Supersymmetric $1+1$d boundary field theory

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1d quantum systems, critical in the bulk

\[ Z = \text{tr} \ e^{-\beta H} \]

for \( L/\beta \gg 1 \)

\[ \ln Z = \ln z(\Lambda \beta) + \frac{\pi c}{6} \frac{L}{\beta} + \ln z' \]

boundary renormalization group flow

\[ \Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = \beta^a(\lambda) \frac{\partial \ln z}{\partial \lambda^a} \]
Gradient formula for boundary entropy

boundary entropy

\[ S = \left( 1 - \beta \frac{\partial}{\partial \beta} \right) \ln Z = s + \frac{\pi c}{3} \frac{L}{\beta} + s' \]

gradient formula

\[ \frac{\partial s}{\partial \lambda^a} = -g_{ab} \beta^b(\lambda) \]

implying second law of boundary thermodynamics

\[ \Lambda \frac{\partial s}{\partial \lambda} = \beta \frac{\partial s}{\partial \beta} = \beta^a \frac{\partial s}{\partial \lambda^a} = -\beta^a g_{ab} \beta^b \leq 0 \]

The boundary behaves as an isolated system.
Supersymmetric 1d systems, critical in the bulk

a conserved fermionic super-charge

\[ H = \hat{Q}^2 \]

(Advertisement: in cond-mat/0505084 and 0505085, I argued that circuits made of bulk-critical quantum wire, joined at boundaries and junctions, would be ideal for asymptotically large-scale quantum computing: the \( c = 24 \) monster system in particular.)
A second gradient formula for supersymmetric systems

[DF & A. Konechny, in preparation]

\[ \frac{\partial \ln z}{\partial \lambda^a} = -g_{ab}^S \beta^b(\lambda) \]

(the \( \lambda^a \) now restricted to the susy coupling constants)

implying positivity of the susy boundary energy

\[ \Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = \beta^a \frac{\partial \ln z}{\partial \lambda^a} = -\beta^a g_{ab}^S \beta^b \leq 0 \]

The boundary behaves as an isolated supersymmetric system.
Here, I will prove directly the positivity of the boundary energy

\[ \Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} \leq 0 \]

equivalently, that \( \ln z \) decreases under the RG flow.
Local densities

energy and super-charge

\[ H = \int_0^L dx \, \mathcal{H}(t, x) \]

\[ \hat{Q} = \int_0^L dx \, \hat{\rho}(t, x) \]

\[ [\hat{Q}, \hat{\rho}(t, x)]_+ = 2\mathcal{H}(t, x) \]

local conservation of super-charge

\[ \partial_t \hat{\rho}(t, x) + \partial_x \hat{j}(t, x) = 0 \]
Boundary energy and super-charge

\[ h(t) = \lim_{\epsilon \to 0} \int_{0}^{\epsilon} dx \ H(t, x) \]

\[ \hat{q}(t) = \lim_{\epsilon \to 0} \int_{0}^{\epsilon} dx \ \hat{\rho}(t, x) \]

\[ [\hat{Q}, \hat{q}(t)]_+ = 2h(t) \]

\[ -\frac{\partial \ln z}{\partial \beta} = \langle h \rangle \]
Separate $\hat{Q}$ into boundary and bulk parts at $x = \epsilon$

\[
\hat{q}_\epsilon(t) = \int_0^\epsilon dx \ \hat{\rho}(t, x) \quad \hat{Q}_{bulk}(t) = \int_\epsilon^L dx \ \hat{\rho}(t, x)
\]

\[
\hat{Q} = \hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t)
\]

Locality implies

\[
[\hat{Q}_{bulk}(0), \hat{q}(0)]_+ = 0
\]

so

\[
\langle 2h \rangle = \langle [Q, \hat{q}(0)]_+ \rangle = \langle [\hat{q}_\epsilon(0), \hat{q}(0)]_+ \rangle
\]

but this equation is useless at $\epsilon = 0$, because

\[
\langle [\hat{q}(t), \hat{q}(0)]_+ \rangle \text{ is uv divergent at } t = 0.
\]

The boundary cannot be separated from the bulk, in general.
Use bulk super-conformal invariance
define

\[ g_\epsilon(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle [\hat{q}_\epsilon(t), \hat{q}(0)]_+ \rangle \]

\[ G_\epsilon^{\pm}(\omega) = \pm \int_{0}^{\pm\infty} dt \, e^{i\omega t} \langle [\hat{Q}_{bulk}(t), \hat{q}(0)]_+ \rangle \]

so

\[ 2\pi\delta(\omega)\langle 2h \rangle = g_\epsilon(\omega) + G_\epsilon^+(\omega) + G_\epsilon^-(\omega) \]

bulk super-conformal invariance implies

\[ G_\epsilon^+(i\pi/\beta) = 0 = G_\epsilon^-(i\pi/\beta) \]

so

\[ \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \, G_\epsilon^{\pm}(\omega) = 0 \]
so

\[ \langle 2h \rangle = \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \ g_\epsilon(\omega) \]

Now take \( \epsilon \to 0 \):

\[ g(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \left\langle [\hat{q}(t), \hat{q}(0)]_+ \right\rangle \]

\[ \beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = -\frac{\beta}{2} \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \ g(\omega) \]

which is uv-finite as long as \( \text{dim}[g(\omega)] < 1 \), i.e., \( \text{dim}[\hat{q}] < 1 \).

We have \( g(\omega) \geq 0 \) and \( g(\omega) = 0 \) iff \( \hat{q} = 0 \), so

\[ \beta \frac{\partial \ln z}{\partial \beta} \leq 0 \]

with equality iff the boundary is critical (superconformal).
The gradient formula

boundary operators

\[ [\hat{Q}, \hat{\phi}_a(t)]_+ = \phi_a(t) \]

\[ \frac{\partial \ln z}{\partial \lambda^a} = \beta \langle \phi_a \rangle \]

boundary beta-functions

\[ \hat{q} = -2 \beta^a \hat{\phi}_a \]

\[ h = \frac{1}{2} [\hat{Q}, \hat{q}]_+ = -\beta^a \phi_a \]

\[ \Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = \beta \langle \beta^a \phi_a \rangle = \beta^a \frac{\partial \ln z}{\partial \lambda^a} \]
\[ \langle \phi_a \rangle = \langle [\hat{Q}, \hat{\phi}_a(0)]_+ \rangle = \langle [\hat{q}_e(t) + \hat{Q}_{bulk}(t), \hat{\phi}_a(0)]_+ \rangle \]

\[
\begin{align*}
g_a(\omega) &= \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle [\hat{q}(t), \hat{\phi}_a(0)]_+ \rangle \\
&= \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle [-2\beta^b \hat{\phi}_b(t), \hat{\phi}_a(0)]_+ \rangle \\
&= -2\beta^b g_{ab}(\omega)
\end{align*}
\]

\[
\begin{align*}
\langle \phi_a \rangle &= \int \frac{d\omega}{2\pi} \ \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \ g_a(\omega) \\
&= -2\beta^b \int \frac{d\omega}{2\pi} \ \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \ g_{ab}(\omega)
\end{align*}
\]
\[
\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab} \beta^b
\]

\[
g_{ab}(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle [\hat{\phi}_b(t), \hat{\phi}_a(0)]_+ \rangle
\]

\[
g^S_{ab} = 2\beta \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} g_{ab}(\omega)
\]

\[
= \pi \int dt \ e^{-\pi|t|/\beta} \langle [\hat{\phi}_b(t), \hat{\phi}_a(0)]_+ \rangle
\]

\[
= 2\pi \int_0^\beta d\tau \ \sin \left( \frac{\pi \tau}{\beta} \right) \langle \hat{\phi}_b(-i\tau), \hat{\phi}_a(0) \rangle
\]
Some questions

1. Why do we need bulk conformal invariance?

2. Why do we need canonical uv behavior in the boundary?
   - no negative dimension boundary operators
   - no strongly irrelevant boundary operators

3. Does the result apply to composite boundaries/junctions?

4. Can $\ln z$ (and/or $s$) be bounded below?
Bulk conformal invariance and zeros of response functions

\[ \partial_t \hat{Q}_{bulk}(t) = \int_\epsilon^L dx \left[ -\partial_x \hat{j}(t, x) \right] = \hat{j}(t, \epsilon) \]

Define response functions

\[ R_{a}^{\pm}(\omega) = \pm \int_{0}^{\pm\infty} dt \ e^{i \omega t - \delta |t|} \langle [i \hat{j}(t, \epsilon), \hat{\phi}_a(0)]_+ \rangle \]

\( R_{a}^{+}(\omega) \) is analytic in the upper half-plane, \( R_{a}^{-}(\omega) \) in the lower.

Use the conservation equation

\[ G_{a, \epsilon}^{\pm}(\omega) = \pm \int_{0}^{\pm\infty} dt \ e^{i \omega t - \delta |t|} \langle [\hat{Q}_{bulk}(t), \hat{\phi}_a(0)]_+ \rangle = \frac{R_{a}^{\pm}(\omega)}{\omega \pm i \delta} \]
\( \tau = it, \ 0 < \tau < \beta \)

\[
\langle \hat{j}(-i\tau, \epsilon) \hat{\phi}_a(0) \rangle = \int \frac{d\omega}{2\pi i} \frac{e^{-\omega \tau}}{1 + e^{-\omega \beta}} \left[ R^+(\omega) + R^-(\omega) \right]
\]

poles at

\[
\omega_n = \frac{2\pi in}{\beta} \quad n \in \frac{1}{2} + \mathbb{Z}
\]

so

\[
\langle \hat{j}(-i\tau, \epsilon) \hat{\phi}_a(0) \rangle = \beta^{-1} \sum_n e^{-\omega_n \tau} \left[ \theta(n)R^+(\omega_n) - \theta(-n)R^-(\omega_n) \right]
\]

but

\[
j(-i\tau, x) = AG(x + i\tau) + \bar{A}G(x - i\tau)
\]

so

\[
R^+_a(i\pi/\beta) = 0 = R^-_a(-i\pi/\beta)
\]