The Space of Conformal Boundary Conditions for the $c = 1$ Gaussian Model

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Abstract

This is a telegraphic summary of work done in 1993 to construct abstractly the space of conformal boundary conditions of an arbitrary two dimensional conformal field theory, and to construct explicitly the spaces of conformal boundary conditions of the $c = 1$ gaussian models. The work was described briefly at a Rutgers group meeting, probably during the academic year 1993-94. This summary is based on a fast re-reading of old notes. Reconstructing the proofs for the $c = 1$ gaussian models would require more serious archeological investigation. No references are given. Speculations which were made about the situation for $c > 1$ are omitted.

1 The space of conformal boundary conditions of a conformal field theory

A is the commutative, associate algebra whose dual space A^* is the linear space of conformally invariant boundary states on the boundary of the unit disk. A basis for A is given by the set of spin 0 conformal primary fields (i.e. primary for both Virasoro algebras). The algebra structure on A is obtained by letting primary fields act near the boundary on conformal boundary states. A spin 0 primary field, approaching the boundary of the unit disk, takes a conformal boundary state to a conformal boundary state. (This is rigorously true in any unitary conformal field theory with central charge $c \leq 1$.) This action of A on A^* is a linear map $A \otimes A^* \to A^*$, which gives the structure constants making A into a commutative, associative algebra. Commutativity follows easily from $SL(2,R)$ invariance on the disk. Associativity is proved by sending two spin 0 primary fields to different points on the boundary, in either order.

The space of conformal boundary conditions is $B = \mathrm{spec}(A)$, the space of multiplicative linear functionals on the algebra A. A linear functional on A is a conformal boundary state. The multiplicative condition is equivalent to cluster decomposition on the disk, or equivalently on the strip. The space of conformal boundary states can now be described as the space of measures on B.

2 The $c = 1$ gaussian model

R is the radius of the target circle, normalized so that $1/R$ is the radius of the dual target circle and $R = 1$ is the $SU(2) \times SU(2)$ invariant model.

The vertex operators are $V_{p,\bar{p}}$, with conformal weights $h = p^2$ and $\bar{h} = \bar{p}^2$, where $p - \bar{p} = R^{-1}k$, k the integer $U(1)$ charge, and $p + \bar{p} = Rn$, n the integer winding number or dual $U(1)$ charge.

The description of the spaces B for all possible values of R is obtained by first doing the case $R = 1$, then the rational values of R, then the irrational values. But I describe the results here in reverse order.

3 Irrational R

For R an irrational number, B is a fiber space over the interval $I = [-1, 1]$. Over the interior of the interval I, each fiber is a single point. Over each endpoint of I, the fiber is a circle. The circle over the endpoint $+1$ is the radius R target circle of the gaussian model, i.e. the Dirichelet boundary conditions. The other circle, the fiber over the endpoint -1, is the radius R^{-1} dual target circle, i.e. the Dirichelet boundary conditions for the dual gaussian model or, equivalently, the Neumann boundary conditions for the original gaussian model.

The $U(1)$ internal symmetry group acts by rotations on the circle over $+1$, leaving the rest of B fixed. The dual $U(1)$ acts by rotations on the circle over

-1, leaving the rest of B fixed. The part of B that lies over the interior of the interval, and which is homeomorphic to the interior of the interval since each fiber there is a single point, is the set of conformal boundary conditions invariant under both $U(1)$ internal symmetry groups.

Topologically, B is not Haussdorff. If you move in the interior of the interval towards an endpoint, you approach every point on the circle over that endpoint. The more precise view is measure theoretic. Each point in B is a conformal state. As you move in the interior of the interval towards an endpoint, the conformal state approaches a decomposable state corresponding to the uniform measure on the circle of boundary conditions that is the fiber over the endpoint.

The Hilbert space of the bulk theory decomposes into irreducible representations of the two chiral $U(1)$ current algebras, characterized by the $U(1)$ charges $(p,\bar{p})$. For R irrational, all of these representations of the $U(1)$ currents are irreducible under the two Virasoro algebras except the neutral sector $p=\bar{p}=0$. The neutral sector decomposes under the Virasoro algebras, with the spin 0 primaries being a collection Φ_l, $l=0,1/2,1,3/2,\cdots$ having conformal weights $h=\bar{h}=l^2$. A basis for the algebra A is given by the Φ_l and the spin 0 vertex operators $V_{p,-p}$, $2p=R^{-1}k$, with integer $U(1)$ charge $k\neq 0$, and the $V_{p,p}$, $2p=Rn$ with integer dual $U(1)$ charge $n\neq 0$.

In the algebra A, the Φ_l generate a sub-algebra A_0 which is isormorphic to the algebra of Legendre polynomials on the interval I, so $\mathrm{spec}(A_0)=I$. The inclusion of algebras $A_0\rightarrow A$ gives the fibering $\mathrm{spec}(A)\rightarrow\mathrm{spec}(A_0)$, i.e. $B\rightarrow I$.

In the algebra A, the product of $V_{p,-p}$ and $V_{-p,p}$ is neutral and so equals a linear combination of the Φ_l (in fact, of infinitely many of them). This linear combination is, in fact, the characteristic function of the endpoint $+1$ in I. Similarly, the product of $V_{p,p}$ and $V_{-p,-p}$ is the characteristic function of the other endpoint -1 in I. The $V_{p,-p}$ and the Φ_l generate an ideal in A, corresponding in B to the inclusion of the dual Dirichelet boundary conditions, the circle fiber over the endpoint -1. The $V_{p,p}$ and the Φ_l generate another ideal in A, corresponding in B to the inclusion of the Dirichelet boundary conditions, the circle fiber over the endpoint $+1$.

4 $R = 1$

For $R = 1$, the space B of conformal boundary conditions is exactly the group manifold $SU(2)$, the Dirichelet boundary conditions for the WZW model at $c = 1$.

5 R rational

When R is rational, some of the charged sectors of the chiral $U(1)$ current algebras are reducible under the two Virasoros. Again, there is a sub-algebra A_0 in A which has as a basis the spin 0 primaries which correspond to small Virasoro representations (where small means smaller than the Verma module, containing null vectors). So, again, there is a fibering $B \to B_0$ where $B_0 = \mathrm{spec}(A)$.

The sub-algebra A_0 can be described explicitly as a sub-algebra of the functions on $SU(2)$. The base space B_0 is a quotient of $SU(2)$ by a discrete group action.

Parametrize each $SU(2)$ matrix U by two complex numbers u, v with $|u|^2 + |v|^2 = 1$. That is, the matrix elements of U are $U_{11} = u$, $U_{12} = v$, $U_{21} = -\bar{v}$, $U_{22} = \bar{u}$. Define an action of the discrete group $Z \times Z$, two copies of the integers, on $SU(2)$ as follows. Let (n_1, n_2) in $Z \times Z$ take $(u, v) \to (e^{2\pi i n_1/R}u, e^{2\pi i n_2 R}v)$.

The sub-algebra A_0 consists of the functions on $SU(2)$ which are invariant under this action of $Z \times Z$.

If R is irrational, then the discrete group acts ergodically on the phases of u and v, so the invariant functions are the functions which depend only on, say, $2|u| - 1$. These are the functions on the interval. (But note that there is much more structure in the quotient space of $SU(2)$ then just the algebra of invariant continuous functions. There is clearly some non-commutative geometry there.)

The "small" spin 0 Virasoro primaries, which form a basis for the sub-algebra A_0, consist of the set $Y_{l,p,l,\bar{p}}$, where the conformal weights are $h = \bar{h} = l^2$, the integer $U(1)$ charge is $k = R(p - \bar{p})$, the integer dual $U(1)$ charge is $n = R^{-1}(p + \bar{p})$, k is restricted to the subgroup of integers for which $R^{-1}k$ is also an integer, n is restricted to the subgroup of integers for which Rn is also an integer, $l - p$ must be an integer, $l \geq |p|$, $l \geq |\bar{p}|$. Each of these "small" Virasoro primaries $Y_{l,p,l,\bar{p}}$ can be identified with a function on $SU(2)$, namely the matrix element in the spin l representation between the state of spin p and the state of spin $\bar{p}$. Moreover, the multiplication in the algebra is exactly the multiplication of the corresponding functions on $SU(2)$. And the selection rules on the values of $l, p, \bar{p}$ are exactly the selection rules imposed by invariance under the action of $Z \times Z$ on $SU(2)$.

The idea of the proof is first to show this at $R = 1$, for the full algebra of functions on $SU(2)$. Then find the selection rules for rational R. Then show that the multiplication law is the same as $R = 1$, for those "small" primaries which survive the selection rules.

The remaining spin 0 conformal primaries are the vertex operators $V_{p,p}$ with $2p \in RZ - Z$ and the $V_{p,-p}$ with $2p \in R^{-1}Z - Z$. The spin 0 vertex operators which have been excluded are those which correspond to representations of the chiral $U(1)$ current algebras which are decomposable under the Virasoro algebras. Those vertex operators are infinite sums of the "small" primaries.

I believe (but my memory definitely needs to be checked in detail) that the full space B of conformal boundary conditions, which is fibered over the base space $B_0 = Z\backslash SU(2)/Z$, has the following structure. Everywhere on the base B_0, except at the subsets $u = 0$ and $v = 0$, the fiber is a single point. The subset of B_0 at $u = 0$ is the circle $|v| = 1$ modulo the action of Z, $v \to e^{2\pi i n_2 R} v$. The subset of B lying above $u = 0$ is the original circle $|v| = 1$. Similarly, the subset of B above $v = 0$ is the circle $|u| = 1$ and the fibering is by the action of Z, $u \to e^{2\pi i n_1 R^{-1}} u$.

The two circles in B are again the Dirichelet and dual Dirichelet boundary conditions.

There is surely an elegant description of B itself, not just of B_0, as a quotient by a group action, the group now acting on a fiber space over $SU(2)$. But I cannot remember the details offhand. It should be reasonably easy to reconstruct.