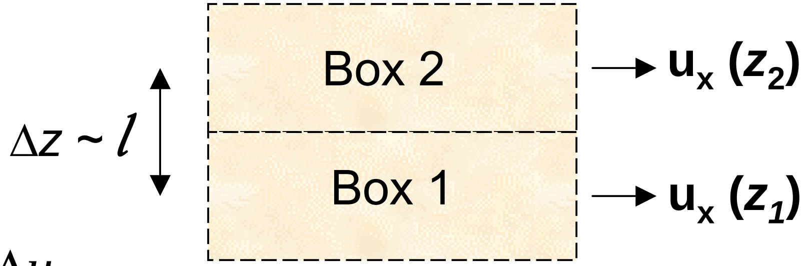
$$\frac{\Delta p_x}{\Delta t} \equiv F_x \propto \frac{A \cdot \left(u_{x,\text{top}} - u_{x,\text{bottom}}\right)}{\Delta z}$$

Laminar flow of a gas (fluid) between two surfaces moving with respect to each other.

$$\boxed{\frac{F_x}{A} = \eta \frac{d\,u_x}{\Delta z}}$$

F_x – the viscous drag force, ***η*** - the coefficient of viscosity
F_x/A – shear stress

Viscosity of an ideal gas (Pr. 1.66): $\Delta z \sim l$



$$\Delta p_z \approx \frac{1}{2} Nmu_x(z_1) - \frac{1}{2} Nmu_x(z_2) = -\frac{1}{2} Nm\,\Delta u_x$$

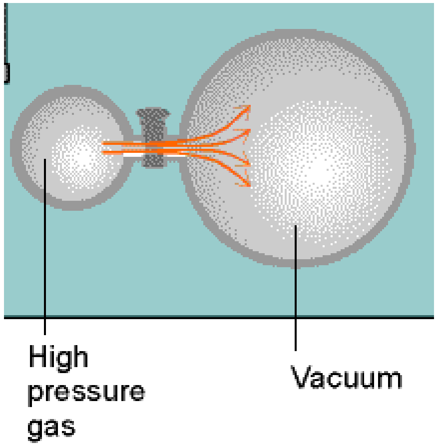
$$\frac{F_x}{A} \approx \frac{1}{A}\frac{\Delta p_z}{\tau} = \frac{Nm\Delta u_x\,\bar{v}}{2Al} \quad \Rightarrow \quad \frac{F_x}{A} \approx \left(\frac{1}{2}\rho\,\bar{v}\,l\right)\frac{du_x}{d\,z}$$

$$\boxed{\eta = \frac{1}{2}\rho\,\bar{v}\,l} \quad \propto \boldsymbol{T^{1/2}}$$

Effusion of an Ideal Gas

- the process of a gas escaping through a small hole (***a*** << ***l***) into a vacuum (Pr. 1.22) – the collisionless regime.

The opposite limit of a very large hole (***a*** >> ***l***) – the hydrodynamic regime.



The number of molecules that escape through a hole of area ***A*** in 1 sec, N_h, in terms of ***P(t)***, ***T*** (how is ***T*** changing in the process?)

$$P = N_h \frac{\Delta p}{\Delta t}\frac{1}{A} = N_h \frac{2m\langle v_x\rangle}{\Delta t}\frac{1}{A}$$

$$N_h = \frac{P\,A\,\Delta t}{2m\,\langle v_x\rangle}$$

$$\langle v_x\rangle \quad \Rightarrow \quad \frac{1}{2}m\langle v_x{}^2\rangle = \frac{1}{2}k_BT, \quad \langle v_x\rangle \approx \sqrt{\langle v_x{}^2\rangle} = \sqrt{\frac{k_BT}{m}}$$

$N_h = -\Delta N$, where ***N*** is the total # of molecules in a system

$$-\Delta N = \frac{P\,A\,\Delta t}{2m}\sqrt{\frac{m}{k_BT}} = \frac{Nk_BT}{V}\frac{A\Delta t}{2m}\sqrt{\frac{m}{k_BT}} = \frac{AN\Delta t}{2V}\sqrt{\frac{k_BT}{m}}$$

$$\frac{\Delta N}{\Delta t} = -\frac{A}{2V}\sqrt{\frac{k_BT}{m}}N = -\frac{1}{\tau}N$$

$$N(t) = N(0)\exp\left(-\frac{t}{\tau}\right), \quad \tau = \frac{2V}{A}\sqrt{\frac{m}{k_BT}}$$

Depressurizing of a space ship,
V - 50m^3, ***A*** of a hole in a wall – 10^{-4} m^2
(clearly, ***a*** << ***l*** does not apply)

$$\tau = \frac{2\times 50\,\text{m}^3}{10^{-4}\,\text{m}^2}\sqrt{\frac{30\times 1.7\cdot 10^{-27}\,\text{kg}}{1.38\cdot 10^{-23}\,\text{J/K}\times 30\,\text{K}}} \approx 10^6\times 10^{-2}\times 0.3\,\text{s} = 3000\,\text{s}$$