Gradient property of the boundary rg flow for supersymmetric 1+1d quantum field theories

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D.F. & A.K., arXiv:0810.0611v1 [hep-th]

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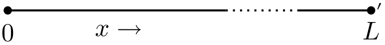
1d quantum systems, critical in the bulk



$$Z = \mathrm{tr}\; e^{-\beta H} = \mathrm{tr}\; e^{-H/T}$$

($\hbar = k_B = v = 1$ in our units.)

for $L/\beta \gg 1$

$$\ln Z = \ln z(\Lambda\beta) + \frac{\pi c}{6}\frac{L}{\beta} + \ln z'$$

The bulk term is determined by bulk conformal invariance (c being the conformal central charge of the bulk system).

The boundary renormalization group flow

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = -T \frac{\partial \ln z}{\partial T} = \beta^a(\lambda) \frac{\partial \ln z}{\partial \lambda^a}$$

The λ^a are the boundary coupling constants. They parametrize the space of all boundary conditions for the given bulk system (the space of all boundary qft's for the given bulk system).

When the boundary is at a critical point, i.e., $\lambda = \lambda_c$ with $\beta^a(\lambda_c) = 0$, then z is a pure number, independent of β.

Affleck & Ludwig (1991) called this number g, the *non-integer degeneracy*. (They called it a "degeneracy" because it can be evaluated at $\beta = \infty$, where the partition function counts the ground states).

Their conjecture: for any nontrivial rg trajectory, g at the UV fixed point ($T = \infty$) is greater than g at the IR fixed point ($T = 0$).

Boundary entropy

entropy of the full system

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z$$

second law

$$\beta\frac{\partial S}{\partial\beta} = -\beta^2\,\frac{\partial^2 \ln Z}{\partial\beta^2} = -\beta^2\left(\langle H^2\rangle - \langle H\rangle^2\right) < 0$$

boundary entropy

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\left(\ln z + \frac{\pi c}{6}\frac{L}{\beta} + \ln z'\right) = s + \frac{\pi c}{3}\frac{L}{\beta} + s'$$

$$s = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln z$$

Gradient formula for the boundary entropy

[D.F. & A.K., 2004]

$$\frac{\partial s}{\partial \lambda^a} = -g_{ab}(\lambda)\,\beta^b(\lambda)$$

$$g_{ab}(\lambda) = \beta \int_0^\beta d\tau\ \left[1 - \cos\left(2\pi\tau/\beta\right)\right]\,\langle \phi_a(-i\tau)\phi_b(0)\rangle_c$$

where the $\phi_a(t)$ are the boundary operators

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta \langle \phi_a \rangle$$

assumptions

- bulk conformal invariance
- canonical UV behavior (as at a UV fixed point)

second law of boundary thermodynamics

$$\beta \frac{\partial s}{\partial \beta} = \Lambda \frac{\partial s}{\partial \Lambda} = \beta^a \frac{\partial s}{\partial \lambda^a} = -\beta^a g_{ab} \beta^b \leq 0$$

s decreases along the rg flow, and is stationary iff $\beta^a = 0$

$\ln g$ decreases from UV fixed point to IR fixed point.

The boundary behaves thermodynamically like an isolated system.

Questions

- Why do we need bulk conformal invariance?
- Why do we need canonical UV behavior?
- Is s bounded below?
  (in general? for a given bulk? for a given system as $T \to 0$?)

Supersymmetric 1d systems, critical in the bulk

a conserved fermionic super-charge

$$H = \hat{Q}^2$$

positive thermodynamic energy

$$-\frac{\partial Z}{\partial \beta} = \mathrm{tr}\left(e^{-\beta H} H\right) > 0$$

(Advertisement: in cond-mat/0505084 and 0505085, I argue that circuits made of bulk-critical quantum wire, joined at boundaries and junctions, would be ideal for asymptotically large-scale quantum computing: the $c = 24$ monster system in particular.

I define an entropy current operator, and derived circuit laws for the flow of entropy, which I suggest are basic constraints on quantum computation in such circuits.)

A second gradient formula for supersymmetric systems

[D.F. & A. K., arXiv:0810.0611v1 [hep-th]]

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}(\lambda)\,\beta^b(\lambda)$$

(the λ^a now restricted to the susy coupling constants)

implying positivity of the susy boundary energy

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = \beta^a \frac{\partial \ln z}{\partial \lambda^a} = -\beta^a g^S_{ab} \beta^b \leq 0$$

The boundary behaves like an isolated supersymmetric system.

$\ln z$ decreases along the rg flow.

Here, I first prove the positivity of the boundary energy

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

then sketch how the gradient formula is proved by the same kind of argument.

Local densities of energy and super-charge

$$H = \int_0^L dx\, \mathcal{H}(t, x)$$

$$\hat{Q} = \int_0^L dx\, \hat{\rho}(t, x)$$

$$\{\hat{Q},\, \hat{\rho}(t, x)\} = 2\mathcal{H}(t, x)$$

local conservation of super-charge

$$\partial_t \hat{\rho}(t, x) + \partial_x \hat{\jmath}(t, x) = 0$$

Boundary energy and super-charge

$$h(t) = \lim_{\epsilon \to 0} \int_0^{\epsilon} dx \, \mathcal{H}(t, x)$$

$$\hat{q}(t) = \lim_{\epsilon \to 0} \int_0^{\epsilon} dx \, \hat{\rho}(t, x)$$

$$\{\hat{Q}, \, \hat{q}(t)\} = 2h(t)$$

bulk conformal invariance implies

$$\langle \mathcal{H}(t, x) \rangle = \frac{\pi c}{6} \frac{1}{\beta} \qquad 0 < x < L$$

so

$$-\frac{\partial \ln z}{\partial \beta} = \langle h \rangle$$

Separate $\hat{Q}$ at $x = \epsilon > 0$

$$\hat{q}_\epsilon(t) = \int_0^\epsilon dx\ \hat{\rho}(t,x) \qquad \hat{Q}_{bulk}(t) = \int_\epsilon^L dx\ \hat{\rho}(t,x)$$

$$\hat{Q} = \hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t)$$

locality implies

$$\{\hat{Q}_{bulk}(0),\ \hat{q}(0)\} = 0$$

so

$$\left\langle 2h \right\rangle = \left\langle \{Q,\ \hat{q}(0)\} \right\rangle = \left\langle \{\hat{q}_\epsilon(0),\ \hat{q}(0)\} \right\rangle$$

but this equation is useless at $\epsilon = 0$, because $\left\langle \{\hat{q}(t),\ \hat{q}(0)\} \right\rangle$ is uv divergent at $t = 0$.

The boundary cannot be separated from the bulk, in general.

Use bulk super-conformal invariance

define

$$g_\epsilon(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t} \left\langle \{\hat{q}_\epsilon(t),\ \hat{q}(0)\} \right\rangle$$

$$G_\epsilon^\pm(\omega) = \pm \int_0^{\pm\infty} dt\ e^{i\omega t} \left\langle \{\hat{Q}_{bulk}(t),\ \hat{q}(0)\} \right\rangle$$

so

$$2\pi\delta(\omega)\langle 2h \rangle = g_\epsilon(\omega) + G_\epsilon^+(\omega) + G_\epsilon^-(\omega)$$

bulk super-conformal invariance implies

$$G_\epsilon^+(i\pi T) = 0 = G_\epsilon^-(-i\pi T)$$

so

$$\int \frac{d\omega}{2\pi}\ \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2}\ G_\epsilon^\pm(\omega) = 0$$

so

$$\langle 2h \rangle = \int \frac{d\omega}{2\pi} \, \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2} \, g_\epsilon(\omega)$$

Now take $\epsilon \to 0$:

$$g(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \left\langle \{ \hat{q}(t), \, \hat{q}(0) \} \right\rangle$$

$$\beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = -\frac{\beta}{2} \int \frac{d\omega}{2\pi} \, \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2} \, g(\omega)$$

which is UV-finite as long as $\dim[g(\omega)] < 1$, i.e., $\dim[\hat{q}] < 1$.

We have $g(\omega) \geq 0$ and $g(\omega) = 0$ iff $\hat{q} = 0$, so

$$\beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

with equality iff the boundary is critical (superconformal).

The susy gradient formula

boundary fermionic operators

$$\{\hat{Q},\, \hat{\phi}_a(t)\} = \phi_a(t)$$

boundary beta-functions

$$\hat{q} = -2\beta^a \hat{\phi}_a$$

$$h = \frac{1}{2}\{\hat{Q},\, \hat{q}\} = -\beta^a \phi_a$$

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = \beta \langle \beta^a \phi_a \rangle = \beta^a \frac{\partial \ln z}{\partial \lambda^a}$$

$$\langle\phi_a\rangle = \left\langle\{\hat{Q},\, \hat{\phi}_a(0)\}\right\rangle = \left\langle\{\hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t),\, \hat{\phi}_a(0)\}\right\rangle$$

$$\begin{aligned} g_a(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t} \left\langle\{\hat{q}(t),\, \hat{\phi}_a(0)\}\right\rangle \\ &= \int_{-\infty}^{\infty} dt\; e^{i\omega t} \left\langle\{-2\beta^b \hat{\phi}_b(t),\, \hat{\phi}_a(0)\}\right\rangle \\ &= -2\beta^b g_{ab}(\omega) \end{aligned}$$

$$\begin{aligned} \langle\phi_a\rangle &= \int \frac{d\omega}{2\pi}\, \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2}\, g_a(\omega) \\ &= -2\beta^b \int \frac{d\omega}{2\pi}\, \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2}\, g_{ab}(\omega) \end{aligned}$$

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b$$

$$g_{ab}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t} \left\langle \{\hat{\phi}_b(t),\, \hat{\phi}_a(0)\} \right\rangle$$

$$\begin{aligned} g^S_{ab} &= 2\beta \int \frac{d\omega}{2\pi}\ \frac{\pi^2 T^2}{\omega^2 + \pi^2 T^2}\ g_{ab}(\omega) \\ &= \pi \int dt\ e^{-\pi|t|T} \left\langle \{\hat{\phi}_b(t),\, \hat{\phi}_a(0)\} \right\rangle \\ &= 2\pi \int_0^\beta d\tau\ \sin\left(\frac{\pi\tau}{\beta}\right) \left\langle \hat{\phi}_b(-i\tau),\, \hat{\phi}_a(0) \right\rangle \end{aligned}$$

Some questions

1. Why do we need bulk conformal invariance?

2. Why do we need canonical uv behavior in the boundary?
   - no negative dimension boundary operators
   - no strongly irrelevant boundary operators

3. Can $\ln z$ (and/or s) be bounded below?

Bulk conformal invariance and zeros of response functions

$$\partial_t \hat{Q}_{bulk}(t) = \int_\epsilon^L dx\, [-\partial_x \hat{\jmath}(t,x)] = \hat{\jmath}(t,\epsilon)$$

Define response functions

$$R_a^\pm(\omega) = \pm \int_0^{\pm\infty} dt\, e^{i\omega t - \delta|t|} \left\langle \{ i\hat{\jmath}(t,\epsilon),\, \hat{\phi}_a(0) \} \right\rangle$$

$R_a^+(\omega)$ is analytic in the upper half-plane, $R_a^-(\omega)$ in the lower.

Use the conservation equation

$$G_{a,\epsilon}^\pm(\omega) = \pm \int_0^{\pm\infty} dt\, e^{i\omega t - \delta|t|} \left\langle \{ \hat{Q}_{bulk}(t),\, \hat{\phi}_a(0) \} \right\rangle = \frac{R_a^\pm(\omega)}{\omega \pm i\delta}$$

$\tau = it$, $0 < \tau < \beta$

$$\left\langle \hat{\jmath}(-i\tau, \epsilon)\, \hat{\phi}_a(0) \right\rangle = \int \frac{d\omega}{2\pi i} \frac{e^{-\omega\tau}}{1 + e^{-\omega\beta}} \left[R^+(\omega) + R^-(\omega) \right]$$

poles at

$$\omega_n = 2\pi i n T \qquad n \in \frac{1}{2} + \mathbb{Z}$$

so

$$\left\langle \hat{\jmath}(-i\tau, \epsilon)\, \hat{\phi}_a(0) \right\rangle = \beta^{-1} \sum_n e^{-\omega_n \tau} \left[\theta(n) R^+(\omega_n) - \theta(-n) R^-(\omega_n) \right]$$

but

$$j(-i\tau, x) = AG(x + i\tau) + \bar{A}G(x - i\tau)$$

so

$$R_a^+(i\pi T) = 0 = R_a^-(-i\pi T)$$