Quasi Riemann surfaces

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draft August 31, 2018

Abstract

This is a naive attempt at describing a mathematical object for use in theoretical physics. The object is an abelian group and metric space with geometric structure analogous to a Riemann surface, suitable for writing an analog of the Cauchy-Riemann equation. Examples are certain abelian groups of singular $(n-1)$-currents in conformal $2n$-manifolds. The hope is that complex analysis on such quasi Riemann surfaces can be developed in analogy with Riemann surfaces. The physics goal is to construct quantum field theories on quasi Riemann surfaces in analogy with 2d conformal field theories on Riemann surfaces, forming a new class of constructable quantum field theories on $2n$-manifolds. The exposition here is unrigorous, aiming to attract interest in building rigorous complex analysis on quasi Riemann surfaces.

Contents

The goal is to formulate an analog of analysis in one complex variable on certain spaces of singular $(n-1)$-currents in a conformal $2n$-manifold, to be used in constructing quantum field theories on those spaces of currents in analogy with 2d conformal field theories on Riemann surfaces [1, 2].

1. Currents in a conformal $2n$-manifold

The basic objects are

$$
\begin{aligned}
M \quad &= \text{a smooth } 2n\text{-manifold with orientation and conformal structure} \\
&\quad\ \ \text{(for simplicity: } n \text{ odd, } M \text{ compact without boundary, } \tilde{H}_{n-1}(M)=0) \\
\Omega_k^{\mathrm{smooth}} M &= \text{the smooth real } k\text{-forms on } M \\
\mathcal{D}_k^{\mathrm{distr}} M &= (\Omega_k^{\mathrm{smooth}} M)^*, \text{ the distributional } k\text{-currents in } M \\
\mathcal{D}_k^{\mathrm{sing}} M &= \text{the abelian group of singular } k\text{-currents in } M, \\
&\quad\ \ \text{generated by the } k\text{-simplices } \Delta^k \to M \text{ in } M
\end{aligned}
$$

The pairing between a k-current ξ and a k-form ω is

$$\int_\xi \omega = \int_M \frac{1}{k!} \omega_{\mu_1\cdots\mu_k}(x)\, \xi^{\mu_1\cdots\mu_k}(x)\, d^{2n}x \tag{1.1}$$

The boundary operator on currents is dual to the exterior derivative on forms

$$\int_{\partial\xi} \omega = \int_\xi d\omega \qquad (\partial\xi)^{\mu_2\cdots\mu_k}(x) = -\partial_{\mu_1}\xi^{\mu_1\cdots\mu_k}(x) \qquad \partial^2 = 0 \tag{1.2}$$

A map σ from the k-simplex to M is represented by a k-current $[\sigma]$

$$\sigma \colon \Delta^k \to M \qquad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^*\omega \tag{1.3}$$

$\mathcal{D}_k^{\mathrm{sing}} M$ is the abelian group of currents generated by the k-simplices in M, i.e., the currents representing the singular k-chains in M. The bilinear intersection form on currents

$$I_M(\xi_1,\xi_2) = \int_M \frac{1}{k_1!k_2!} \xi_1^{\mu_1\cdots\mu_{k_1}}\, \epsilon_{\mu_1\cdots\mu_{k_1}\nu_1\cdots\nu_{k_2}}\, \xi_2^{\nu_1\cdots\nu_{k_2}}(x)\, d^{2n}x \qquad k_1+k_2 = 2n \tag{1.4}$$

is defined almost everywhere, vanishes unless $k_1+k_2=2n$, and depends only on the orientation $\epsilon_{1\cdots 2n} dx^1\cdots dx^{2n}\,(d^{2n}x)^{-1}$. It satisfies

$$I_M(\xi_2,\xi_1) = (-1)^{k_1k_2} I_M(\xi_1,\xi_2) \qquad I_M(\partial\xi_1,\xi_2) = (-1)^{k_1} I_M(\xi_1,\partial\xi_2) \tag{1.5}$$

On singular currents $I_M(\xi_1,\xi_2)$ gives the integer intersection number (where defined). The Hodge $*$-operator acting on n-forms and on n-currents is conformally invariant

$$*\omega_{\mu_1\cdots\mu_n}(x) = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n}(x)\omega_{\nu_1\cdots\nu_n}(x) \qquad \int_{*\xi}\omega = \int_\xi *\omega \qquad *^2 = (-1)^n \tag{1.6}$$

$$I_M(\xi,*\xi') = I_M(\xi',*\xi) \qquad I_M(\xi,*\xi) > 0 \quad \xi \neq 0 \qquad \deg(\xi) = \deg(\xi') = n \tag{1.7}$$

For n odd, the n-currents form a Hilbert space with complex structure $J = *$ and hermitian inner product $\langle\, \xi,\xi'\, \rangle = I_M(\xi,*\xi')$ (where defined).

2. j-currents in a Riemann surface, $j = 0, 1, 2$

When $n = 1$, M is a Riemann surface. Write Σ instead of M. The augmented chain complex of j-currents in Σ is

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{\partial} & \mathcal{D}_2^{\text{sing}}\Sigma & \xrightarrow{\partial} & \mathcal{D}_1^{\text{sing}}\Sigma & \xrightarrow{\partial} & \mathcal{D}_0^{\text{sing}}\Sigma & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \xrightarrow{\partial} & \mathcal{D}_2^{\text{distr}}\Sigma & \xrightarrow{\partial} & \mathcal{D}_1^{\text{distr}}\Sigma & \xrightarrow{\partial} & \mathcal{D}_0^{\text{distr}}\Sigma & \xrightarrow{\partial} & \mathbb{R} & \xrightarrow{\partial} & 0 \end{array} \tag{2.1}$$

$$\xi \in \mathcal{D}_0^{\text{distr}}\Sigma \quad \overset{\partial}{\longmapsto} \quad \int_\xi 1 \tag{2.2}$$

The intersection form and the conformal Hodge $*$-operator on j-currents satisfy

$$I_\Sigma(\xi_1, \xi_2) = 0 \qquad \text{unless } j_1 + j_2 = 2 \tag{2.3}$$

$$I_\Sigma(\xi_2, \xi_1) = (-1)^{j_1 j_2} I_\Sigma(\xi_1, \xi_2) \qquad I_\Sigma(\partial\xi_1, \xi_2) = (-1)^{j_1} I_\Sigma(\xi_1, \partial\xi_2) \tag{2.4}$$

$$J = * \quad \text{on } \mathcal{D}_1^{\text{distr}}\Sigma \qquad J^2 = -1 \tag{2.5}$$

$$I_\Sigma(\xi, J\xi') = I_\Sigma(\xi', J\xi) \qquad I_\Sigma(\xi, J\xi) > 0 \quad \xi \neq 0 \qquad \deg(\xi) = \deg(\xi') = 1 \tag{2.6}$$

3. Cauchy-Riemann equation in terms of j-currents

A fundamental solution

$$G(z, z')dzdz' \qquad G(z, z') = \frac{1}{(z - z')^2} + \text{ holomorphic} \tag{3.1}$$

of the Cauchy-Riemann equation

$$\partial_{\bar{z}} G(z, z') = -\pi \partial_z \delta^2(z - z') \qquad G(z, z') = G(z', z) \tag{3.2}$$

can be considered as a complex bilinear form on 1-currents

$$G(\xi, \xi') = \int_\xi \int_{\xi'} G(z, z')dzdz' \tag{3.3}$$

and the Cauchy-Riemann equation can be written in terms of J and $I_\Sigma(\xi_1, \xi_2)$

$$G(\xi, \partial\xi_2) = 2\pi i I_\Sigma(\partial\xi, \xi_2) \qquad G(\xi, \xi') = G(\xi', \xi) \qquad G(J\xi, \xi') = iG(\xi, \xi') \tag{3.4}$$

Taking ξ_2 to be a disk in Σ and ξ to be a 1-simplex, this is the residue formula.

4. $(n{-}1{+}j)$-currents in a conformal $2n$-manifold (n odd)

For simplicity, suppose n is odd. The general case is discussed in Appendix A. For each singular $(n{-}2)$-boundary $\partial\xi_0 \in \partial\mathcal{D}_{n-1}^{\text{sing}}M$ let

$$Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0} = \left\{ \xi \in \mathcal{D}_{n-1}^{\text{sing}} M \colon \partial\xi \in \mathbb{Z}\partial\xi_0 \right\} \tag{4.1}$$

This abelian group Q is to have structure analogous to a Riemann surface — a *quasi Riemann surface*. For each $\partial\xi_0$ there is an augmented chain complex

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{\partial} & Q_2 & \xrightarrow{\partial} & Q_1 & \xrightarrow{\partial} & Q & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \xrightarrow{\partial} & Q_2^{\mathbb{R}} & \xrightarrow{\partial} & Q_1^{\mathbb{R}} & \xrightarrow{\partial} & Q^{\mathbb{R}} & \xrightarrow{\partial} & \mathbb{R} & \xrightarrow{\partial} & 0 \end{array} \tag{4.2}$$

$$Q_{-1} = \mathbb{Z}\partial\xi_0 \qquad Q_0 = Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0} \qquad Q_1 = \mathcal{D}_n^{\text{sing}} M \tag{4.3}$$

$$Q_2 = \mathcal{D}_{n+1}^{\text{sing}} M / Q_0^\perp \qquad Q_0^\perp = \left\{ \xi' \in \mathcal{D}_{n+1}^{\text{sing}} M \colon I_M(\xi, \xi') = 0 \; \forall \xi \in Q \right\}$$

The $Q_j^{\mathbb{R}}$ are the analogously defined real vector spaces of distributional currents.

The intersection form $I_M(\xi_1,\xi_2)$ descends by construction to a bilinear form $I_Q(\xi_1,\xi_2)$, $\xi_1 \in Q_{j_1}$, $\xi_2 \in Q_{j_2}$. So there is the same structure as on the j-currents in a Riemann surface

$$I_Q(\xi_1,\xi_2) = 0 \qquad \text{unless } j_1 + j_2 = 2 \tag{4.4}$$

$$I_Q(\xi_2,\xi_1) = (-1)^{j_1 j_2} I_Q(\xi_1,\xi_2) \qquad I_Q(\partial\xi_1,\xi_2) = (-1)^{j_1} I_Q(\xi_1,\partial\xi_2) \tag{4.5}$$

$$J = * \in \mathrm{End}(Q_1^{\mathbb{R}}) \qquad J^2 = -1 \tag{4.6}$$

$$I_Q(\xi, J\xi') = I_Q(\xi', J\xi) \qquad I_Q(\xi, J\xi) > 0 \quad \xi \neq 0 \qquad \deg(\xi) = \deg(\xi') = 1 \tag{4.7}$$

5. j-currents in an abelian group of $(n-1)$-currents

The natural equivalence $\Delta^j \times \Delta^k \simeq \Delta^{j+k}$ will give rise to a natural map

$$\Pi_{j,k} \colon \mathcal{D}^{\mathrm{sing}}_j(\mathcal{D}^{\mathrm{sing}}_k M) \to \mathcal{D}^{\mathrm{sing}}_{j+k} M \tag{5.1}$$

once given a construction of currents in the space $\mathcal{D}^{\mathrm{sing}}_k M$. In particular, the maps

$$\Pi_{j,n-1} \colon \mathcal{D}^{\mathrm{sing}}_j(\mathcal{D}^{\mathrm{sing}}_{n-1} M) \to \mathcal{D}^{\mathrm{sing}}_{n-1+j} M \tag{5.2}$$

will descend to maps

$$\Pi_j \colon \mathcal{D}^{\mathrm{sing}}_j Q \to Q_j \tag{5.3}$$

which can be used to pull back the structure on the Q_j to give a structure on the j-currents in Q analogous to the structure on the j-currents in a Riemann surface. The project then becomes to develop calculus on Q from this structure on its j-currents in analogy with complex analysis on a Riemann surface.

At least one construction of currents in $\mathcal{D}^{\mathrm{sing}}_k M$ is available. A metric is put on $\mathcal{D}^{\mathrm{sing}}_k M$, in particular the flat metric of [3]. Then the general construction of currents in a metric space [4] gives currents in $\mathcal{D}^{\mathrm{sing}}_k M$. This construction is adopted tentatively.

The metric completion of $\mathcal{D}^{\mathrm{sing}}_k M$ in the flat metric is the space of integral k-currents

$$(\mathcal{D}^{\mathrm{sing}}_k M)' = \mathcal{D}^{\mathrm{int}}_k M \qquad \mathcal{D}^{\mathrm{sing}}_k M \subset \mathcal{D}^{\mathrm{int}}_k M \subset \mathcal{D}^{\mathrm{distr}}_k M \tag{5.4}$$

Currents in a metric space are defined in [4] not as linear functionals on differential forms but rather as multilinear functionals of Lipschitz functions. There is no presumption of a smooth structure on the metric space. The j-forms will then be defined as functions of j-currents, reversing the usual construction.

Part of the initial impetus to consider this mathematical material came from comments on spaces of cycles in section 5 of [5] which referred to [6] where the maps $\Pi_{j,k}$ originate.

6. Currents in an abelian group

The metric spaces of interest here are abelian groups with metric completion,

$$A = \mathcal{D}^{\mathrm{sing}}_k M \quad A' = \mathcal{D}^{\mathrm{int}}_k M \quad \text{or} \quad A = \mathcal{D}^{\mathrm{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0} \quad A' = \mathcal{D}^{\mathrm{int}}_{n-1} M_{\mathbb{Z}\partial\xi_0} \tag{6.1}$$

The general construction of currents in a metric space [4] gives $\mathcal{D}^{\mathrm{distr}}_j A$ and $\mathcal{D}^{\mathrm{sing}}_j A$, the vector space of distributional j-currents in A and the abelian subgroup of singular j-currents in A. The latter is generated by the j-simplices $[\sigma]$ representing maps $\sigma \colon \Delta^j \to A'$.

The following discussion of the currents in an abelian group A is entirely unrigorous. Topologies, regularity conditions, and domains of definition are left unspecified. Appendix B remarks on needing a weak topology in addition to the metric topology.

6.1. **Tangent j-vectors as infinitesimal j-simplices**

The singular j-currents are generated by the j-simplices. Each simplex can be subdivided into a sum of arbitrarily small simplices, so the singular j-currents are generated by the infinitesimal j-simplices,

$$\sigma\colon \Delta^j \to A' \qquad [\sigma] = \int_{\Delta^j} \delta_{\sigma(t)} D\sigma(t)\, d^j t \tag{6.2}$$

$$\delta_{\sigma(t)} D\sigma(t) = \lim_{\epsilon\downarrow 0} \epsilon^{-j} [\sigma_{\epsilon,t}] \qquad \sigma_{\epsilon,t}(t') = \sigma(t+\epsilon t') \qquad \Delta^j = [0,1]^j \tag{6.3}$$

$\delta_{\sigma(t)}$ being the 0-current representing the point $\sigma(t) \in A'$. Appendix B comments on the nature of the limit, on the need to use a weak topology.

The vector space of tangent j-vectors to A at ξ is the set of infinitesimal j-simplices

$$T_j(A,\xi) = \{D\sigma(0)\colon \sigma(0) = \xi\} \tag{6.4}$$

The $T_j(A,\xi)$ are the same for all ξ by translation in the abelian group A.

$$T_j(A,\xi) = T_j(A,0) \tag{6.5}$$

$T_j(A,0)$ is a vector space, not merely a cone, because A is an abelian group.

$$D(\sigma_1 + \sigma_2) = D\sigma_1 + D\sigma_2 \qquad \sigma_1(0) = \sigma_2(0) = 0 \tag{6.6}$$

Define

$$\mathcal{D}_j A = \mathcal{D}_0^{\mathrm{distr}} A \otimes T_j(A,0) \tag{6.7}$$

The derivatives $D\sigma(t)$ in the integral (6.2) all lie in $T_j(A,0)$ so

$$\mathcal{D}_j^{\mathrm{sing}} A \subset \mathcal{D}_j A \tag{6.8}$$

$\mathcal{D}_j A$ is the useful vector space of j-currents because it is constructed from infinitesimal j-simplices. In the examples, the map $\Pi_{j,k}$ will act on the $\mathcal{D}_j A$ via its action on the j-simplices.

6.2. **j-forms on A**

The j-forms on A are defined as the homomorphisms of abelian groups

$$\Omega_j A = \mathrm{Hom}(\mathcal{D}_j^{\mathrm{sing}} A, \mathbb{R}) \tag{6.9}$$

continuous in the same topology as used to define tangent j-vectors in equation (6.3). A homomorphism is determined by its action on the infinitesimal j-simplices, so the j-forms can be regarded as functions on A with values in the dual vector space $T_j(A,0)^*$. The j-forms are the sections of the bundle of cotangent j-vectors over A

$$T_j^* A = A \times T_j(A,0)^* \qquad \Omega_j A = \Gamma(T_j^* A, A) \tag{6.10}$$

6.3. **Translations in A**

Let T^ξ be translation by ξ in the abelian group A. The translations act on $\mathcal{D}_j^{\mathrm{distr}} A$ and on $\mathcal{D}_j^{\mathrm{sing}} A$ by the push-forward T_*^ξ. The translations act on $\mathcal{D}_j A$ via translation of 0-currents

$$\mathcal{D}_0^{\mathrm{distr}} A \otimes T_j(A,0) \xrightarrow{T_*^\xi \otimes 1} \mathcal{D}_0^{\mathrm{distr}} A \otimes T_j(A,0) \tag{6.11}$$

Construct a morphism of chain complexes of vector spaces and of abelian subgroups

$$
\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial} & \mathcal{D}_j A & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{D}_2 A & \xrightarrow{\partial} & \mathcal{D}_1 A & \xrightarrow{\partial} & \mathcal{D}_0 A & \xrightarrow{\partial} & 0 \\
 & & \big\downarrow{\scriptstyle \tilde\Pi_j} & & & & \big\downarrow{\scriptstyle \tilde\Pi_2} & & \big\downarrow{\scriptstyle \tilde\Pi_1} & & \big\downarrow{\scriptstyle \tilde\Pi_0} & & \\
\cdots & \xrightarrow{\partial} & \tilde A_j^{\mathbb{R}} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \tilde A_2^{\mathbb{R}} & \xrightarrow{\partial} & \tilde A_1^{\mathbb{R}} & \xrightarrow{\partial} & A^{\mathbb{R}} & \xrightarrow{\partial} & 0 \\
 \\
\cdots & \xrightarrow{\partial} & \mathcal{D}_j^{\text{sing}} A & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{D}_2^{\text{sing}} A & \xrightarrow{\partial} & \mathcal{D}_1^{\text{sing}} A & \xrightarrow{\partial} & \mathcal{D}_0^{\text{sing}} A & \xrightarrow{\partial} & 0 \\
 & & \big\downarrow{\scriptstyle \tilde\Pi_j} & & & & \big\downarrow{\scriptstyle \tilde\Pi_2} & & \big\downarrow{\scriptstyle \tilde\Pi_1} & & \big\downarrow{\scriptstyle \tilde\Pi_0} & & \\
\cdots & \xrightarrow{\partial} & \tilde A_j & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \tilde A_2 & \xrightarrow{\partial} & \tilde A_1 & \xrightarrow{\partial} & A & \xrightarrow{\partial} & 0
\end{array}
\tag{6.12}
$$

When A has an augmentation $A \xrightarrow{\partial} \mathbb{Z}$, these chain complexes are augmented by

$$
\mathcal{D}_0 A \xrightarrow{\partial} \mathbb{R} \qquad \mathcal{D}_0^{\text{sing}} A \xrightarrow{\partial} \mathbb{Z} \qquad \delta_\xi \overset{\partial}{\longmapsto} \partial\xi \tag{6.13}
$$

For $j \geq 1$ define

$$
\begin{gathered}
\tilde\Pi_j = 1^* \otimes 1 \colon \mathcal{D}_0^{\text{distr}} A \otimes T_j(A,0) \to \tilde A_j^{\mathbb{R}} \qquad 1^* \colon \mathcal{D}_0^{\text{distr}} A \to \mathbb{R} \qquad 1^*\xi = \int_\xi 1 \\
\tilde A_j^{\mathbb{R}} = \text{a completion of } T_j(A,0) \qquad \tilde A_j = \tilde\Pi_j(\mathcal{D}_j^{\text{sing}} A)
\end{gathered}
\tag{6.14}
$$

The completion $\tilde A_j^{\mathbb{R}}$ is to contain the integrals (6.14) of elements of $T_j(A,0)$. The map $\tilde\Pi_j$ identifies $\tilde A_j^{\mathbb{R}}$ with $\mathcal{D}_j A$ modulo translations, and $\tilde A_j$ with $\mathcal{D}_j^{\text{sing}} A$ modulo translations.

For $j = 0$, the problem is to construct a natural vector space $A^{\mathbb{R}}$ containing A with a linear operator $\tilde\Pi_0$ extending the map $\delta_\xi \mapsto \xi$ from $\mathcal{D}_0^{\text{sing}} A$ to A. $\tilde\Pi_0$ should act on the boundary of an infinitesimal 1-simplex by

$$
\lim_{\epsilon\to 0} \epsilon^{-1}(\delta_{\sigma(\epsilon)} - \delta_0) \mapsto \lim_{\epsilon\to 0} \epsilon^{-1}\sigma(\epsilon) \tag{6.15}
$$

so $A^{\mathbb{R}}$ should contain the vector space A_{inf} of infinitesimal elements of A. A weak topology on A is needed for the limits. The infinitesimal elements generate the connected component of the identity $A_{\text{cc}} \subset A$. Therefore $A^{\mathbb{R}} = A_{\text{inf}}$ when A is connected, as in the examples $A = \mathcal{D}_k^{\text{sing}} M$. More generally, assume $A/A_{\text{cc}} \cong \mathbb{Z}^r$. Then $A^{\mathbb{R}}$ is an extension of A_{inf} by $\mathbb{R}^r$.

The examples $A = Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0}$ come with an augmentation $A \xrightarrow{\partial} \mathbb{Z}$. The connectedness condition $A_{\text{cc}} = \operatorname{Ker}\partial$ implies $A/A_{\text{cc}} = \mathbb{Z}$ so $A^{\mathbb{R}}$ is an extension of A_{inf} by $\mathbb{R}$. In the examples, the connectedness condition is equivalent to $H_{n-1}(M) = 0$.

7. j-currents in an abelian group of $(n-1)$-currents (2)

7.1. Maps $\Pi_{j,k} \colon \mathcal{D}_j^{\text{sing}}(\mathcal{D}_k^{\text{sing}} M) \to \mathcal{D}_{j+k}^{\text{int}} M$

Consider the abelian group $A = \mathcal{D}_k^{\text{sing}} M$ with metric completion $\mathcal{D}_k^{\text{int}} M$. The equivalence $\Delta^j \times \Delta^k \simeq \Delta^{j+k}$ gives

$$
\Pi_{j,k} \colon \mathcal{D}_j^{\text{sing}}(\mathcal{D}_k^{\text{sing}} M) \to \mathcal{D}_{j+k}^{\text{int}} M \tag{7.1}
$$

In particular,

$$
\Pi_{0,k} = \tilde\Pi_0 \qquad \Pi_{0,k}\delta_\xi = \xi \tag{7.2}
$$

The $\Pi_{j,k}$ are invariant under translations in $\mathcal{D}_k^{\text{sing}} M$

$$
\Pi_{j,k} \mathrm{T}_*^\xi = \mathrm{T}^\xi \Pi_{j,k} \tag{7.3}
$$

where the translations T^ξ act trivially on the $\mathcal{D}^{\mathrm{int}}_{j+k}M$, $j \geq 1$. From

$$\partial(\Delta^j \times \Delta^k) = \partial\Delta^j \times \Delta^k + (-1)^j \Delta^j \times \partial\Delta^k \tag{7.4}$$

it follows that

$$\partial\Pi_{j,k} = \Pi_{j-1,k}\partial + (-1)^j \Pi_{j,k-1}\partial_* \tag{7.5}$$

where ∂_* is the push-forward of the boundary map $\partial\colon \mathcal{D}^{\mathrm{int}}_k M \to \mathcal{D}^{\mathrm{int}}_{k-1}M$

$$\partial_*\colon \mathcal{D}^{\mathrm{sing}}_j(\mathcal{D}^{\mathrm{sing}}_k M) \to \mathcal{D}^{\mathrm{sing}}_j(\mathcal{D}^{\mathrm{sing}}_{k-1}M) \tag{7.6}$$

7.2. **Derivatives** $d\Pi_{j,k}\colon T_j(\mathcal{D}^{\mathrm{sing}}_k M, 0) \to \mathcal{D}^{\mathrm{distr}}_{j+k}M$

$\Pi_{j,k}$ acts on infinitesimal j-simplices as a linear function

$$d\Pi_{j,k}\colon T_j(\mathcal{D}^{\mathrm{sing}}_k M, 0) \to \mathcal{D}^{\mathrm{distr}}_{j+k}M \tag{7.7}$$

Define $\mathcal{V}_{j,k}$ to be the image

$$\mathcal{V}_{j,k} = d\Pi_{j,k}(T_j(\mathcal{D}^{\mathrm{sing}}_k M, 0)) \subset \mathcal{D}^{\mathrm{distr}}_{j+k}M \tag{7.8}$$

$\mathcal{V}_{j,k}$ is, roughly, the subspace of $(j+k)$-currents supported on integral k-currents. The boundary operator ∂ is injective on $\mathcal{V}_{j,k}$

$$(\mathrm{Ker}\,\partial) \cap \mathcal{V}_{j,k} = \{0\} \tag{7.9}$$

There is also a linear function

$$T_j(\mathcal{D}^{\mathrm{sing}}_k M, 0) \xrightarrow{d\partial^*} T_j(\mathcal{D}^{\mathrm{sing}}_{k-1}M, 0) \xrightarrow{d\Pi_{j,k-1}} \mathcal{D}^{\mathrm{distr}}_{j+k-1}M \tag{7.10}$$

The two linear functions $d\Pi_{j,k}$ and $d\Pi_{j,k-1} \circ d\partial^*$ identify

$$T_j(\mathcal{D}^{\mathrm{sing}}_k M, 0) = \partial(\mathcal{V}_{j,k}) \oplus \mathcal{V}_{j,k-1} \tag{7.11}$$

7.3. **Maps** $\Pi_j\colon \mathcal{D}^{\mathrm{sing}}_j Q \to Q'_j$

Now take $A = Q = \mathcal{D}^{\mathrm{sing}}_{n-1}M_{\mathbb{Z}\partial\xi_0}$ with metric completion $Q' = \mathcal{D}^{\mathrm{int}}_{n-1}M_{\mathbb{Z}\partial\xi_0}$. Define

$$\begin{array}{ccccccccccccc}
\mathcal{D}^{\mathrm{sing}}_4 Q & \xrightarrow{\partial} & \mathcal{D}^{\mathrm{sing}}_3 Q & \xrightarrow{\partial} & \mathcal{D}^{\mathrm{sing}}_2 Q & \xrightarrow{\partial} & \mathcal{D}^{\mathrm{sing}}_1 Q & \xrightarrow{\partial} & \mathcal{D}^{\mathrm{sing}}_0 Q & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Pi_3} & & \downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_1} & & \downarrow{\scriptstyle \Pi_0} & & \downarrow{\scriptstyle 1} & & \\
0 & \xrightarrow{\quad\partial\quad} & \mathbb{Z} & \xrightarrow{\quad\partial\quad} & Q'_2 & \xrightarrow{\quad\partial\quad} & Q'_1 & \xrightarrow{\quad\partial\quad} & Q' & \xrightarrow{\quad\partial\quad} & \mathbb{Z} & \xrightarrow{\partial} & 0
\end{array} \tag{7.12}$$

$$\Pi_0 = \tilde{\Pi}_0 \qquad \Pi_j = \mathcal{D}^{\mathrm{sing}}_j Q \xrightarrow{\Pi_{j,n-1}} \mathcal{D}^{\mathrm{int}}_{j+n-1}M \xrightarrow{p_j} Q'_j \quad j \geq 1$$

where the Q_j are defined in (4.3) and p_j is projection on the quotient. The Π_j comprise a morphism of chain complexes, because the operator ∂_* in (7.5) vanishes on $\mathcal{D}^{\mathrm{int}}_j Q$ for $j \geq 1$. The Π_j extend to linear maps $\Pi_j\colon \mathcal{D}_j Q \to Q^{\mathbb{R}}_j$ forming a morphism of the linear chain complexes. The Π_j are translation invariant in the sense that they factor through the $\tilde{\Pi}_j$ of (6.12). The bilinear form on the Q_j now lifts to the $\mathcal{D}^{\mathrm{sing}}_j Q$

$$I = \Pi^* I_Q \qquad I(\xi_1, \xi_2) = I_Q(\Pi_{j_1}\xi_1, \Pi_{j_2}\xi_2) \tag{7.13}$$

The derivative $d\Pi_1$ identifies the tangent space $T_1(Q, 0)$ with $\mathcal{V}_{1,n-1}$

$$T_1(Q, 0) = \mathcal{V}_{1,n-1} \subset \mathcal{D}^{\mathrm{distr}}_n M \tag{7.14}$$

A crucial point is that $J = *$ should act on $T_1(Q, 0)$

$$J\mathcal{V}_{1,n-1} = \mathcal{V}_{1,n-1} \tag{7.15}$$

a germ of a proof of which is given in Appendix 1 of [2]. Assuming (7.15), J lifts to

$$J\colon \mathcal{D}_1 Q \to \mathcal{D}_1 Q \qquad J\Pi_1 = \Pi_1 J \tag{7.16}$$

Now the structure on the currents $\mathcal{D}_j^{\text{sing}} Q$ is analogous to the $\mathcal{D}_j^{\text{sing}}\Sigma$ in a Riemann surface

$$I(\xi_1, \xi_2) = 0 \qquad \text{unless } j_1 + j_2 = 2 \tag{7.17}$$

$$I(\xi_2, \xi_1) = (-1)^{j_1 j_2} I(\xi_1, \xi_2) \qquad I(\partial\xi_1, \xi_2) = (-1)^{j_1} I(\xi_1, \partial\xi_2) \tag{7.18}$$

$$J \in \operatorname{End}(\mathcal{D}_1 Q) \qquad J^2 = -1 \tag{7.19}$$

$$I(\xi, J\xi') = I(\xi', J\xi) \qquad I(\xi, J\xi) \geq 0 \qquad \deg(\xi) = \deg(\xi') = 1 \tag{7.20}$$

8. First definition

A first definition of *quasi Riemann surface* recapitulates the examples $\mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0}$.

(1) an abelian group Q with augmentation $Q \xrightarrow{\partial} \mathbb{Z}$ and metric completion Q'.

(2) a bilinear form $I(\xi_1, \xi_2)$ on the $\mathcal{D}_j^{\text{sing}} Q$ (defined almost everywhere).

  (a) $I(\xi_1, \xi_2) \in \mathbb{Z}$ (where defined)

  (b) $I(\xi_1, \xi_2) = 0$ unless $j_1 + j_2 = 2$

  (c) $I(\xi_2, \xi_1) = (-1)^{j_1 j_2} I(\xi_1, \xi_2)$

  (d) $I(\partial\xi_1, \xi_2) = (-1)^{j_1} I(\xi_1, \partial\xi_2)$

  (e) $I(T_*^{\xi}\xi_1, \xi_2) = I(\xi_1, \xi_2)$, with a modified action of translations on 0-currents

$$T_*^{\xi}\delta_{\xi'} = \delta_{\xi+\xi'} - \delta_{\xi} \tag{8.1}$$

  so that $I(\xi_1, \xi_2)$ on 0-currents factors through $\tilde{\Pi}_0\colon \delta_{\xi'} \mapsto \xi'$.

(3) A linear function $J\colon \mathcal{D}_1 Q \to \mathcal{D}_1 Q$

  (a) $J^2 = -1$

  (b) $JT_*^{\xi} = T_*^{\xi} J$

  (c) $I(\xi, J\xi') = I(\xi', J\xi)$

  (d) $I(\xi, J\xi) \geq 0$

This structure is sufficient to write a Cauchy-Riemann equation on Q analogous to (3.4) on a Riemann surface

$$G\colon \mathcal{D}_1 Q \otimes \mathcal{D}_1 Q \to \mathbb{C} \tag{8.2}$$

$$G(\xi, \xi') = G(\xi', \xi) \qquad G(J\xi, \xi') = G(\xi, J\xi') = iG(\xi, \xi') \tag{8.3}$$

$$G(\xi, \partial\xi_2) = 2\pi i I(\partial\xi, \xi_2) \tag{8.4}$$

In addition, translation invariance of the fundamental solution can be required

$$G(T_*^{\xi}\xi', \xi'') = G(\xi', \xi'') \tag{8.5}$$

$Q_j^{\mathbb{R}}$ is the quotient of $\mathcal{D}_j Q$ by the null space of $I(\xi_1, \xi_2)$, Q_j the quotient of $\mathcal{D}_j^{\text{sing}} Q$, with bilinear form $I_Q(\xi_1, \xi_2) = I(\xi_1, \xi_2)$.

9. Second definition

A second definition is

(1) an abelian group Q with augmentation $Q \xrightarrow{\partial} \mathbb{Z}$ and metric completion Q'.

(2) a complex structure J and hermitian inner product $\langle\, \xi_1, \xi_2 \,\rangle$ on $T_1(Q,0)$ satisfying

$$(\operatorname{Im}\partial) \cap J(\operatorname{Im}\partial) = \{0\} \qquad (\operatorname{Ker}\partial) \cup J(\operatorname{Ker}\partial) = T_1(Q,0) \tag{9.1}$$

(3) a 2-form $\omega_{\xi_0} \in T_2(Q,0)^*$ associated to $\xi_0 \in Q$, $\partial\xi_0 = 1$, satisfying

$$\omega_{\xi_0\,/\operatorname{Ker}\partial} \neq 0 \qquad \omega_{\xi_0+\partial\xi_1}(\xi_2) = \omega_{\xi_0}(\xi_2) + \operatorname{Im}\langle\, \xi_1, \partial\xi_2 \,\rangle \tag{9.2}$$

(4) integrality conditions

(a) $\operatorname{Im}\langle\, \xi, \xi' \,\rangle \in \mathbb{Z}$ for almost all $\xi, \xi' \in \tilde{Q}_1 = \tilde{\Pi}_1(\mathcal{D}_1^{\text{sing}} Q)$

(b) $\omega_{\xi_0}(\Pi_2(\mathcal{D}_2^{\text{sing}} Q)) = \mathbb{Z}$

To recover the first definition, set

$$\begin{aligned} I(\xi_1,\xi_2) &= \operatorname{Im}\langle\, \tilde{\Pi}_1\xi_1, \tilde{\Pi}_1\xi_2 \,\rangle && \xi_1,\xi_2 \in \mathcal{D}_1 Q \\ I(\xi_1,\xi_2) &= \omega_{\tilde{\Pi}_0\xi_1}(\tilde{\Pi}_2\xi_2) && \xi_2 \in \mathcal{D}_2 Q,\ \xi_1 \in \mathcal{D}_0 Q,\ \partial\xi_1 = 1 \end{aligned} \tag{9.3}$$

This definition makes two simplifying assumptions:

(1) the nondegeneracy condition $\partial Q = \mathbb{Z}$, i.e., that there exists ξ_0 with $\partial\xi_0 = 1$
(2) the connectedness condition $\tilde{H}_0(Q) = 0$, i.e. $\partial\xi = 0$ implies $\xi = \partial\xi_1$ for $\xi \in \mathcal{D}_0^{\text{sing}} Q$.

Connectedness ensures that $I(\xi_1,\xi_2)$ is completely determined by (9.3). In the examples, connectedness is the condition $\tilde{H}_{n-1}(M) = 0$. For $n = 1$, the Riemann surface is connected. For $n > 1$, the $2n$-manifold satisfies $H_{n-1}(M) = 0$.

9.1. **Basic consequences**

The Hilbert space $Q_1^{\mathbb{R}} = T_1(Q,0)$ decomposes into orthogonal subspaces

$$\begin{gathered} Q_1^{\mathbb{R}} = Q_{1,\text{exact}}^{\mathbb{R}} \oplus J(Q_{1,\text{exact}}^{\mathbb{R}}) \oplus Q_{1,\text{harm}}^{\mathbb{R}} \\ Q_{1,\text{exact}}^{\mathbb{R}} = \operatorname{Im}\partial \qquad Q_{1,\text{harm}}^{\mathbb{R}} = (\operatorname{Ker}\partial) \cap J(\operatorname{Ker}\partial) \end{gathered} \tag{9.4}$$

The 2-form ω_{ξ_0} restricted to $\operatorname{Ker}\partial$ does not depend on the choice of ξ_0, and descends to the one-dimensional space $\operatorname{Ker}\partial \subset Q_2^{\mathbb{R}}$ so there is an element $[Q] \in Q_2$ uniquely defined by

$$[Q] \in Q_2 \qquad \partial[Q] = 0 \qquad \omega_{\xi_0}([Q]) = 1 \tag{9.5}$$

which therefore has the property

$$I_Q(\xi, [Q]) = \partial\xi \qquad \xi \in Q \tag{9.6}$$

Choosing $\xi_0 \in Q$, $\partial\xi_0 = 1$ gives decompositions

$$Q_0^{\mathbb{R}} = \mathbb{R}\xi_0 \oplus (\operatorname{Im}\partial) \qquad Q_2^{\mathbb{R}} = \mathbb{R}[Q] \oplus \xi_0^{\perp} \qquad \operatorname{Im}\partial \cong J(Q_{1,\text{exact}}^{\mathbb{R}}) \qquad \xi_0^{\perp} \cong Q_{1,\text{exact}}^{\mathbb{R}} \tag{9.7}$$

In the examples for $n = 1$, $[Q]$ is the 2-current representing the Riemann surface Σ. For $n > 1$, $[Q]$ is a certain equivalence class of $(n+1)$-boundaries in M

$$Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0} \qquad [Q] = \Big\{ \partial\xi \in \partial\mathcal{D}_{n+2}^{\text{sing}} M \colon I_M(\xi_0, \partial\xi) = 1 \Big\} \tag{9.8}$$

10. Morphisms

A morphism $F\colon Q_{(1)} \to Q_{(2)}$ between quasi Riemann surfaces consists of linear functions $F_j\colon Q^{\mathbb{R}}_{(1)j} \to Q^{\mathbb{R}}_{(2)j}$ which preserve the linear structures ∂, J, and $I_Q(\xi_1, \xi_2)$, and which satisfy some additional integrality and regularity conditions. Assume the nondegeneracy condition

$$F_{-1} = 1\colon \mathbb{R} \to \mathbb{R} \tag{10.1}$$

10.1. **Linear morphisms**

By the decomposition (9.4), F_1 consists of two partial unitary operators

$$\begin{aligned} &F_{1,\mathrm{exact}}\colon Q^{\mathbb{R}}_{(1)1,\mathrm{exact}} \to Q^{\mathbb{R}}_{(2)1,\mathrm{exact}} \qquad F^{\dagger}_{1,\mathrm{exact}} F_{1,\mathrm{exact}} = 1 \\ &F_{1,\mathrm{harm}}\colon Q^{\mathbb{R}}_{(1)1,\mathrm{harm}} \to Q^{\mathbb{R}}_{(2)1,\mathrm{harm}} \qquad F^{\dagger}_{1,\mathrm{harm}} F_{1,\mathrm{harm}} = 1 \end{aligned} \tag{10.2}$$

F_0 on $\operatorname{Im}\partial \subset Q^{\mathbb{R}}_{(1)0}$ is determined by $F_{1,\mathrm{exact}}$. Choose $\xi_{(1)} \in Q_{(1)}$, $\partial\xi_{(1)} = 1$. Then $\xi_{(2)} = F_0\xi_{(1)}$ and $F_0\partial\xi = \partial F_1\xi$ determine F_0 completely. F_2 must satisfy

$$F_2(\xi^{\perp}_{(1)}) \subset \xi^{\perp}_{(2)} \qquad F_2[Q_{(1)}] - [Q_{(2)}] \in \xi^{\perp}_{(2)} \qquad I_{Q_{(2)}}(F_1\xi_1, \partial F_2\xi_2) = I_{Q_{(1)}}(\xi_1, \partial\xi_2) \tag{10.3}$$

the last of which is equivalent to

$$\partial(F_2\xi_2) - F_1(\partial\xi_2) \in (F_1 Q_{(1)1})^{\perp} \tag{10.4}$$

so F_2 is determined up to an arbitrary linear operator

$$F_{2,0}\colon Q_{(1)2} \to (F_1 Q_{(1)1})^{\perp} \cap (\partial Q_{(2)2}) \tag{10.5}$$

Thus a linear morphism is given by

(1) the partial unitary operators $F_{1,\mathrm{exact}}$ and $F_{1,\mathrm{harm}}$
(2) $\xi_{(2)} \in Q^{\mathbb{R}}_{(2)}$, $\partial\xi_{(2)} = 1$
(3) the linear operator $F_{2,0}$

A linear isomorphism is a morphism with $F_{1,\mathrm{exact}}$ and $F_{1,\mathrm{harm}}$ each unitary, implying $F_{2,0} = 0$. The linear automorphism group is the semi-direct product

$$\operatorname{Aut}(Q) = Q_{0,0} \rtimes (UQ^{\mathbb{R}}_{1,\mathrm{harm}} \times UQ^{\mathbb{R}}_{1,\mathrm{exact}}) \qquad Q_{0,0} = \operatorname{Ker}\partial = \operatorname{Im}\partial \subset Q \tag{10.6}$$

where $Q_{0,0}$ acts on Q by translation, i.e., $\xi_{(2)} = \xi_{(1)} + \partial\xi'_{(1)}$, $\partial\xi'_{(1)} \in Q_{0,0}$.

10.2. **Integrality and regularity conditions**

A morphism might be defined as a linear morphism that preserves the integral structure $Q_j \subset Q^{\mathbb{R}}_j$ and also the metric structure on Q. However, the Cauchy Riemann equation sees only ∂, J, and $I(\xi_1, \xi_2)$. These are preserved by every linear morphism. So it might be reasonable to impose only one additional condition: that the linear morphism preserve the period lattice $L \subset Q^{\mathbb{R}}_{1,\mathrm{harm}}$, i.e., $F_{1,\mathrm{harm}}(L_{(1)}) \subset L_{(2)}$. With this definition of morphism, a quasi Riemann surface is classified up to isomorphism by the genus $g = \dim_{\mathbb{C}}(Q^{\mathbb{R}}_{1,\mathrm{harm}})$ and the rank $2g$ lattice L in the complex Hilbert space $Q^{\mathbb{R}}_{1,\mathrm{harm}} \cong \mathbb{C}^g$.

10.3. **Quasi holomorphic curves**

A quasi holomorphic curve is a morphism $C\colon Q(\Sigma) \to Q$ where Σ is a Riemann surface and $Q(\Sigma) = \mathcal{D}_0^{\mathrm{sing}}\Sigma$ is the quasi Riemann surface associated to Σ. A solution of the Cauchy Riemann equation on Q pulls back along C to an ordinary solution on the Riemann surface Σ. A local quasi holomorphic curve is one where Σ is the open complex disk.

Complex analysis on Q should be equivalent to ordinary analysis in one complex variable on each of a suitable collection of (local) quasi holomorphic curves, subject to compatibility conditions when quasi holomorphic curves overlap. The physics application would be to construct a quantum field theory on Q as an ordinary two-dimensional conformal field theory on each of those quasi holomorphic curves.

Acknowledgments

This work was supported by the New High Energy Theory Center (NHETC) of Rutgers, The State University of New Jersey.

Appendix A. General case (n even or odd)

Now consider the general case of M a conformal $2m$-manifold with n even or odd. For n even, the conformal Hodge $*$-operator on n-forms satisfies $*^2 = (-1)^n = 1$, and the intersection form $I_M(\xi,\xi')$ on n-currents is symmetric. So the construction of a quasi Riemann surface for general n requires the complex currents

$$\mathcal{D}^{\mathrm{distr}}_{k,\mathbb{C}}M = (\mathcal{D}^{\mathrm{distr}}_{k}M)\otimes\mathbb{C} \tag{A.1}$$

Write

$$\deg(\xi) = n-1+\deg'(\xi) \qquad k = n-1+j \qquad \xi\in\mathcal{D}^{\mathrm{distr}}_{k,\mathbb{C}}M \tag{A.2}$$

Choose a root ϵ_n of the equation

$$\epsilon_n^2 = (-1)^{n-1} \tag{A.3}$$

then define

$$J = \epsilon_n * \quad \text{on } \mathcal{D}^{\mathrm{distr}}_{n,\mathbb{C}}M \tag{A.4}$$

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = \epsilon_n^{-1}\,(-1)^{(n-1)k_2}\,I_M(\bar\xi_1,\xi_2) \quad \text{on } \overline{\mathcal{D}^{\mathrm{distr}}_{k_1,\mathbb{C}}M}\otimes\mathcal{D}^{\mathrm{distr}}_{k_2,\mathbb{C}}M \tag{A.5}$$

satisfying

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = 0 \;\text{ unless } j_1+j_2 = 2 \tag{A.6}$$

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = (-1)^{j_1j_2}\overline{I_M\langle\,\bar\xi_2,\xi_1\,\rangle} \qquad I(\partial\bar\xi_1,\xi_2) = (-1)^{j_1}I(\bar\xi_1,\partial\xi_2) \tag{A.7}$$

$$J\in\mathrm{End}(\mathcal{D}^{\mathrm{distr}}_{n,\mathbb{C}}M) \qquad J^2=-1 \tag{A.8}$$

$$I_M\langle\,\bar\xi,J\xi'\,\rangle = \overline{I_M\langle\,\bar\xi',J\xi\,\rangle} \qquad I_M\langle\,\bar\xi,J\xi\,\rangle>0 \quad \xi\neq 0 \qquad \deg'(\xi)=\deg'(\xi')=1 \tag{A.9}$$

For each $\partial\xi_0 = \partial\mathcal{D}^{\mathrm{sing}}_{n-1}M$ there is the augmented chain complex of abelian groups/complex vector spaces

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial} & Q_2 & \xrightarrow{\partial} & Q_1 & \xrightarrow{\partial} & Q & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \xrightarrow{\partial} & Q_2^{\mathbb{C}} & \xrightarrow{\partial} & Q_1^{\mathbb{C}} & \xrightarrow{\partial} & Q^{\mathbb{C}} & \xrightarrow{\partial} & \mathbb{C} & \xrightarrow{\partial} & 0
\end{array} \tag{A.10}$$

$$\begin{aligned}
Q_{-1} &= \mathbb{Z}\partial\xi_0 & Q^{\mathbb{C}}_{-1} &= \mathbb{C}\\
Q = Q_0 &= (\mathcal{D}^{\mathrm{sing}}_{n-1}M_{\mathbb{Z}\partial\xi_0})\oplus i(\partial\mathcal{D}^{\mathrm{sing}}_{n}M) & Q^{\mathbb{C}}_0 &= \mathcal{D}^{\mathrm{distr}}_{n-1,\mathbb{C}}M_{\mathbb{C}\partial\xi_0}\\
Q_1 &= (\mathcal{D}^{\mathrm{sing}}_{n}M)\oplus i(\mathcal{D}^{\mathrm{sing}}_{n}M) & Q^{\mathbb{C}}_1 &= \mathcal{D}^{\mathrm{distr}}_{n,\mathbb{C}}M\\
Q_2 &= [(\mathcal{D}^{\mathrm{sing}}_{n+1}M)\oplus i(\mathcal{D}^{\mathrm{sing}}_{n+1}M)]/Q_0^{\perp} & Q^{\mathbb{C}}_2 &= (\mathcal{D}^{\mathrm{distr}}_{n+1,\mathbb{C}}M)/(Q^{\mathbb{C}}_0)^{\perp}
\end{aligned} \tag{A.11}$$

Q is defined so that $\partial(Q) \subset \mathbb{Z}$, so that $\tilde{H}_0(Q) = 0$ in the connected case, and so that $T_0(Q, 0)$ is the complex vector space $\mathcal{V}_{1,n-1} \otimes \mathbb{C}$ on which $J = \epsilon_n *$ acts.

A *complex* quasi Riemann surface is a abelian group Q with augmentation and metric completion and also with an involution $\xi \mapsto \bar{\xi}$, $\partial\bar{\xi} = \partial\xi$, called complex conjugation. The definition is as before, but with $\mathbb{C}$ in place of $\mathbb{R}$ and with a sesquilinear form $I_Q\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ in place of the bilinear form $I_Q(\xi_1, \xi_2)$.

Appendix B. Topologies for infinitesimal j-simplices

The construction of an infinitesimal j-simplex as a derivative in (6.3) requires taking a limit of j-currents $\lim_{\epsilon\downarrow 0} \epsilon^{-j}[\sigma_{\epsilon,t}]$. Suppose $A = \mathbb{R}^d$ as a simple example. In the ordinary sense of derivative, $D\sigma(t)$ is a j-vector in $\mathbb{R}^d$ at $\sigma(t)$. The j-currents $\epsilon^{-j}[\sigma_{\epsilon,t}]$ converge weakly to $\delta_{\sigma(t)} D\sigma(t)$ but they do not converge in the metric topology. The limit in the metric topology is a 0-current $|v|\delta_{\hat{v}}$ in the unit sphere of j-vectors at $\sigma(t)$, where $v = D\sigma(t)$, $\hat{v} = v/|v|$. The linear map $a\delta_{\hat{v}} \mapsto a\hat{v}$ projects the metric j-tangent space down to the weak j-tangent space. This example suggests that the infinitesimal j-simplices in a general abelian A must be constructed as weak limits of currents. The same weak topology would be used to define j-forms as homomorphisms from $\mathcal{D}_j^{\mathrm{sing}} A$ to $\mathbb{R}$ as in (6.9). In the same vein, a weak topology is needed for the vector space of infinitesimal elements of A so that in the examples $A = \mathcal{D}_k^{\mathrm{sing}} M$ the result will be a subspace of $\mathcal{D}_k^{\mathrm{distr}} M$. The metric topology gives much too large a space of infinitesimals.

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