Properties of the boundary rg flow

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Geometry $\rightarrow$ 1+1d quantum field theory

- the space of Riemannian manifolds $\rightarrow$ the space of 1+1d qfts
- Ricci flow $\rightarrow$ renormalization group (rg) flow
- the space of submanifolds of Ricci-flat manifolds (and Ricci-solitons) $\rightarrow$ the space of boundary qfts
- mean curvature flow of submanifolds $\rightarrow$ boundary rg flow

The boundary rg is a gradient flow. (A. Konechny & DF, 2003)

I hope to convey some of what quantum field theories are and what we can say about them, in general.

The analogous result for the mean curvature flow is trivial (gradient of submanifold volume).

Will the connection between geometric flows and rg flows prove useful for either?

Example: the 2d nonlinear model

the geometry:

- a manifold M with Riemannian metric g

the 1+1d quantum field theory (formally):

- a Hilbert space $\mathcal{H} = L_2(\mathrm{Maps}\ \phi\colon \mathbb{R} \to M)$ or $(\mathrm{Maps}\ S^1 \to M)$
- a hamiltonian operator H on $\mathcal{H}$

$$H = \int_{\mathbb{R}} dx \ \left[\Delta_{\phi(x)} + g(\partial_x \phi, \partial_x \phi)\right]$$

- an algebra of local operators $\mathcal{O}(x)$, e.g.

$$\mathcal{O}_f(x) = f(\phi(x)) \qquad f \in C^\infty(M)$$

time translation: $\mathcal{O}(x,t) = e^{itH}\mathcal{O}(x)e^{-itH}$

Path integral formulation as 2d euclidean qft

$$\tau = it \qquad e^{-itH} = e^{-\tau H}$$

path integral for $e^{-\beta H}$:

$$e^{-\beta H}(\phi_\beta, \phi_0) \quad = \int_{\phi\colon [0,\beta]\times\mathbb{R}\to M} \mathcal{D}\phi\ e^{-S(\phi)}$$

$$S(\phi) = \int_0^\beta d\tau \int_{-\infty}^{\infty} dx\ \left[g(\partial_\tau\phi, \partial_\tau\phi) + g(\partial_x\phi, \partial_x\phi)\right]$$

partition function:

$$Z = \mathrm{tr}\ e^{-\beta H} = \int_{\phi\colon S^1\times\mathbb{R}\to M} \mathcal{D}\phi\ e^{-S(\phi)}$$

Less formally: cutoff qft

Choose a unit $\Lambda_0^{-1}$of 2d distance: $(ds)^2 = \Lambda_0^2[(dx)^2 + (d\tau)^2]$.

Take $x \in \mathbb{Z}/\Lambda_0$ instead of $\mathbb{R}$ (or $\mathbb{Z}_N/\Lambda_0$ instead of S^1)

$$\mathcal{H} = \bigotimes_{x \in \mathbb{Z}/\Lambda_0} L_2(M)$$

$$H = \sum_x \Lambda_0^{-1} \left(\Delta_{\phi(x)} + \Lambda_0^2 \operatorname{dist}_g^2 \left[\phi(x), \phi(x + \Lambda^{-1}) \right] \right)$$

We can take $\Lambda_0^{-1} \to 0$, at least perturbatively: replacing the metric g with $\alpha'^{-1} g$, and expanding the path integral in powers of α'.

$$S(\phi) = \int_0^\beta d\tau \int_{-\infty}^{\infty} dx \ \left[\alpha'^{-1} g(\partial_\tau \phi, \partial_\tau \phi) + \alpha'^{-1} g(\partial_x \phi, \partial_x \phi) \right]$$

The path integral is then concentrated near the constant maps. Transverse to the constants, it is a gaussian measure plus a small perturbation (evaluated by Feynman diagram technology).

The renormalization group (rg)

To take the limit $\Lambda_0^{-1} \to 0$, we need to let $\alpha'^{-1} g$ depend on Λ_0,

$$-\Lambda_0 \frac{d}{d\Lambda_0} \left(\alpha'^{-1} g_{\Lambda_0}\right) = \beta \left(\alpha'^{-1} g_{\Lambda_0}\right) = -\operatorname{Ricci}\left(\alpha'^{-1} g_{\Lambda_0}\right) + O(\alpha')$$

β is a vector field on the space of Riemannian manifolds (a formal power series in α'). It generates the (perturbative) rg flow.

Fix a unit of distance Λ^{-1}. Hold $\alpha'^{-1} g_\Lambda$ fixed while $\Lambda_0^{-1} \to 0$. (We need stability in the past/ultraviolet to get a well-defined qft.)

The result is a continuum (perturbative) qft depending on

- $(M, \alpha'^{-1} g_\Lambda)$
- the 2d metric $ds^2 = \Lambda^2 \left[(dx)^2 + (d\tau)^2\right]$

and is invariant under the rg = simultaneous increase of the unit of distance Λ^{-1} and flow of the target metric along β.

The space of qfts

(cf. the space of Riemannian manifolds)

Parametrized locally by coordinates λ^i, the *coupling constants*,

$$H(\lambda) = H(0) - \int dx \, \Lambda \, \lambda^i \mathcal{O}_i(x)$$

Finitely many λ^i because unstable manifolds are finite dimensional.

Partition function

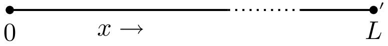
$$Z(\Lambda\beta, \lambda) = \mathrm{tr}\, e^{-\beta H(\lambda)}$$

Invariant under the rg flow generated by $\beta^i(\lambda)$, a vector field on the space of qfts:

$$\left(-\Lambda \frac{\partial}{\partial \Lambda} + \beta^i(\lambda) \frac{\partial}{\partial \lambda^i} \right) Z = 0$$

Boundary conditions

Quantum field theory on a 1d space with boundary (the half-line)



$(L \to \infty)$

The hamiltonian needs a boundary condition at $x = 0$
(and at $x = L$, but that boundary decouples in the limit $L \to \infty$).

Example: d-branes in the nonlinear model

Choose a submanifold $N \subset M$. Boundary condition at $x = 0$:
require $\phi(0) \in N$.

N parametrizes boundary qfts associated with the given bulk qft parametrized by $(M, \alpha'^{-1} g)$.

N flows under the rg by a vector field on N in the normal bundle: the mean curvature vector field $k^I(\phi)$ plus $O(\alpha')$ corrections.

The space of qfts with boundary condition

For a given bulk qft, the space of possible boundary conditions is parametrized by the boundary coupling constants λ^a that couple to operators $\mathcal{O}_a$ localized at the boundary, $x = 0$.

$$H = H(0) - \lambda^a \mathcal{O}_a$$

Under the rg, the λ^a flow along a vector field β^a.

$$\begin{aligned} \delta\lambda^a &\sim \text{normal vector fields } v^I(\phi) \text{ on } N \\ \delta\lambda^a \mathcal{O}_a &\sim v^I(\phi(0))\, \alpha'^{-1} g_{IJ} \partial_x \phi^J(0) \\ \beta^a &\sim k^I(\phi) + O(\alpha') \end{aligned}$$

The space of boundary qfts forms a bundle over the space of bulk qfts. The rg flow on the boundary qfts is a lift of the rg flow on the bulk qfts.

The space of boundary qfts

Specialize to bulk fixed point qfts, $\beta^i = 0$, the bulk *conformal field theories* (cf. the Ricci-flat manifolds and Ricci-solitons).

The space of boundary conditions are the *boundary qfts*.

Only the boundary coupling constants λ^a flow.

The boundary qfts describe a certain class of quantum wires – bulk critical – for use in quantum circuits. Boundaries are the simplest form of circuit junction.

Proposal (DF, cond-mat/0505084, 0505085)

Circuits made of bulk-critical quantum wires, joined at boundaries and junctions, would be ideal for asymptotically large-scale quantum computers (esp. the $c = 24$ Monster-symmetric bulk cft).

Equilibrium at temperature $T = 1/\beta$

equilibrium density matrix: $\rho_\beta = Z^{-1} e^{-\beta H}$ $\quad Z = \mathrm{tr}\ e^{-\beta H}$

entropy: $S = \mathrm{tr}\ (-\rho_\beta \ln \rho_\beta) = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z$

expectation values: $\langle \mathcal{O} \rangle = \mathrm{tr}\ \rho_\beta \mathcal{O}$ $\quad \langle \mathcal{O}(t) \rangle = \langle \mathcal{O} \rangle$

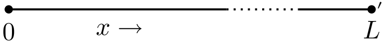
correlation functions:

$$\langle \mathcal{O}_1(t)\, \mathcal{O}_2(0) \rangle = Z^{-1}\ \mathrm{tr}\left(e^{-\beta H} e^{itH} \mathcal{O}_1 e^{-itH} \mathcal{O}_2\right)$$

analytic in $\tau = it$, $0 < \mathbf{Re}\ \tau < \beta$

KMS condition: $\langle \mathcal{O}_1(t - i\beta)\, \mathcal{O}_2(0) \rangle = \langle \mathcal{O}_2(0)\, \mathcal{O}_1(t) \rangle$

connected correlation functions: $\langle \mathcal{O}_1\, \mathcal{O}_2 \rangle_c = \langle \mathcal{O}_1\, \mathcal{O}_2 \rangle - \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$



$(L \to \infty)$

Bulk conformal invariance (i.e., bulk rg fixed point) implies

$$\ln Z = \ln z(\Lambda\beta, \lambda) + \frac{\pi c}{6}\frac{L}{\beta} + \ln z' \qquad \frac{L}{\beta} \gg 1$$

(c is the central charge of the Virasoro algebra of the bulk cft.)

So

$$S = s(\Lambda\beta, \lambda) + \frac{\pi c}{3}\frac{L}{\beta} + s' \qquad \frac{L}{\beta} \gg 1$$

s is the *boundary entropy*. It carries all the dependence on the boundary coupling constants λ^a (the boundary condition at $x = 0$),

$$\frac{\partial S}{\partial \lambda^a} = \frac{\partial s}{\partial \lambda^a}$$

Boundary gradient formula

$$\frac{\partial s}{\partial \lambda^a} = -g_{ab}(\lambda)\beta^b(\lambda)$$

$g_{ab}(\lambda)$ a positive-definite metric on the space of boundary qfts

$$g_{ab}(\lambda) = \beta \int_0^\beta d\tau \ \left[1 - \cos\left(2\pi\tau/\beta\right)\right] \langle \mathcal{O}_a(-i\tau)\mathcal{O}_b(0)\rangle_c$$

s decreases along the flow (2nd law of boundary thermodynamics)

$$\beta\frac{\partial s}{\partial \beta} = \Lambda\frac{\partial s}{\partial \Lambda} = \beta^a\frac{\partial s}{\partial \lambda^a} = -\beta^a g_{ab}\beta^b \leq 0$$

Comments

s is *not* the entropy of a density matrix ($\exists$ examples with $s < 0$).

We have not succeeded in putting a lower bound on s:

- no universal lower bound
- no lower bound per bulk cft
- no lower bound for a given boundary qft (rg trajectory)

The proof of the gradient formula depends on a strong assumption: essentially that the rg trajectory emerges from an ultraviolet (ancient) fixed point. This assumption is physically reasonable, but we do not understand why it should be necessary.

rg-covariance of the metric seems to need a slightly stronger assumption (maybe not physically reasonable).

The proof

$$
\begin{aligned}
\partial_a s = \partial_a S &= \partial_a \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \beta \langle \mathcal{O}_a \rangle \\
&= -\beta^2 \frac{\partial}{\partial \beta} \langle \phi_a \rangle = \beta^2 \langle H \, \mathcal{O}_a \rangle_c
\end{aligned}
$$

The hamiltonian is local:

$$
H = \int_0^\infty dx \, \mathcal{E}(x, t) \, .
$$

Separate into boundary and bulk energies: $\quad H = h_\epsilon(t) + H_\epsilon(t)$

$$
h_\epsilon(t) = \int_0^\epsilon dx \, \mathcal{E}(x, t) \qquad H_\epsilon(t) = \int_\epsilon^\infty dx \, \mathcal{E}(x, t)
$$

$$
\beta^{-2} \partial_a s = \langle h_\epsilon(t) \, \mathcal{O}_a(0) \rangle_c + \langle H_\epsilon(t) \, \mathcal{O}_a(0) \rangle_c
$$

Fourier transform:

$$
\begin{aligned}
f_a(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t} \langle h_\epsilon(t)\, \mathcal{O}_a(0)\rangle_c \\
F_a(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t} \langle H_\epsilon(t)\, \mathcal{O}_a(0)\rangle_c
\end{aligned}
$$

$$
2\pi\delta(\omega)\beta^{-2}\partial_a s = f_a(\omega) + F_a(\omega)
$$

Bulk conformal invariance implies a sum rule:

$$
\int \frac{d\omega}{2\pi}\; \left[\frac{(2\pi/\beta)^2}{\omega^2 + (2\pi/\beta)^2}\right] \left(\frac{1-e^{-\beta\omega}}{\beta\omega}\right) F_a(\omega) = 0
$$

from which

$$
\frac{\partial s}{\partial \lambda^a} = \int \frac{d\omega}{2\pi}\; \left[\frac{(2\pi/\beta)^2}{\omega^2 + (2\pi/\beta)^2}\right] \left(\frac{1-e^{-\beta\omega}}{\beta\omega}\right) f_a(\omega)
$$

Invert the F.T. and analytically continue to $\tau = it$ to get

$$\frac{\partial s}{\partial \lambda^a} = \beta \int_0^\beta d\tau \ \left[1 - \cos\left(2\pi\tau/\beta\right)\right] \langle h_\epsilon(-i\tau)\mathcal{O}_a(0)\rangle_c$$

We can substitute $h_\epsilon(t) = -\beta^a \mathcal{O}_a(t)$ as suggested by

$$\beta \frac{\partial \ln z}{\partial \beta} = -\langle \beta h_\epsilon \rangle = \Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta^a \frac{\partial \ln z}{\partial \lambda^a} = \beta^a \langle \beta \mathcal{O}_a \rangle$$

with this substitution

$$\frac{\partial s}{\partial \lambda^a} = \beta \int_0^\beta d\tau \ \left[1 - \cos\left(2\pi\tau/\beta\right)\right] \langle -\beta^b \mathcal{O}_b(-i\tau)\mathcal{O}_a(0)\rangle_c = -g_{ab}\beta^b$$

The sum rule

Bulk conformal invariance implies chiral energy flow:

$$\mathcal{E}(x,t) = \mathcal{E}_R(x - vt) + \mathcal{E}_L(x + vt) \quad x > 0 \qquad (v = 1)$$

Locality:

$$[\mathcal{E}_R(x{-}t),\, \mathcal{O}_a(0)] = 0 \quad x{-}t > 0, \quad [\mathcal{E}_L(x{+}t),\, \mathcal{O}_a(0)] = 0 \quad x{+}t > 0$$

so we can construct analytic *response functions*

$$R_a^+(\omega) = \int dt\; e^{i\omega(t-x)} \langle [i\mathcal{E}_R(x{-}t),\, \mathcal{O}_a(0)] \rangle \quad \text{analytic for } \mathrm{Im}\,\omega \geq 0$$

$$R_a^-(\omega) = \int dt\; e^{i\omega(t+x)} \langle [i\mathcal{E}_L(x{+}t),\, \mathcal{O}_a(0)] \rangle \quad \text{analytic for } \mathrm{Im}\,\omega \leq 0$$

The F.T. of the KMS condition gives

$$F_a(\omega) = \frac{1}{i(1 - e^{-\beta\omega})} \left[\frac{R_a^+(\omega)}{\omega + i0^+} + \frac{R_a^-(\omega)}{\omega - i0^+} \right] \quad (R_a^\pm(0) = 0).$$

So the sum rule is

$$\int \frac{d\omega}{2\pi} \left[\frac{(2\pi/\beta)^2}{\omega^2 + (2\pi/\beta)^2}\right] \frac{1}{i\beta\omega} \left[\frac{R_a^+(\omega)}{\omega + i0^+} + \frac{R_a^-(\omega)}{\omega - i0^+}\right] = 0$$

and follows from the vanishing formulas

$$R^+(2\pi i/\beta) = R^-(-2\pi i/\beta) = 0$$

which we get by using first KMS

$$\begin{aligned} \langle \mathcal{E}_R(x-t)\, \mathcal{O}_a(0)\rangle_c &= \int \frac{d\omega}{2\pi i}\, e^{-i\omega(t-x)} \frac{R^+(\omega)}{1 - e^{-\beta\omega}} \\ &= \sum_{n=1}^{\infty} e^{n(t-x)2\pi/\beta} R_a^+(2\pi i n/\beta) \end{aligned}$$

then the *bulk quantization* (where $-x$ is imaginary time, τ space)

$$\begin{aligned} \langle \mathcal{E}_R(x+i\tau)\, \mathcal{O}_a(0)\rangle_c &= \langle B|\mathcal{O}_a(0)\mathcal{E}_R(x+i\tau)|0\rangle \\ &= \langle B|\mathcal{O}_a(0) \sum_{n=2}^{\infty} e^{n(-i\tau-x)2\pi/\beta} L_n|0\rangle \end{aligned}$$

The chiral energy currents

The qft depends on the (arbitrary) 2d metric

$$(ds)^2 = \gamma_{\mu\nu}(x,t)dx^\mu dx^\nu = \gamma_{tt}(dt)^2 + \gamma_{xt}dxdt + \gamma_{tx}dtdx + \gamma_{xx}(dx)^2$$

The energy-momentum (stress-energy) tensor

$$\langle T^{\mu\nu}(x,t)\cdots\rangle_c = 2\frac{\partial}{\partial\gamma_{\mu\nu}(x,t)}\langle\cdots\rangle_c$$

2d coordinate-independence

$$\partial_\mu T^{\mu\nu} = 0$$

bulk conformal invariance is $T^\mu_\mu(x,t) = 0,\ x > 0$

$$H = \int dx\ T_{tt}(x,t)$$

$$\mathcal{E}_R(x-t) = \frac{1}{2}(T_{tt} - T_{xt}) \qquad \mathcal{E}_L(x+vt) = \frac{1}{2}(T_{tt} + T_{xt})$$