Introduction to the 2d Nonlinear Model and the Renormalization Group Flow

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Aside: a question about the Yang-Mills and Ricci flows

I'm interested in stable 2-spheres for the Ricci flow or the Yang-Mills flow on $\mathbb{R}^4$ or S^4.

That is, let the flow act on a 2-sphere in the space of metrics mod diffeomorphisms or in the space of connections mod gauge transformations. Is the 2-sphere driven to a stable 2-sphere, that is fixed under the flow, mod $\mathrm{Diff}(S^2)$?

Each of these spaces has non-trivial $\pi_2 = \mathbb{Z}_2$, so a stable 2-sphere seems possible. Let the flow act on a homotopically non-trivial 2-sphere of metrics or connections. What is its ultimate fate?

I've been looking at the nontrivial $SU(3)$ bundle $G_2 \to G_2/SU(3) = S^6$. Pulled back along a map $S^2 \times S^4 \to S^6$, this gives a homotopically nontrivial 2-sphere of $SU(3)$ connections on S^4. I've learned a little bit about its fate under the Yang-Mills flow.

0+1 dim quantum field theory (quantum mechanics)

Quantum mechanics in general:

- a Hilbert space $\mathcal{H}$
- a self-adjoint hamiltonian operator $H \geq 0$ acting on $\mathcal{H}$
- an algebra of self-adjoint operators $\mathcal{O}$ (the observables)
- H generates time translation:

$$t \mapsto e^{itH} \qquad \mathcal{O}(t) = e^{itH} \mathcal{O}\, e^{-itH}$$

Geometric examples:

- data: a manifold M with riemannian metric g
- $\mathcal{H} = L_2(M)$
- $H = \Delta_g$, the laplacian

The data (M, g) does not uniquely define this quantum mechanics. There are many other natural self-adjoint operators.

1+1 dim quantum field theories: the nonlinear models

$1{+}n$ dim quantum field theory $=$ a quantum system in an n-dimensional space.

Here, $n = 1$. Take space to be, e.g., a circle S^1 of length L.

The nonlinear model is parametrized by the same data (M, g).

Formally:

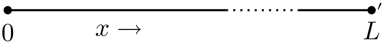
- $\mathcal{H} = L_2(\Omega(M))$

  $\Omega(M) =$ the space of maps $\phi : S^1 \to M$ (an infinite-dimensional space)

- $H = \Delta_\phi + V(\phi) \qquad V(\phi) = \int_0^L dx\, g(d\phi(x), d\phi(x))$

A cutoff theory

Replace S^1 with a discrete approximation (Z_N)



$$x \in \{0, 1\epsilon, 2\epsilon, 3\epsilon, \ldots, N\epsilon = L\} \qquad N \gg 1 \qquad x = 0 \sim x = L$$

Now we can write a well-defined quantum mechanics:

- $\Omega(M) = \prod_x M = M^N$
- $\mathcal{H} = \bigotimes_x L_2(M) = L_2(M^N)$
- $H = \sum_x \ \epsilon \left[\epsilon^{-2}\Delta_{\phi(x)} + \ \epsilon^{-2}\mathrm{dist}^2\left(\phi(x), \phi(x+\epsilon)\right)\right]$

This generalizes the Heisenberg model, which has $(M, g) = (S^2, \text{a round metric})$.

Formally, in the limit $\epsilon \to 0$,

$$H = \int dx \ \left[g^{-1}(\pi(x), \pi(x)) + g(d\phi(x), d\phi(x))\right]$$

where

$$\pi(x) = \epsilon^{-1}\nabla(x) \qquad [\pi(x),\, \phi(x')] = \epsilon^{-1}\delta_{x,x'} \to \delta(x - x')$$

No inconvenient powers of ϵ^{-1} appear when we take the formal limit.
We say that g is naively *dimensionless*.

This formal limit is valid only for M asymptotically large, $g \to \infty$.

Write $\frac{1}{\alpha'}g$ in place of g.

The quantum field theory is constructed as a formal power series in α'.

To make the qft independent of ϵ, we need $\frac{1}{\alpha'}g$ to depend on ϵ according to

$$\epsilon\frac{\partial}{\partial\epsilon}\left(\frac{1}{\alpha'}g\right) = \beta\left(\frac{1}{\alpha'}g\right) = -\mathrm{Ricci}(g) + \alpha' R^2(g) + \cdots$$

This is the renormalization group flow.
The continuum limit $\epsilon \to 0$ requires stability in the far past (e.g., round S^n).

Path integral

Quantum mechanics in M as integral over paths $\phi : [0, T] \to M$

$$e^{-T(\Delta + V(\phi))} = \int_{\text{paths } \phi(\tau)} \mathcal{D}\phi \; e^{-S(\phi)} \qquad S(\phi) = \int_0^T d\tau \; [g(\partial_\tau \phi, \partial_\tau \phi) + V(\phi(\tau))]$$

Nonlinear model as integral over paths $\phi : [0, T] \to \Omega(M) = M^N$

$$e^{-TH} = \int_{\text{paths } \phi : [0,T] \to M^N} \mathcal{D}\phi \; e^{-S(\phi)}$$

$$\begin{aligned} S(\phi) &= \int_0^T d\tau \sum_x \epsilon \left[g(\partial_\tau \phi(x,\tau), \partial_\tau \phi(x,\tau)) + \epsilon^{-2} \mathrm{dist}^2 \left(\phi(x,\tau), \phi(x+\epsilon,\tau) \right) \right] \\ &\sim \int_0^T d\tau \int_0^L dx \; [g(\partial_\tau \phi, \partial_\tau \phi) + g(\partial_x \phi, \partial_x \phi)] \\ &\sim \int d^2x \; \delta^{\mu\nu} g(\partial_\mu \phi, \partial_\nu \phi) \qquad (x^1, x^2) = (x, \tau) \end{aligned}$$

Take $L, T \to \infty$, so $(x^1, x^2) \in \mathbb{R}^2$

$$S(\phi) = \int d^2x \; \delta^{\mu\nu} g(\partial_\mu \phi, \partial_\nu \phi)$$

invariant under the euclidean group (*relativistic* quantum field theory).

Easy generalization:

For Σ a 2-dimensional manifold with riemannian metric γ,

$$Z(\Sigma, \gamma) = \int_{\text{maps } \phi:\Sigma \to M} \mathcal{D}\phi \; e^{-S(\phi)}$$

$$S(\phi) = \int_\Sigma d^2x \; \sqrt{\det \gamma} \; \gamma^{\mu\nu}(x) \; g(\partial_\mu \phi, \partial_\nu \phi)$$

Discretize Σ to turn this into a finite dimensional integral (for Σ compact).

e.g., $\quad \tau \in \{0, 1\epsilon, 2\epsilon, 3\epsilon, \dots, N'\epsilon = T\}$.

$S(\phi)$ is (locally) scale-invariant, but $\mathcal{D}\phi$ is not. The cutoff distance ϵ breaks the scale-invariance.

Locality

$$Z(\Sigma, \gamma) = \int \mathcal{D}\phi \; e^{-S(\phi)}$$

$$\mathcal{D}\phi = \prod_{x \in \Sigma} \mathrm{dvol}_g(\phi(x)) \qquad S(\phi) = \int_\Sigma d^2x \, \sqrt{\det \gamma} \; \gamma^{\mu\nu}(x) \, g(\partial_\mu \phi, \partial_\nu \phi)$$

Locality:

- The functional integral can be done patch-wise over Σ.

  The degrees of freedom (integration variables) are distributed locally on Σ. The action S is an integral (sum) over Σ of a local expression.

- $H = \sum_x \epsilon \mathcal{H}(x)$ with $\mathcal{H}(x)$ depending only on the $\phi(x')$ for x' near x.

The measure is encoded in its integrals (measurements), e.g.

$$\int \mathcal{D}\phi \; e^{-S(\phi)} \; F_1(\phi(x_1)) \cdots F_n(\phi(x_n)) \qquad F_k \in C^\infty(M) \quad x_k \in \Sigma \,.$$

Think of these *correlation functions* as distributions in the x_k, smeared against functions which are essentially constant over distances of the order of the cutoff distance ϵ.

The renormalization group (conceptual version)

Integrate out a tiny fraction $\delta\epsilon/\epsilon$ of the $\phi(x)$, distributed evenly over Σ.

The average spacing between points becomes slightly larger: $\epsilon \to \epsilon + \delta\epsilon$.

$$\int \mathcal{D}_\epsilon \phi \; e^{-S(\phi)} = \int \mathcal{D}_{\epsilon+\delta\epsilon}\phi \int \mathcal{D}_{\delta\epsilon}\phi \; e^{-S(\phi)} = \int \mathcal{D}_{\epsilon+\delta\epsilon}\phi \; e^{-S(\phi)-\delta S(\phi)}$$

The same expectation values are now given by a functional integral with slightly larger cutoff ϵ and a slightly modified action $S + \delta S$.

Locality ensures that the change in the action has the form

$$\delta S(\phi) = -\frac{\delta\epsilon}{\epsilon} \int_\Sigma d^2x \, \sqrt{\det\gamma} \; \left[\gamma^{\mu\nu}(x)\, \beta(\partial_\mu\phi, \partial_\nu\phi) + O(\epsilon^2)\right]$$

for some symmetric 2-tensor field $\beta(g)$ on M. The negligible error terms contain 4 or more derivatives ∂_μ.

So nothing observable changes if

$$\epsilon \to \epsilon + \delta\epsilon \qquad g \to g - \frac{\delta\epsilon}{\epsilon}\beta(g)$$

that is, nothing observable depends on ϵ if g depends on ϵ according to

$$\epsilon\frac{\partial}{\partial\epsilon}g(\epsilon) = -\beta\left(g(\epsilon)\right)$$

Comments:

- The renormalization group is a semi-group.
- Information seems to be destroyed, in some sense.
- The rg flow does not end.

The renormalized metric

Consider euclidean 2-d metrics, $\gamma_{\mu\nu} = \mu^2 \delta_{\mu\nu}$.

μ^{-1} is our unit of distance.

The cutoff distance is unobservably small: $\mu\epsilon << 1$.

Let R_t be the rg flow

$$\frac{d}{dt} R_t(g) = -\beta(R_t(g))$$

Define the *renormalized* metric

$$g_r = R_{\ln(1/\mu\epsilon)}(g(\epsilon))$$

which is independent of ϵ.
Everything observable depends only on μ and g_r, invariant under the rg

$$\mu \to \mu + \delta\mu \qquad g_r \to g_r + \frac{\delta\mu}{\mu} \beta(g_r)$$

The formal 2d scale invariance — independence of μ — is gone, because we had to introduce a scale, the cutoff ϵ, in order to make sense of the theory.

The Ricci flow from the rg flow

The rg flow

$$\frac{d}{dt}\left(\frac{1}{\alpha'}g\right) = -\mathrm{Ricci}(g) + O(\alpha')$$

does not become the Ricci flow in the limit $\alpha' \to 0$.

We have to re-scale the rg "time" to $s = \alpha' t$ to get

$$\frac{d}{ds}\, g = -\mathrm{Ricci}(g) + O(\alpha')$$

A solution of this re-scaled rg equation is of the form

$$\tilde{g}(s) + \sum_{n=1}^{\infty} \alpha'^n \Delta\tilde{g}^n(s)$$

where $\tilde{g}(s)$ is a Ricci flow and $\Delta\tilde{g}^n(0) = 0$. Then the perturbative solution of the rg equation would be

$$\frac{1}{\alpha'}\tilde{g}_{ij}(0) + t\tilde{g}'_{ij}(0) + \alpha'\left[\frac{1}{2}t^2(\tilde{g}''_{ij}(0) + t(\Delta\tilde{g}^1_{ij})'(0)\right] + \cdots$$

At each order in α' the rg flow is polynomial in t.

Perturbation theory

For $\alpha' \sim 0$, the measure is dominated by the constant maps $\phi(x) = \phi_0 \in M$. Around each $\phi_0 \in M$, choose coordinates ϕ^i in $T_{\phi_0}M$:

$$\phi^i(\phi_0) = \phi_0^i \qquad \phi^i(\phi(x)) = \phi_0^i + \pi^i(x)$$

The integral is now over the constants $\phi_0 \in M$ and the fluctuations $\pi^i(x)$

$$S(\phi) = S(\phi_0; \pi) = \int_\Sigma \mathrm{dvol}_\gamma(x)\, \frac{1}{\alpha'} g_{ij}(\phi_0 + \pi(x))\, \partial^\mu \pi^i(x)\, \partial_\mu \pi^j(x)$$

$$\int \mathcal{D}\phi\, e^{-S(\phi)}\,(\cdots) = \int_M \mathrm{dvol}_g(\phi_0) \int_V \mathcal{D}\pi\, e^{-S(\phi_0;\pi)}\,(\cdots)$$

V is the vector space of maps $\pi : \Sigma \to T_{\phi_0}M$ (modulo the constant maps). The integral over V is very close to a gaussian integral:

$$\tilde{\pi}^i(x) = (\alpha')^{-1/2}\pi^i(x)$$

$$S(\phi_0, \pi) = \int_\Sigma \mathrm{dvol}_\gamma(x)\, g_{ij}(\phi_0)\, \partial^\mu \tilde{\pi}^i(x)\, \partial_\mu \tilde{\pi}^j(x) \;+\; O((\alpha')^{1/2})$$

The Feynman diagrams organize the perturbative calculation of nearly gaussian integrals on vector spaces.

Regularize

Approximate the integration space V by the subspace V_{t_0} on which

$$-\nabla^\mu \partial_\mu = \Delta < e^{-2t_0} \qquad t_0 \ll 0 \qquad e^{t_0} \ll 1$$

We would like to take the limit $t_0 \to -\infty$.
Label the metric g_{t_0}. The regularized (cutoff) measure is

$$\int d\rho_{t_0}(\gamma, g_{t_0}; \phi)\,(\cdots) = \int_M \mathrm{dvol}_{g_{t_0}}(\phi_0) \int_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0},\phi_0;\pi_0)}\,(\cdots)$$

The integration space V_{t_0} is finite dimensional if Σ is compact.
If $\Sigma = \mathbb{R}^2$ then V_{t_0} is still infinite dimensional. This is the *infrared* problem. We won't actually have to face it.

Renormalize

Take $\delta > 0$ very small. Let $V_{t_0,t_0+\delta}$ be the subspace of short-distance fluctuations

$$e^{-2(t_0+\delta)} < \Delta < e^{-2t_0}$$

The integration space decomposes: $\quad V_{t_0} = V_{t_0+\delta} \oplus V_{t_0,t_0+\delta}$

$$\pi_0(x) = \pi(x) + \pi'(x) \qquad \pi_0 \in V_{t_0} \quad \pi \in V_{t_0+\delta} \quad \pi' \in V_{t_0,t_0+\delta}$$

We can integrate out the the short-distance fluctuations as long as the functions being integrated depend only on the $\pi \in V_{t_0+\delta}$. (We only take measurements at 2d distances larger than e^{t_0}.)

$$\begin{aligned} \int_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0},\phi_0;\pi_0)} (\cdots) &= \int_{V_{t_0+\delta}} \mathcal{D}\pi \int_{V_{t_0,t_0+\delta}} \mathcal{D}\pi' \; e^{-S(g_{t_0},\phi_0;\pi+\pi')} (\cdots) \\ &= \int_{V_{t_0+\delta}} \mathcal{D}\pi \; e^{-S'(g_{t_0},\phi_0;\pi)} (\cdots) \end{aligned}$$

where

$$e^{-S'(g_{t_0},\phi_0;\pi)} = \int_{V_{t_0,t_0+\delta}} \mathcal{D}\pi' \; e^{-S(g_{t_0},\phi_0;\pi+\pi')}$$

Next, we argue that the new action takes the same form as the old

$$S'(g_{t_0}, \phi_0; \pi) = S(g_{t_0+\delta}, \phi_0; \pi) + O(e^{t_0})$$

for some slightly changed metric on M

$$g_{t_0+\delta} = g_{t_0} - \delta \cdot \beta(g_{t_0})$$

$g_{t_0+\delta}$ is calculated in the form of a Taylor series around $\phi_0 \in M$. We do this for each ϕ_0. Then we show that the resulting Taylor series all come from a single metric $g_{t_0+\delta}$ on M. This means that we need only calculate explicitly the leading term in the Taylor series.
So we have

$$\int_M d\phi_0 \int_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0}, \phi_0; \pi_0)} (\cdots) = \int_M d\phi_0 \int_{V_{t_0+\delta}} \mathcal{D}\pi \; e^{-S(g_{t_0+\delta}, \phi_0; \pi)} (\cdots)$$

- $\Delta \approx e^{-2t_0} \gg 1$ so Σ might as well be euclidean $\mathbb{R}^2$
- The integrating out can be done effectively, order by order in α', as a sum of Feynman diagrams, each a bounded integral of a bounded function.
- $\beta(g_{t_0})$ depends only on g_{t_0}, not on Σ or $\gamma_{\mu\nu}$ (since $e^{2t_0}\Delta \approx 1$).
- The new metric $g_{t_0+\delta}$ is constructed covariantly wrt $\mathit{Diff}(M)$, so $\beta(g)$ is a covariant function of g. Choosing natural coordinate systems on M gives the perturbation series for $\beta(g)$ manifestly in terms of the curvature tensor and its covariant derivatives.
- $\beta(g)$ does not depend on any of the arbitrary choices, such as coordinate systems or method of regularization, up to equivalence under $\mathit{Diff}(M)$. Changing these can only change β by a vertical vector field.

Construction of QFTs

Perturbatively, the rg flow is polynomial in time, at each order in α', so we can run the rg back to time $-\infty$, so we can remove the cutoff and construct continuum correlation functioins.

But we want to construct honest QFTs, not just perturbative ones. We need flows that exist for $t \to -\infty$ and we need control over the behavior as $t \to -\infty$.

The only obvious way to find rg trajectories that go to $\alpha' = 0$ as $t \to -\infty$.

There are fixed points $\frac{1}{\alpha'} g_{ij}$ at $\alpha' = 0$ for

$$R_{ij} - \lambda g_{ij} = (\mathcal{L}_v g)_{ij} = \nabla_i v_j + \nabla_j v_i$$

for some vector field v on M. The rhs expresses the fact that β_{ij} is defined only up to infinitesimal diffeomorphisms of M, that the rg flow actually acts on the space of metrics modulo $Diff(M)$.

For $\lambda > 0$, the α'-direction is unstable, so there is at least one rg trajectory that can be run backwards forever.
For $\lambda = 0$, the same is true because of the $O(\alpha' R^2)$ term in β.
For $\lambda < 0$, the fixed point is attractive in the α' direction. These describe limits of the RG flow as $t \to +\infty$.

Ancient solutions of the Ricci flow might give new QFTs, if there is enough control of the limit $t \to -\infty$. It would be necessary to show stability against the terms at higher order in α'.

Further topics:

- supersymmetric nonlinear models (Kahler-Ricci flow)
- boundary conditions and the boundary rg flow (mean curvature flow, Yang-Mills flow, Calabi flow?)
- axiomatic 2-d qft: the *c*-theorem (a function on the space of qfts that decreases under the rg flow)
- lack of a gradient formula for the rg flow on the space of qfts
- gradient formulas for the boundary rg flow