A loop of $SU(2)$ gauge fields on $S^4$
stable under the Yang-Mills flow

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The space of connections

\[ G \text{ a compact Lie group} \quad \text{— say } G = SU(2) \]
\[ M \text{ a riemannian 4-manifold} \quad \text{— say } M = S^4 \]
\[ B \text{ a principle } G\text{-bundle over } M \quad \text{— say } B = S^4 \times SU(2) \]

\[ \mathcal{A} = \text{the space of connections in } B \]
\[ D = d + A \text{ the covariant derivative} \]
\[ A \text{ an } su(2)\text{-valued 1-form on } S^4 \]
\[ F = D^2 \text{ the curvature, an } su(2)\text{-valued 2-form} \]

\[ \mathcal{G} = \text{the group of automorphisms of } B \text{ (gauge transformations)} \]
\[ \mathcal{G} = Maps(S^4 \to SU(2)) \]

\[ \mathcal{A}/\mathcal{G} = \text{the gauge equivalence classes of connections} \]
The Y-M flow on the unparametrized loops of connections

The Yang-Mills action

\[ S_{YM}(A) = \frac{1}{2\pi^2} \int_M \frac{1}{4} \text{tr}(-F \ast F) \]
(normalized so that the instanton has \( S_{YM} = 1 \).)

The metric on \( \mathcal{A} \)

\[ (ds)^2_{\mathcal{A}} = \frac{1}{2\pi^2} \int_M \frac{1}{2} \text{tr}(-\delta A \ast \delta A) \]

The gradient flow

\[ \frac{\partial A}{\partial t} = -\nabla S_{YM} = \ast D \ast F(A) \]

\[ (\partial_t A_\mu(x) = D^\nu F_{\nu \mu}(x) = \partial^\nu \partial_\nu A_\mu(x) + \cdots) \]

The Y-M flow is \( \mathcal{G} \)-invariant, so it acts:

- on the gauge equivalence classes, \( \mathcal{A}/\mathcal{G} \)
- pointwise on loops of connections, \( \text{Maps}(S^1 \to \mathcal{A}/\mathcal{G}) \)
- on unparametrized loops, \( \mathcal{L} = \text{Maps}(S^1 \to \mathcal{A}/\mathcal{G})/\text{Diff}(S^1) \)
The problem

What is the generic long-time behavior of the Y-M flow on the space $\mathcal{L}$ of unparametrized loops?

A stable loop would be a parametrized loop $\sigma \mapsto A(\sigma)$ on which the flow acts by reparametrization: $\partial_t A(\sigma) = v(\sigma) \partial_\sigma A(\sigma)$.

Are there nontrivial stable loops for all the nontrivial elements in $\pi_0 \mathcal{L} = \pi_1 (A/G)$?

$A$ is contractible, so

$$\pi_1 (A/G) = \pi_0 G = \pi_0 Maps(S^4 \to G) = \pi_4 G$$

$$\pi_4 SU(2) = \mathbb{Z}_2$$

$SU(2)$ is the only compact Lie group with nontrivial $\pi_4$. 
Personal motivation

Hypothetical application in a speculative physics theory [DF, 2003].

The lambda model is a modified 2-d nonlinear model whose target space is the space of space-time fields, e.g., $\mathcal{A}/\mathcal{G}$.

The modification consists of interspersing the ordinary dilation operator of the nonlinear model with the gradient flow of the space-time action, e.g., $S_{YM}$.

The dilation operator of the lambda model generates a measure on the target manifold — a space-time quantum field theory.

A stable loop for the $SU(2)$ Yang-Mills flow is a classical winding mode for the lambda model.

If the stable loop can be quantized in the lambda model, it might give low energy states that are not in lagrangian $SU(2)$ quantum gauge field theory, and that might be observable.
A plausible outcome of the Y-M flow would be for one point on the loop to flow to a saddle point with a one-dimensional unstable manifold. The outgoing flows in both directions along the unstable manifold would end at the flat connection. The unstable manifold would thus form a stable loop.

Here, I find such a saddle point, with $S_{YM} = 2$, and show that its unstable manifold is one-dimensional, so forms a stable loop.

Actually, the saddle point is not a point, but rather a loop of fixed points.

This is naive mathematics: entirely explicit, elementary calculations.
Previous work

Sibner, Sibner and Uhlenbeck (1989) constructed nontrivial loops of connections in the trivial bundle over $S^4$, with certain given $U(1)$ symmetry.

Then they minimized over loops with the given $U(1)$ symmetry the maximum of $S_{YM}$ on the loop, and showed that this min-max was realized by a solution of the Yang-Mills equations.

As far as I can tell, these solutions have $S_{YM} > 2$. Each of these solutions should have a one-dimensional unstable manifold within the space of connections of the given $U(1)$ symmetry, but the full unstable manifold, within the space of all connections, presumably has dimension $> 1$.

They conjectured the existence of an additional solution, not given by their construction. Presumably, their missing solution is the saddle point described here.
**SU(3)/SU(2) = S^5 \subset \mathbb{C}^3**

Lacking intuition, I looked for an explicit nontrivial loop of SU(2) connections, planning to run the Y-M flow numerically to see what would happen.

The SU(2) bundles over S^5 are classified by π_4 SU(2), because they are made by gluing together trivial bundles on the north and south hemispheres using a map from the equator, S^4, to SU(2). So there is a unique nontrivial SU(2) bundle over S^5.

SU(3) → SU(3)/SU(2) = S^5 \subset \mathbb{C}^3 is a homogeneous model of the nontrivial bundle, with a canonical connection invariant under SU(3)_L \times U(1)_R.

Pull back along a suitable map S^1 \times S^4 → S^5 to get a nontrivial loop of SU(2) connections on S^4,

\[ \sigma \in [0, 2\pi] \mapsto D_\sigma \]

with D_0 and D_{2\pi} nontrivially gauge equivalent flat connections.
Parametrizations

\[ S^3 \subset \mathbb{C}^2 : \quad z = (z_1, z_2) \quad |z_1|^2 + |z_2|^2 = 1 \]

\[ S^4 \subset \mathbb{R} \oplus \mathbb{C}^2 : \quad (\cos \theta, z_1 \sin \theta, z_2 \sin \theta) \quad 0 \leq \theta \leq \pi \]

\[ r = e^x = \tan \frac{\theta}{2} \]

**U(2) symmetry**

The map \( S^1 \times S^4 \rightarrow S^5 \) can be chosen so that each connection \( D_\sigma \) is invariant under the action of \( U(2) \) on \( S^3 \subset \mathbb{C}^2 \).

For example, take

\[(\sigma, \theta, z_1, z_2) \mapsto \left(\cos \theta + i \sin \theta \cos \frac{\sigma}{2}, z_1 \sin \theta \sin \frac{\sigma}{2}, z_2 \sin \theta \sin \frac{\sigma}{2}\right)\]
An additional $\mathbb{Z}_2$ symmetry exchanges $D_\sigma \leftrightarrow D_{2\pi - \sigma}$.

So the midpoint $D_\pi$ has an enhanced $U(2) \rtimes \mathbb{Z}_2$ symmetry.

$D_\pi$ should flow to the saddle point.

The $\mathbb{Z}_2$ symmetry would exchange the two branches of the unstable manifold.

Instead of running the Y-M flow numerically on the loop, it is considerably easier just to minimize $S_{YM}$ within the space of $U(2) \rtimes \mathbb{Z}_2$-invariant connections.

$U(2)$ acts transitively on $S^3$, so the $U(2)$-invariant connection forms $A_\sigma$ are functions only of $\theta$, or $x$.

The additional $\mathbb{Z}_2$ symmetry reflects $\theta \leftrightarrow \pi - \theta$, $x \leftrightarrow -x$, so the invariant connection forms satisfy reflection symmetry conditions.
Numerical investigations

The connection 1-form $A(\theta)$ has two independent components, satisfying certain symmetry conditions under $\theta \rightarrow \pi - \theta$.

Write them as suitable polynomials in $\cos \theta$. $S_{YM}$ is a quartic polynomial in the coefficients of the polynomials.

Minimize $S_{YM}$ numerically on this $2N$ dimensional submanifold of the space of invariant connections, using Sage, Mathematica, and Maple. The programs misbehaved for $N > 15$.

The numerics suggest an absolute minimum at $S_{YM} = 2$.

$S_{YM}$ a small integer suggests topology.

$SU(3)$: $S_{YM} = 2.4$

\[
\begin{array}{ll}
N & \min(S_{YM}) \\
1 & 2.15627 \\
2 & 2.06011 \\
3 & 2.03019 \\
4 & 2.01735 \\
5 & 2.01086 \\
6 & 2.00723 \\
7 & 2.00504 \\
8 & 2.00368 \\
9 & 2.00346 \\
10 & 2.00313 \\
11 & 2.00286 \\
12 & 2.00251 \\
13 & 2.00202 \\
14 & 2.00186 \\
15 & 2.00147 \\
\end{array}
\]
(anti-)self-duality

\[ F_\pm = \frac{1}{2}(F \pm \ast F) \quad \ast \text{ the Hodge operator} \]

\[ S_\pm = \frac{1}{8\pi^2} \int_{S^4} (-F_\pm \ast F_\pm) = \int_{-\infty}^{\infty} dx \ L_\pm(x) \]

\[ S_{YM} = S_+ + S_- \]

\( S_+ - S_- \in \mathbb{Z} \) is the instanton number

The instanton: \( F = \ast F \quad S_+ = 1 \)

\[ F_- = 0 \quad L_-(x) = 0 \quad S_- = 0 \]

The \( U(2) \)-invariant instanton centered at the south pole \( (x = \infty) \) will be written explicitly later. For now, its action density is

\[ L_+(x) = L_{\text{inst}}(x) = .75 \cosh^{-4}(x - x_+) \]

where

\[ r_+ = e^{-x_+} \]

is the instanton size.
The numerics suggest that $S_{YM} = 2$ is attained as a zero-size instanton at the south pole (at $x=\infty$) combined with a zero-size anti-instanton at the north pole (at $x=-\infty$).
The $U(2)$-invariant $su(2)$-valued forms on $S^3$

$U(2)$ invariance:

$$\omega(hz) = h\omega(z)h^{-1}, \quad h \in U(2)$$

Identify $S^3$ with $SU(2)$,

$$\hat{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z = g\hat{e}, \quad g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

There is a single invariant 0-form:

$$\phi(z) = i(P - Q) = g \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g^{-1}, \quad P = zz^\dagger, \quad Q = 1 - P$$

There are three invariant 1-forms:

$$\nu_+ = PdP, \quad \nu_- = (dP)P, \quad \nu_3 = (z^\dagger dz)(Q - P)$$

The Maurer-Cartan form on $SU(2)$ is

$$\Omega = gd(g^{-1}) = \nu_+ - \nu_- + \nu_3$$
The instanton and the anti-instanton

The basic instanton of size $r_+$ located at the south pole ($x = \infty$)

\[
D_+ = d + f_+(x)\Omega \\
f_+ = \frac{1}{1 + e^{2(x-x_+)}}, \quad r_+ = e^{-x_+}
\]

The basic anti-instanton of size $r_-$ at the north pole ($x = -\infty$)

\[
D_- = d + f_-(x)\Omega \\
f_- = \frac{1}{1 + e^{-2(x-x_-)}}, \quad r_- = e^{x_-}
\]

The instanton and anti-instanton twisted by $h_{\pm} \in SU(2)/\{\pm \mathbf{1}\}$

\[
(gh_+g^{-1})D_+(gh_+g^{-1})^{-1}, \quad (gh_-g^{-1})D_-(gh_-g^{-1})^{-1}
\]

8 (anti-)instanton moduli: 4 the location in $S^4$  
1 the size $r_\pm$  
3 the twist $h_\pm$
The twisted-pair

Send \( r_\pm \to 0 \). Then, away from the poles, \( f_{\pm 1} = 1 \), so

\[
D_\pm = gdg^{-1}, \quad \text{so} \quad (gh_{\pm} g^{-1}) D_\pm (gh_{\pm} g^{-1})^{-1} = gdg^{-1}
\]

so it makes sense to patch the twisted pair at the equator, defining

\[
D(h_-, h_+) = \begin{cases} 
(gh_- g^{-1}) D_- (gh_- g^{-1})^{-1} & \text{if } x \leq 0 \\
(gh_+ g^{-1}) D_+ (gh_+ g^{-1})^{-1} & \text{if } x \geq 0
\end{cases}
\]

The 15 parameter conformal group of \( S^4 \) absorbs 14 of the 16 moduli. Use 8 to put the instanton and the anti-instanton at the poles, and use dilation to scale \( r_+ \) and \( r_-^{-1} \), to make \( r_+ = r_- \).

This leaves the 6 parameters of \( O(4) = SU(2)_L \times SU(2)_R / \{\pm 1\} \).
$SU(2)_L \times SU(2)_R/\{\pm 1\}$ acts on the twists by

$$(h_+, h_-) \mapsto (g_R h_+ g_L^{-1}, g_R h_- g_L^{-1})$$

Absorb $h_+$ by taking $g_L = g_R h_+$, leaving $SU(2)_R/\{\pm 1\}$ acting by conjugation on $h_-$

$$(1, h_-) \mapsto (1, g_R h_- g_R^{-1})$$

Thus the zero-size twisted-pair has 14 conformal moduli, plus a 1 parameter symmetry, plus 1 additional parameter

$$\cos \frac{\sigma}{2} = \frac{1}{2} \text{tr}(h_- h_+^{-1}) \quad 0 \leq \sigma \leq 2\pi$$

with $\sigma = 0 \sim \sigma = 2\pi$ because the twists act by conjugation.
The loop of zero-size twisted pairs

The relative twist \( h_- h_+^{-1}/\{\pm 1\} \) lies in \( SU(2)/\{\pm 1\} = SO(3) \).

\[ \pi_1 SO(3) = \mathbb{Z}_2. \]
We want to show that the nontrivial loops of zero-size twisted pairs are nontrivial loops in \( \mathcal{A}/\mathcal{G} \).

A convenient nontrivial loop of zero-size twisted pairs is

\[
D_\sigma = \begin{cases} 
D^-_\sigma = e^{\frac{1}{4}\sigma\phi} D_- e^{-\frac{1}{4}\sigma\phi} & x \leq 0 \\
D^+_\sigma = e^{-\frac{1}{4}\sigma\phi} D_+ e^{\frac{1}{4}\sigma\phi} & x \geq 0 
\end{cases}
\]

\[ gh_+ g^{-1} = e^{\frac{1}{4}\sigma\phi} = g \begin{pmatrix} e^{\frac{i\sigma}{4}} & 0 \\
0 & e^{-\frac{i\sigma}{4}} \end{pmatrix} g^{-1} \]

\[ h_- h_+^{-1} = \begin{pmatrix} e^{\frac{i\sigma}{2}} & 0 \\
0 & e^{-\frac{i\sigma}{2}} \end{pmatrix} \]
Non-triviality of the loop of twisted pairs in $\mathcal{A}/\mathcal{G}$

The twisted pairs, as written, are singular at the poles (before the limit $r_\pm \to 0$). The connections can be made regular everywhere on $S^4$ by a gauge transformation, giving

$$D_\sigma = \begin{cases} D^-_\sigma = \Phi^-_\sigma D_-(\Phi^-_\sigma)^{-1} & x \leq 0 \\ D^+_\sigma = \Phi^+_\sigma D_+(\Phi^+_\sigma)^{-1} & x \geq 0 \end{cases}$$

$$\Phi^-_\sigma(x, g) = e^{\frac{1}{2}\sigma k_-(x)} \phi \quad \Phi^+_\sigma(x, g) = e^{(\pi - \frac{1}{2}\sigma k_+(x))} \phi$$

$$k_+(-\infty) = 0 \quad k_+(-\infty) = 1 \quad k_- + k_+ = 1.$$ 

$$D_{2\pi} = \Phi D_0 \Phi^{-1}$$

$$\Phi = e^{-\pi k_+(x)} \phi = g \begin{pmatrix} e^{-i\pi k_+(x)} & 0 \\ 0 & e^{i\pi k_+(x)} \end{pmatrix} g^{-1}$$

$\Phi = \Sigma H : S^4 \to S^3$, the suspension of the Hopf map $H : S^3 \to S^2$, representing the nontrivial element in $\pi_4 S^3$. 
Stability

The zero-size twisted-pairs are all critical points of the Y-M action, so the loop of zero-size twisted pairs is pointwise fixed under the Y-M flow.

Is the loop of zero-size twisted-pairs stable under the flow?

The instanton and the anti-instanton are individually stable, so we need only calculate the flow in the two zero-modes $r_+$ and $\sigma$, in the limit where $r_+$ is asymptotically small.

To determine the topology of the flow, it should be enough to calculate $\dot{r}_+$ and $\dot{\sigma}$ as functions of $r_+$ and $\sigma$, at least to leading order in $r_+$. 
Calculation of $\dot{r}_+$ and $\dot{\sigma}$

Write the general asymptotically small $U(2)$-invariant perturbation

$$D_\sigma = \begin{cases} e^{\frac{1}{4}\sigma \phi} (D_- + \delta A_-) e^{-\frac{1}{4}\sigma \phi} & x \leq 0 \\ e^{-\frac{1}{4}\sigma \phi} (D_+ + \delta A_+) e^{\frac{1}{4}\sigma \phi} & x \geq 0 \end{cases}$$

(1) Enforce the flow equation to leading order

$$\dot{r}_- \partial_{r_-} D_- + \frac{1}{4} \dot{\sigma} [\phi, D_-] = *D_- * D_- \delta A_-$$

$$\dot{r}_+ \partial_{r_+} D_+ - \frac{1}{4} \dot{\sigma} [\phi, D_+] = *D_+ * D_+ \delta A_+$$

(2) Require $D_\sigma$ to be regular at $x = 0$,

$$e^{\frac{1}{4}\sigma \phi} (D_- + \delta A_-) e^{-\frac{1}{4}\sigma \phi} - e^{-\frac{1}{4}\sigma \phi} (D_+ + \delta A_+) e^{\frac{1}{4}\sigma \phi} = 0 + O(x^2)$$

Together, (1) and (2) ensure that $D_\sigma$ satisfies the flow equation.
Working in $A_x = 0$ gauge, expand in the invariant 1-forms on $S^3$

$$\delta A_\pm = \delta A^\pm_+(x) v_+ + \delta A^\pm_-(x) v_- + \delta A^3_\pm(x) v_3$$

The flow equations are thus second order inhomogeneous linear ordinary differential equations in $x$, with non-constant $3 \times 3$ matrix coefficients. They can be diagonalized and integrated exactly.

Then there are 6 patching equations: on the values of the coefficients of $v_+, v_-, v_3$ at $x = 0$, and the first derivatives.

After elementary but laborious calculations, I get

$$\dot{r}_+ = r_+^3(1 + 2 \cos \sigma) + O(r_+^5)$$

$$\dot{\sigma} = -8 r_+^2 \sin \sigma + O(r_+^4)$$

The topology of the flow is easiest understood by looking at the flow lines.
The stable loop

There is a stable loop consisting of two branches.

One branch consists of the line of fixed points at $r_+ = 0$ from $\sigma = \pi$ to $\sigma = 0$, then the outgoing trajectory along the vertical axis at $\sigma = 0$.

The second branch consists of the line of fixed points at $r_+ = 0$ from $\sigma = \pi$ to $\sigma = 2\pi$, then out along the vertical axis at $\sigma = 2\pi$.

Recall that the two vertical axes, $\sigma = 0$ and $\sigma = 2\pi$ are gauge equivalent, under a non-trivial gauge transformation.

I strongly suspect that the outgoing flows at $\sigma = 0$ and $\sigma = 2\pi$ end at the flat connection.
Geometry

The metric inherited from the space of connections $\mathcal{A}$ is, to leading order in $r_+$,

$$(ds)^2 = 64 \left[ (dr_+)^2 + r_+^2 (d\sigma')^2 \right] \quad \sigma' = \frac{1}{4}\sigma \quad 0 \leq \sigma' \leq \frac{\pi}{2}$$

The flow is the gradient flow wrt this metric for

$$S_{YM} = 2 - 16r_+^4(1 + 2 \cos \sigma) + O(r_+^6)$$

$\mathcal{A}/\mathcal{G}$ is a cone: the upper right quadrant of the plane, the positive $x$-axis identified with the positive $y$-axis.

The space of zero-size twisted pairs lives at the vertex of the cone, at $r_+ = 0$. 
The outgoing trajectory

The outgoing trajectory at $\sigma = 0$ (or $\sigma = 2\pi$) is the flow of connections of the form

$$D = d + f(x)\Omega \quad f(\pm \infty) = 0, \quad f(x) = f(-x)$$

generated by

$$\frac{df}{dt} = \partial_x^2 f - 4f(1 - f)(2f - 1)$$

with initial conditions

$$f(x) \to \frac{1}{1 + r_+(t)^2 e^{2|x|}} \quad t \to -\infty$$

$$r_+(t)^2 = (-6t)^{-1} + O(t^{-2})$$
\[
\frac{df}{dt} = \partial_x^2 f - 4f(1-f)(2f-1)
\]
is a nonlinear diffusion equation, a special case of the FitzHugh–Nagumo equation

\[
\partial_t u = \partial_x^2 u - u(u-1)(u-a)
\]
at \(a = \frac{1}{2}\).

This is not integrable, but some exact solutions are known, including travelling shock waves which become static when \(a = \frac{1}{2}\). These are our instanton and anti-instanton.

The calculation described above guarantees an early time solution that starts as a widely separated shock-anti-shock pair moving slowly towards each other with a speed that goes as \(\frac{1}{2t}\).

Does this solution exist for all \(t\)? As \(t \to +\infty\), does the shock-anti-shock pair annihilate to \(f = 0\) (the flat connection)?
Questions

- What is the outgoing trajectory from the loop of twisted pairs located at separation $R$ in euclidean $\mathbb{R}^4$? How does it approach the flat connection?

  I expect that any low energy states in the lambda model would come from this asymptotic approach to the flat connection on $\mathbb{R}^4$.

- Global stability? Is the attracting basin open and dense? Are there any other locally stable loops?
Stable 2-spheres (lambda instantons)?

\[ \pi_2(A/G) = \pi_5 G \]

\[ \pi_5 SU(3) = \mathbb{Z}. \quad G_2/SU(3) = S^6 \] is a model. Numerics are essentially the same [DF, Pisa Workshop on Geometric Flows, June 2009]. The space of twisted pairs in \( SU(3) \) contains a \( \mathbb{CP}^1 \). The stability calculation has not yet been done.

\[ \pi_5 SU(2) = \mathbb{Z}_2. \quad \] I have no idea what a stable 2-sphere might look like. Is there a homogeneous model? The model in the literature,

\[ \Sigma H \circ \Sigma^2 H : S^5 \to S^4 \to S^3 \]

does not seem useful.

Ricci flow

\[ \pi_0 \text{Diff}(S^4) = 0? \quad (\text{no exotic 5-spheres?}) \]

\[ \pi_2(\text{Metrics}/\text{Diff}(S^4)) = 0? \]