Quasi Riemann surfaces

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Abstract

A *quasi Riemann surface* is defined to be a certain kind of complete metric space Q whose integral currents are analogous to the integral currents of a Riemann surface. In particular, they have properties sufficient to express Cauchy-Riemann equations on Q. The prototypes are the spaces $\mathcal{D}_0^{\mathrm{int}}(\Sigma)_m$ of integral 0-currents of total mass m in a Riemann surface Σ (usually called the integral 0-cycles of degree m).

For M an oriented conformal $2n$-manifold, there is a bundle $\mathcal{Q}(M) \rightarrow \mathcal{B}(M)$ of quasi Riemann surfaces naturally associated to M. For n odd, this is the bundle $\mathcal{D}_{n-1}^{\mathrm{int}}(M) \xrightarrow{\partial} \partial\mathcal{D}_{n-1}^{\mathrm{int}}(M)$ of integral $(n-1)$-currents in M fibered over the integral $(n-2)$-boundaries in M. For n even, the examples $\mathcal{Q}(M)$ are slightly more complicated.

I suggest that complex analysis on quasi Rieman surfaces be developed by analogy with classical complex analysis on Riemann surfaces, based on the Cauchy-Riemann equations. I want to use complex analysis on quasi Rieman surfaces to construct a new class of quantum field theories in spacetimes M. The new quantum field theories are to be constructed on the quasi Riemann surfaces $\mathcal{Q}(M)$ by analogy with the construction of 2d conformal field theories on Riemann surfaces. The quasi Rieman surfaces $\mathcal{Q}(M)$ might also be useful in the study of the manifolds M.

Contents

1 Introduction

This note is taken from the paper [1] where the proposed definition of quasi Riemann surface is motivated by considerations from quantum field theory. The motivations are summarized in [2]. The goal is to construct quantum field theories on quasi Riemann surfaces by imitating the construction of 2d conformal field theories on Riemann surfaces. The latter was based on elementary principles of analysis in one complex variable, especially the Laurent expansions of meromorphic functions and the Cauchy integral formula. Analogous basic principles of complex analysis are needed for quasi Riemann surfaces.

Here, the notion of quasi Riemann surface is presented on its own, without the quantum field theory motivation. The presentation is entirely formal. In particular, no attempt is made to be particular about topologies or domains of definition. A companion note [3] will collect some questions, comments, and speculations about quasi Riemann surfaces.

The main elements are, roughly:

1. There is an analogy

$$\mathcal{D}^{\mathrm{int}}_{j+n-1}(M) \;\longleftrightarrow\; \mathcal{D}^{\mathrm{int}}_{j}(\Sigma)\,, \qquad j=0,1,2 \tag{1.1}$$

   between the integral $(j{+}n{-}1)$-currents in a $2n$-dimensional conformal manifold M and the integral j-currents in a Riemann surface Σ. In particular, each has a bilinear form, the *intersection form*, which gives the intersection number of two currents with $j_1 + j_2 = 2$, and each has a conformally invariant Hodge $*$-operator in the middle dimension, $j = 1$, acting on the differential forms dual to the currents.

2. The Cauchy-Riemann equations on a Riemann surface Σ can be expressed in terms of the integral currents in Σ, the boundary operator ∂, the conformal Hodge $*$-operator, and the intersection form.

3. There are natural maps

$$\Pi_{j,n-1}\colon \mathcal{D}^{\mathrm{int}}_{j}(\mathcal{D}^{\mathrm{int}}_{n-1}(M)) \to \mathcal{D}^{\mathrm{int}}_{j+n-1}(M) \tag{1.2}$$

   which take integral j-currents in the complete metric space of integral $(n{-}1)$-currents in M to integral $(j{+}n{-}1)$-currents in M.

4. In each fiber

$$Q = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n-1}(M)\colon \partial\xi = \partial\xi_0\right\}, \tag{1.3}$$

   of the bundle

$$\mathcal{D}^{\mathrm{int}}_{n-1}(M) \xrightarrow{\partial} \partial\mathcal{D}^{\mathrm{int}}_{n-1}(M)\,, \tag{1.4}$$

   the intersection form and Hodge $*$-operator of M pull back along the maps $\Pi_{j,n-1}$ to act on the integral currents $\mathcal{D}^{\mathrm{int}}_{j}(Q)$ in Q. The resulting structure on the integral currents in Q is sufficient to express Cauchy-Riemann equations on Q.

5. A *quasi Riemann surface* is a complete metric space Q whose integral currents have this structure.

Sections 2 and 3 sketch mathematical background on integral currents from Geometric Measure Theory and establish notation. Some basic references are [4–6]. Section 3 is essentially a restatement of parts of section 5 of [7], from which came part of the inspiration to use this mathematical material, especially the maps $\Pi_{j,n-1}$ mentioned above.

2 Integral currents in an oriented conformal $2n$-manifold M

2.1 The manifold M

Let M be an oriented conformal manifold of even dimension $2n \geq 2$. For simplicity, take M to be compact and without boundary. The Hodge $*$-operator acting on n-forms depends only on the conformal structure,

$$*\colon \Omega_n(M) \to \Omega_n(M)\,, \qquad *^2 = (-1)^n \tag{2.1}$$

$$*\omega_{\mu_1\ldots\mu_n}(x) = \frac{1}{n!}\,\epsilon_{\mu_1\ldots\mu_n}{}^{\nu_1\ldots\nu_n}(x)\,\omega_{\nu_1\ldots\nu_n}(x) \tag{2.2}$$

The Hodge $*$-operator on n-forms is all that will be used of the conformal structure on M.

2.2 Currents and boundaries

A k-current ξ in M is a distribution on the smooth k-forms ω,

$$\xi\colon \omega \mapsto \int_\xi \omega = \int_M d^dx\, \frac{1}{k!}\xi^{\mu_1\cdots\mu_k}(x)\,\omega_{\mu_1\cdots\mu_k}(x)\,, \qquad \deg(\xi) = k\,. \tag{2.3}$$

$\mathcal{D}_k^{\mathrm{distr}}(M)$ is the real vector space of k-currents in M. The boundary operator on currents is dual to the exterior derivative on differential forms

$$\partial\colon \mathcal{D}_k^{\mathrm{distr}}(M) \to \mathcal{D}_{k-1}^{\mathrm{distr}}(M)\,, \qquad \int_{\partial\xi}\omega = \int_\xi d\omega\,, \qquad \partial^2 = 0\,, \tag{2.4}$$

$$(\partial\xi)^{\mu_2\cdots\mu_k}(x) = -\partial_{\mu_1}\xi^{\mu_1\cdots\mu_k}(x)\,. \tag{2.5}$$

The Hodge $*$-operator acts on distributional n-currents by

$$\int_{*\xi}\omega = \int_\xi *\omega\,. \tag{2.6}$$

2.3 Singular currents

A k-simplex in M is represented by a k-current

$$\sigma\colon \Delta^k \to M\,, \qquad \int_{[\sigma]}\omega = \int_{\Delta^k}\sigma^*\omega\,. \tag{2.7}$$

The space $\mathcal{D}_k^{\mathrm{sing}}(M)$ of *singular* k-currents in M is the abelian group of currents generated by the k-simplices in M, i.e., the currents representing the singular k-chains in M,

$$\sigma = \sum_i m_i\sigma_i\,, \quad m_i \in \mathbb{Z}\,, \qquad [\sigma] = \sum_i m_i[\sigma_i]\,, \qquad \int_{[\sigma]}\omega = \sum_i m_i \int_{\Delta^k}\sigma_i^*\omega\,. \tag{2.8}$$

The boundary operator on singular currents is compatible with the boundary operator on singular chains,

$$\partial\mathcal{D}_k^{\text{sing}}(M) \subset \mathcal{D}_{k-1}^{\text{sing}}(M)\,, \qquad \partial[\sigma] = [\partial\sigma]\,. \tag{2.9}$$

2.4 Integral currents

$\mathcal{D}_k^{\text{int}}(M)$ is the metric abelian group of *integral k-currents* in M. It is the metric completion of $\mathcal{D}_k^{\text{sing}}(M)$ with respect to the *flat metric* which is induced from the *flat norm* which is defined in terms of the *mass* which is the k-volume of a singular k-current,

$$\text{mass}_k(\xi) = k\text{-volume}(\xi)\,, \qquad \xi \in \mathcal{D}_k^{\text{sing}}(M)\,, \tag{2.10}$$

$$\|\xi\|_{\text{flat}} = \inf\left\{\text{mass}_k(\xi - \partial\xi') + \text{mass}_{k+1}(\xi') \colon\ \xi' \in \mathcal{D}_{k+1}^{\text{sing}}(M)\right\}\,, \tag{2.11}$$

$$\text{dist}(\xi_1,\xi_2)_{\text{flat}} = \|\xi_1 - \xi_2\|_{\text{flat}}\,. \tag{2.12}$$

The flat metric measures the ease of deforming one current into another. The flat metric depends on a choice of riemannian metric on M in order to define the k-volume, but all the resulting metric completions $\mathcal{D}_k^{\text{int}}(M)$ are equivalent.

The boundary operator takes integral currents to integral currents,

$$\partial\mathcal{D}_k^{\text{int}}(M) \subset \mathcal{D}_{k-1}^{\text{int}}(M)\,. \tag{2.13}$$

2.5 The fiber bundle of integral currents

Regard

$$\mathcal{D}_k^{\text{int}}(M) \xrightarrow{\partial} \partial\mathcal{D}_k^{\text{int}}(M) \tag{2.14}$$

as a fiber bundle, with fibers

$$\mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}_k^{\text{int}}(M)\colon \partial\xi = \partial\xi_0\right\}\,. \tag{2.15}$$

$\mathcal{D}_k^{\text{int}}(M)_0$ is the metric abelian group of k-cycles. $\mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0}$ is the space of integral k-cycles relative to $\partial\xi_0$, which is an affine space for $\mathcal{D}_k^{\text{int}}(M)_0$,

$$\mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0} = \xi_0 + \mathcal{D}_k^{\text{int}}(M)_0\,, \quad \forall\xi_0 \in \mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0} \tag{2.16}$$

The fiber bundle is an exact sequence of metric abelian groups

$$0 \longrightarrow \mathcal{D}_k^{\text{int}}(M)_0 \xrightarrow{\partial} \mathcal{D}_k^{\text{int}}(M) \xrightarrow{\partial} \partial\mathcal{D}_k^{\text{int}}(M) \longrightarrow 0\,. \tag{2.17}$$

2.6 The intersection form

There is a bilinear *intersection form* $I_M(\xi_1,\xi_2)$ defined almost everywhere on currents in M, which vanishes unless the degrees of the two currents, $\deg(\xi_1) = k_1$, $\deg(\xi_2) = k_2$, add up to the dimension of M,

$$I_M(\xi_1,\xi_2) = \int_M d^dx\, \epsilon_{\mu_1\cdots\mu_{k_1}\nu_1\cdots\nu_{k_2}}(x) \frac{1}{k_1!}\xi_1^{\mu_1\cdots\mu_{k_1}}(x)\,\frac{1}{k_2!}\xi_2^{\nu_1\cdots\nu_{k_2}}(x)\,, \qquad k_1+k_2 = 2n \tag{2.18}$$

$$I_M(\xi_1,\xi_2) = 0\,, \quad k_1 + k_2 \neq 2n\,. \tag{2.19}$$

The definition of the intersection form depends only on the orientation of M. On integral currents the intersection form gives the integer intersection number

$$I_M(\xi_1,\xi_2) \in \mathbb{Z} \text{ (where defined)}\,, \qquad \xi_1 \in \mathcal{D}^{\mathrm{int}}_{k_1}(M)\,, \quad \xi_2 \in \mathcal{D}^{\mathrm{int}}_{k_2}(M)\,. \tag{2.20}$$

On the integral cycles, $\xi_1 \in \mathcal{D}^{\mathrm{int}}_{k_1}(M)_0$, $\xi_2 \in \mathcal{D}^{\mathrm{int}}_{k_2}(M)_0$, the intersection form is everywhere defined and depends only on the homology classes of ξ_1 and ξ_2.

2.7 The chain complex of integral currents

For $n \geq 2$, we will use only the portion

$$\mathcal{D}^{\mathrm{int}}_{n+2}(M) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{n+1}(M) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{n}(M) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{n-1}(M) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{n-2}(M) \tag{2.21}$$

of the chain complex of integral currents.

For $n = 1$, M is a two-dimensional conformal manifold, i.e., a Riemann surface, which we write Σ instead of M. We will use the integral chain complex augmented at both ends,

$$0 \longrightarrow \mathcal{D}^{\mathrm{int}}_{3}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{2}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{1}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{0}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{-1}(\Sigma) \longrightarrow 0 \tag{2.22}$$

where

$$\mathcal{D}^{\mathrm{int}}_{3}(\Sigma) = \mathbb{Z}\,, \qquad \mathcal{D}^{\mathrm{int}}_{-1}(\Sigma) = \mathbb{Z}\,, \tag{2.23}$$

and the augmentation is

$$1 \in \mathcal{D}^{\mathrm{int}}_{3}(\Sigma) \;\stackrel{\partial}{\longmapsto}\; \Sigma \in \mathcal{D}^{\mathrm{int}}_{2}(\Sigma)\,, \qquad \eta \in \mathcal{D}^{\mathrm{int}}_{0}(\Sigma) \;\stackrel{\partial}{\longmapsto}\; \int_\eta 1 \in \mathcal{D}^{\mathrm{int}}_{-1}(\Sigma)\,. \tag{2.24}$$

In particular,

$$\partial \delta_z = 1 \tag{2.25}$$

where δ_z is the 0-current representing the point $z \in \Sigma$, the Dirac delta-function at z.

This terminology in the case $n = 1$ is perhaps nonstandard. The chain complex is usually not augmented, so $\mathcal{D}^{\mathrm{int}}_{0}(\Sigma)$ is usually called the space of integral 0-cycles. The augmentation $\partial\colon \mathcal{D}^{\mathrm{int}}_{0}(\Sigma) \to \mathbb{Z}$ is the degree of the 0-cycle. The kernel of ∂, the space $\mathcal{D}^{\mathrm{int}}_{0}(\Sigma)_0$, is the space of 0-cycles of degree 0. Here, to maintain a uniform terminology over all n, $\mathcal{D}^{\mathrm{int}}_{0}(\Sigma)$ is called the space of integral 0-currents and $\operatorname{Ker}\partial = \mathcal{D}^{\mathrm{int}}_{0}(\Sigma)_0$ is called the space of integral 0-cycles.

3 Integral currents in spaces of integral currents

The space $\mathcal{D}^{\mathrm{int}}_{k}(M)$ is a complete metric space. Currents and integral currents can be constructed in any complete metric space [6]. So there is a space $\mathcal{D}^{\mathrm{int}}_{j}(\mathcal{D}^{\mathrm{int}}_{k}(M))$ of integral j-currents in the space of integral k-currents in M. These j-currents in $\mathcal{D}^{\mathrm{int}}_{k}(M)$ are not constructed as distributions on j-forms, but rather as linear functionals on $(j{+}1)$-tuplets of Lipschitz functions. We take the integral j-currents on a complete metric space as given. Later we will define the j-forms as linear functions on the integral j-currents.

The product $\Delta^j \times \Delta^k$ of a j-simplex with a k-simplex is a singular $(j{+}k)$-chain, so there is a natural map

$$\Pi\colon \mathcal{D}^{\mathrm{int}}_{j}(\mathcal{D}^{\mathrm{int}}_{k}(M)) \to \mathcal{D}^{\mathrm{int}}_{j+k}(M) \tag{3.1}$$

which we also write $\Pi_{j,k}$. We will be using especially the

$$\Pi_{j,n-1}\colon \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_{n-1}(M)) \to \mathcal{D}^{\text{int}}_{j+n-1}(M) \tag{3.2}$$

Let T^ξ be translation by ξ in the abelian group $\mathcal{D}^{\text{int}}_k(M)$,

$$T^\xi\colon \mathcal{D}^{\text{int}}_k(M) \to \mathcal{D}^{\text{int}}_k(M)\,, \qquad T^\xi\colon \xi' \mapsto \xi+\xi'\,. \tag{3.3}$$

T^ξ acts on currents in $\mathcal{D}^{\text{int}}_k(M)$ by pushing forward,

$$T^\xi_*\colon \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_k(M)) \to \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_k(M))\,. \tag{3.4}$$

The maps $\Pi_{j,k}$ are translation-invariant,

$$\Pi_{0,k}T^\xi_* = T^\xi\Pi_{0,k} \quad \text{and} \quad \Pi_{j,k}T^\xi_* = \Pi_{j,k}\,, \quad j\ge 1\,. \tag{3.5}$$

The first follows from

$$\Pi_{0,k}\delta_\xi = \xi\,, \qquad \xi\in\mathcal{D}^{\text{int}}_k(M) \tag{3.6}$$

where δ_ξ is the 0-current representing the point ξ. For $j\ge 1$, translation invariance follows from the fact that a map from $\Delta^j\times\Delta^k$ to M which is constant on Δ^k is represented by 0 as a $(j+k)$-current in M, if $j\ge 1$.

From

$$\partial(\Delta^j\times\Delta^k) = \partial\Delta^j\times\Delta^k + \Delta^j\times\partial\Delta^k \tag{3.7}$$

it follows that

$$\partial\Pi = \Pi\partial + \Pi\partial_* \tag{3.8}$$

where

$$\partial_*\colon \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_k(M)) \to \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_{k-1}(M)) \tag{3.9}$$

is the push-forward of the boundary map $\partial\colon \mathcal{D}^{\text{int}}_k(M)\to\mathcal{D}^{\text{int}}_{k-1}(M)$. The map ∂_* will also be written $\partial_{*j,k}$. Equation (3.8) is then written

$$\partial\Pi_{j,k} = \Pi_{j-1,k}\partial + \Pi_{j,k-1}\partial_{*j,k}\,. \tag{3.10}$$

4 Modify conformal Hodge-$*$ and the intersection form to be independent of n

Some trivial modifications have to be made to the conformal Hodge $*$-operator and to the intersection form $I_M(\xi_1,\xi_2)$ so that their properties on $(j+n-1)$-currents will be independent of n, so that when pulled back along $\Pi_{j,n-1}$ to $\mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_{n-1}(M))$ their properties on j-currents will be independent of n.

To accomplish this for n both even and odd requires going to the complex currents $\mathcal{D}^{\text{distr}}_k(M,\mathbb{C}) = \mathcal{D}^{\text{distr}}_k(M)\otimes\mathbb{C}$. When n is odd, the modifications will be invariant under complex conjugation so the currents can all stay real.

Choose a root ϵ_n of the equation

$$\epsilon_n^2 = (-1)^{n-1}\,, \qquad \epsilon_1 = 1 \tag{4.1}$$

and define

$$J = \epsilon_n * \tag{4.2}$$

$$I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = \epsilon_n^{-1}\,(-1)^{\frac{1}{2}(k_2-n)(k_2+n+1)} I_M(\bar{\xi}_1,\xi_2)\,,\qquad k_2=\deg(\xi_2)\,, \tag{4.3}$$

$$\deg'(\xi) = \deg(\xi) - n + 1\,. \tag{4.4}$$

These satisfy (where defined)

$$I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = 0 \qquad\qquad \text{unless } \deg'(\xi_1)+\deg'(\xi_2)=2 \tag{4.5}$$

$$I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = -\overline{I_M\langle\,\bar{\xi}_2,\xi_1\,\rangle} \tag{4.6}$$

$$I_M\langle\,\overline{\partial\xi_1},\xi_2\,\rangle = -I_M\langle\,\bar{\xi}_1,\partial\xi_2\,\rangle \tag{4.7}$$

and, on the subspace where $\deg'(\xi)=1$, which is $\mathcal{D}_n^{\mathrm{distr}}(M,\mathbb{C})$,

$$J^2 = -1 \qquad\qquad \text{on } \mathcal{D}_n^{\mathrm{distr}}(M,\mathbb{C}) \tag{4.8}$$

$$I_M\langle\,\overline{J\xi_1},J\xi_2\,\rangle = I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle \qquad \xi_1,\xi_2\in\mathcal{D}_n^{\mathrm{distr}}(M,\mathbb{C}) \tag{4.9}$$

$$I_M\langle\,\bar{\xi},J\xi\,\rangle > 0 \qquad\qquad \xi\neq 0\in\mathcal{D}_n^{\mathrm{distr}}(M,\mathbb{C})\,. \tag{4.10}$$

Because of (4.6), we call $I_M\langle\,\bar{\xi}_2,\xi_1\,\rangle$ the *skew-hermitian intersection form*. When n is odd, the number ϵ_n is real, so we can restrict to the real currents. Then we call $I_M\langle\,\xi_1,\xi_2\,\rangle$ the *skew intersection form*.

When M is a Riemann surface Σ, the skew intersection form has a unique extension to the bi-augmented chain complex (2.22). $I_\Sigma\langle\,\xi_1,\xi_2\,\rangle$ is given on $\mathcal{D}_3^{\mathrm{int}}(\Sigma)\times\mathcal{D}_{-1}^{\mathrm{int}}(\Sigma)$ by

$$I_\Sigma\langle\,1,1\,\rangle = I_\Sigma\langle\,1,\partial\delta_z\,\rangle = -I_\Sigma\langle\,\partial 1,\delta_z\,\rangle = -I_\Sigma\langle\,\Sigma,\delta_z\,\rangle = I_\Sigma\langle\,\delta_z,\Sigma\,\rangle = I_\Sigma(\delta_z,\Sigma) = 1\,. \tag{4.11}$$

5 Forms and tangents on a metric abelian group

The integral currents in a complete metric space Q are constructed in [6] without reference to forms. But in order to do tensor analysis on Q we will need forms on Q. Given that the integral currents in Q are constructed in [6], we will try to use the integral currents as the basis for all geometry on Q. The most economical definition of the real vector space of j-forms on Q would seem to be

$$\Omega_j(Q) = \mathrm{Hom}(\mathcal{D}_j^{\mathrm{int}}(Q),\mathbb{R}) \tag{5.1}$$

where these are the homomorphisms of metric abelian groups. The exterior derivative d on forms is the dual of the boundary operator ∂. The real vector space of j-currents on Q is defined as the vector space dual of the j-forms,

$$\mathcal{D}_j(Q) = \Omega_j(Q)^*\,. \tag{5.2}$$

This version of the j-currents $\mathcal{D}_j(Q)$, derived from the integral currents, might not give the same vector space of j-currents as originally constructed in [6], but for present purposes it seems more appropriate to take the integral currents as fundamental, and then derive the real currents from the integral currents.

We are interested in complete metric spaces Q which are metric abelian groups or affine spaces of metric abelian groups. We are interested in a category of metric abelian groups that have the property that the connected component of 0 is generated by an arbitrarily small ϵ-ball around 0. The metric abelian group of integral k-currents in a manifold M belongs to this category. The metric abelian group of integral j-currents in a metric space in this category should also be a metric space in this category.

When Q is an affine space for a metric abelian group $\mathcal{G}$, there is a quick way to construct the tangent space $T_{\xi_0}Q$. We can speak of *the* tangent space, because the tangent space at any point in Q is identical to the tangent space at any other point, by translation in $\mathcal{G}$. And the tangent space of Q is identical to the tangent space $T_0\mathcal{G}$ of $\mathcal{G}$,

$$T_{\xi_0}Q = T_\xi \mathcal{G} = T_0\mathcal{G}\,, \qquad \xi_0 \in Q\,, \quad \xi \in \mathcal{G}\,. \tag{5.3}$$

Let T^ξ be translation by ξ,

$$T^\xi \colon Q \to Q\,. \tag{5.4}$$

Then T^ξ acts on the integral j-currents in Q by pushing forward and on the j-forms by pulling back,

$$T^\xi_* \colon \mathcal{D}^{\text{int}}_j(Q) \to \mathcal{D}^{\text{int}}_j(Q)\,, \qquad T^{\xi*} \colon \Omega_j(Q) \to \Omega_j(Q) \tag{5.5}$$

The translation invariant j-forms

$$\Omega_j(Q)_{inv} = \left\{\omega \in \Omega_j(Q)\colon T^{\xi*}\omega = \omega \ \ \forall \xi \in \mathcal{G}\right\} \tag{5.6}$$

can be identified with the j-forms at $\xi_0 \in Q$, so the cotangent space is

$$T^*_{\xi_0}Q = \Omega_1(Q)_{inv} \tag{5.7}$$

and the tangent space at ξ_0 is the dual space,

$$T_{\xi_0}Q = (\Omega_1(Q)_{inv})^*\,. \tag{5.8}$$

These tangent vectors at ξ_0 can be pictured more concretely as the infinitesimal 1-simplices starting from $\xi_0 \in Q$

$$\sigma_\epsilon \colon [0,\epsilon] \to Q\,, \qquad \sigma_\epsilon(0) = \xi_0 \tag{5.9}$$

acting on the 1-forms by

$$\omega \mapsto \frac{\omega([\sigma_\epsilon])}{\epsilon}\,, \qquad \omega \in \Omega_1(Q)\,. \tag{5.10}$$

This is an equivalent construction of the tangent space as long as the metric abelian group $\mathcal{D}^{\text{int}}_1(Q)$ is generated by an arbitrarily small ϵ-ball around 0, so the 1-forms are determined by their actions on the infinitesimally small integral 1-currents in Q.

More generally, we suppose that all the metric abelian groups $\mathcal{D}^{\text{int}}_j(Q)$ have this property, so the j-forms are determined by their actions on the infinitesimally small integral j-currents in Q.

Another approach might be to identify the 0-forms with an algebra of (Lipschitz) functions on Q, show that the j-forms are modules for this algebra, then construct the sections of the tangent bundle as the module homomorphisms $\Omega_1(Q) \to \Omega_0(Q)$.

6 Cauchy-Riemann equations in terms of Hodge-$*$ and the intersection form

The Cauchy-Riemann equations on a Riemann surface Σ (more precisely, on a local neighborhood Σ in a Riemann surface) can be written

$$F_\Sigma(\bar{\xi}_0, \partial\xi_2) = 2\pi i I_\Sigma\langle\, \bar{\xi}_0, \xi_2\,\rangle\,, \qquad \xi_0 \in \mathcal{D}_0^{\mathrm{int}}(\Sigma)\,, \quad \xi_2 \in \mathcal{D}_2^{\mathrm{int}}(\Sigma)\,. \tag{6.1}$$

where the fundamental solution $F_\Sigma(\bar{\xi}_0, \xi_1)$ is a homomorphism (almost everywhere defined)

$$F_\Sigma \colon\ \overline{\mathcal{D}_0^{\mathrm{int}}(\Sigma)} \times \mathcal{D}_1^{\mathrm{int}}(\Sigma) \to \mathbb{C} \tag{6.2}$$

whose extension to $\overline{\mathcal{D}_0(\Sigma)} \times \mathcal{D}_1(\Sigma)$ satisfies

$$F_\Sigma(\bar{\xi}_0, (J-i)\xi_1) = 0\,. \tag{6.3}$$

The point is that the Cauchy-Riemann equations are expressed entirely in terms of currents, the J operator, and the skew-hermitian intersection form,

To see that equation (6.1) expresses the Cauchy-Riemann equations, let $\xi_1 = \delta_w$, the 0-current representing a point $w \in \Sigma$. With w fixed, F_Σ is a 1-form on Σ,

$$F_\Sigma = F(x)_\mu dx^\mu\,, \qquad F_\Sigma(\bar{\delta}_w, \xi_1) = \int_\Sigma d^2x\; G(x)_\mu\, \xi_1^\mu(x)\,. \tag{6.4}$$

J acts on 1-forms by

$$Jdz = idz\,, \qquad Jd\bar{z} = -id\bar{z} \tag{6.5}$$

where $z = x^1 + ix^2$ is a complex coordinate on Σ. So condition (6.3) says that F is a $(1,0)$-form

$$F_\Sigma = F(z,\bar{z})dz \tag{6.6}$$

Then, by (2.18) and (4.3), equation (6.1) is the Cauchy-Riemann equation

$$\frac{\partial}{\partial\bar{z}} F = \pi\delta^2(z-w)\,, \qquad F(z,\bar{z}) = \frac{1}{z-w} + \cdots\,. \tag{6.7}$$

One might say that equation (6.1) expresses the Cauchy-Riemann equations by the residue formula.

7 The real examples $\mathcal{Q}(M)$ for n odd

A quasi Riemann surface is a complete metric space Q with some additional structure modeled on the space $\mathcal{D}_0^{\mathrm{int}}(\Sigma)$ of integral 0-currents in a Riemann surface Σ. This structure is sufficient to express Cauchy-Riemann equations on Q.

What might make quasi Riemann surfaces interesting is a class of examples constructed from the integral $(n-1)$-currents in a compact oriented conformal manifold M of dimension $2n$. These examples were motivated in [1] by consideration of quantum field theory on the quasi Riemann surfaces. The quantum field theory motivations are summarized in [2].

The general quasi Riemann surface is *complex.* The real quasi Riemann surfaces are the complex quasi Riemann surfaces which are invariant under an involutive automorphism, "complex conjugation".

This section describes real quasi Riemann surfaces naturally associated to $2n$-manifolds M with n odd. The following section describes complex quasi Riemann surfaces associated to $2n$-manifolds for all n, even and odd.

7.1 The fiber bundle $\mathcal{Q}(M) \to \mathcal{B}(M)$ of quasi Riemann surfaces

Assume now that n is odd. All currents will be real

The "bundle" of quasi Riemann surfaces associated to the manifold M will be the exact sequence of metric abelian groups

$$0 \longrightarrow \mathcal{D}^{\mathrm{int}}_{n-1}(M)_0 \xrightarrow{\partial} \mathcal{D}^{\mathrm{int}}_{n-1}(M) \xrightarrow{\partial} \partial\mathcal{D}^{\mathrm{int}}_{n-1}(M) \longrightarrow 0 \tag{7.1}$$

renamed

$$0 \longrightarrow \mathcal{G}(M) \xrightarrow{\partial} \mathcal{Q}(M) \xrightarrow{\partial} \mathcal{B}(M) \longrightarrow 0\,. \tag{7.2}$$

We are regarding

$$\mathcal{Q}(M) \to \mathcal{B}(M) \tag{7.3}$$

as a fiber bundle. The metric abelian group $\mathcal{G}(M) = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_0$, i.e., the group of integral $(n{-}1)$-cycles, is the "gauge group" of the bundle. The fibers

$$\mathcal{Q}(M)_{\partial\xi_0} = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n-1}(M):\ \partial\xi = \partial\xi_0\right\}\,, \qquad \partial\xi_0 \in \partial\mathcal{D}^{\mathrm{int}}_{n-1}(M) \tag{7.4}$$

are the spaces of integral relative $(n{-}1)$-cycles. Each fiber $\mathcal{Q}(M)_{\partial\xi_0}$ is an affine space for the metric abelian group $\mathcal{G}(M)$. Each fiber is to be a quasi Riemann surface.

Now fix $\partial\xi_0$ and consider the fiber $\mathcal{Q}(M)_{\partial\xi_0}$. To save space, write

$$Q = \mathcal{Q}(M)_{\partial\xi_0}\,, \qquad \mathcal{G} = \mathcal{G}(M) = \mathcal{Q}(M)_0\,. \tag{7.5}$$

7.2 The analogous chain complex to $\oplus_j \mathcal{D}^{\mathrm{int}}_j(\Sigma)$, Σ a Riemann surface

The first step is to construct a chain complex of metric abelian groups

$$0 \longrightarrow Q_3 \xrightarrow{\partial} Q_2 \xrightarrow{\partial} Q_1 \xrightarrow{\partial} Q_0 \xrightarrow{\partial} Q_{-1} \longrightarrow 0 \tag{7.6}$$

analogous to the chain complex (2.22) of integral currents $\mathcal{D}^{\mathrm{int}}_j(\Sigma)$ in a Riemann surface Σ. Define metric abelian groups

$$Q_{-1} = \mathbb{Z}\partial\xi_0 \qquad\qquad \subset \mathcal{D}^{\mathrm{int}}_{n-2}(M) \tag{7.7}$$

$$Q_0 = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n-1}(M):\ \partial\xi \in \mathbb{Z}\partial\xi_0\right\} \qquad \subset \mathcal{D}^{\mathrm{int}}_{n-1}(M) \tag{7.8}$$

$$Q_1 = \mathcal{D}^{\mathrm{int}}_{n}(M) \tag{7.9}$$

$$Q_2 = \mathcal{D}^{\mathrm{int}}_{n+1}(M)/Q_0^{\perp} \tag{7.10}$$

$$Q_3 = \mathcal{D}^{\mathrm{int}}_{n+2}(M)/Q_{-1}^{\perp} \tag{7.11}$$

where $Q_0^{\perp}$ and $Q_{-1}^{\perp}$ are the orthogonal complements of Q_0 and Q_{-1} in the skew intersection form of M,

$$Q_0^{\perp} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n+1}(M):\ I_M\langle\, \xi', \xi\,\rangle = 0 \quad \forall \xi' \in Q_0\right\} \tag{7.12}$$

$$Q_{-1}^{\perp} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n+2}(M):\ I_M\langle\, \partial\xi_0, \xi\,\rangle = 0\right\}. \tag{7.13}$$

$Q_{-1} = Q_3 = \mathbb{Z}$ when $\partial\xi_0 \neq 0$. At $\partial\xi_0 = 0$ they degenerate to $Q_{-1} = Q_3 = 0$. The Q_j form a chain complex (7.6) because $Q_{-1} = \partial Q_0$ so $\partial Q_{-1}^{\perp} = Q_0^{\perp}$ by property (4.7) of $I_M\langle\, \xi_1, \xi_2\,\rangle$.

7.3 The skew form $I\langle \xi_1, \xi_2 \rangle$ on $\oplus_j Q_j$

By construction, $I_M\langle \xi_1, \xi_2 \rangle$ defines (almost everywhere) an integer-valued form

$$I\langle \xi_1, \xi_2 \rangle \in \mathrm{Hom}(\bigoplus_j Q_j \times \bigoplus_j Q_j, \mathbb{Z}) \tag{7.14}$$

satisfying (where defined)

$$I\langle \xi_1, \xi_2 \rangle = 0 \qquad \text{unless } \deg'(\xi_1) + \deg'(\xi_2) = 2, \quad \text{where } \deg'(\xi) = j \text{ for } \xi \in Q_j \tag{7.15}$$

$$I\langle \xi_1, \xi_2 \rangle = -I\langle \xi_2, \xi_1 \rangle \tag{7.16}$$

$$I\langle \partial\xi_1, \xi_2 \rangle = -I\langle \xi_1, \partial\xi_2 \rangle \tag{7.17}$$

$$I\langle \xi_1, \xi_2 \rangle \text{ is nondegenerate.} \tag{7.18}$$

7.4 The J operator

Define real vector spaces

$$\Omega_j = \mathrm{Hom}(Q_j, \mathbb{R}) , \qquad \mathcal{D}_j = \Omega_j^* \tag{7.19}$$

as in section 5 above. In particular,

$$\Omega_1 = \mathrm{Hom}(\mathcal{D}_n^{\mathrm{int}}(M), \mathbb{R}) \tag{7.20}$$

so

$$\mathcal{D}_1 \subset \mathcal{D}_n^{\mathrm{distr}}(M) . \tag{7.21}$$

To complete the analogy with currents in a Riemann surface, $J = \epsilon_n *$ should act on $\mathcal{D}_1$. However, it does not seem obvious that Hodge-$*$ takes $\mathcal{D}_1$ to itself, i.e., that Hodge-$*$ acts on the infinitesimal integral n-currents. This is more or less proved — or at least the germ of a proof is given — in Appendix 1 of [1]. The argument makes essential use of fractal integral currents that are limits of Cauchy sequences of singular currents in the flat metric on currents. This is one of the motivations for adopting the integral currents, along with the existence of integral currents in the complete metric spaces of integral currents.

Assuming that Hodge-$*$ does act on $\mathcal{D}_1$, we have the final elements of the analogy:

$$J^2 = -1 \quad \text{on } \mathcal{D}_1 \tag{7.22}$$

$$I\langle J\xi_1, J\xi_2 \rangle = I\langle \xi_1, \xi_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{D}_1 \tag{7.23}$$

$$I\langle \xi, J\xi \rangle > 0, \quad \xi \neq 0 \in \mathcal{D}_1 \tag{7.24}$$

where $I\langle \xi_1, \xi_2 \rangle$ is extended from $\mathcal{Q}_1$ to $\mathcal{D}_1$ by linearity.

7.5 Pull up the structure from $\oplus_j Q_j$ to $\oplus_j \mathcal{D}_j^{\text{int}}(\mathcal{Q})$

These structures, $I\langle \xi_1, \xi_2 \rangle$ and J, are pulled up to the currents in Q via a morphism of chain complexes of metric abelian groups,

$$\begin{tikzcd}
\mathcal{D}_4^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_4"] & \mathcal{D}_3^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_3"] & \mathcal{D}_2^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_2"] & \mathcal{D}_1^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_1"] & \mathcal{D}_0^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_0"] & \mathcal{D}_{-1}^{\text{int}}(Q) \arrow[r] \arrow[d, "\Pi_{-1}"] & 0 \\
0 \arrow[r] & Q_3 \arrow[r, "\partial"] & Q_2 \arrow[r, "\partial"] & Q_1 \arrow[r, "\partial"] & Q_0 \arrow[r, "\partial"] & Q_{-1} \arrow[r] & 0
\end{tikzcd} \tag{7.25}$$

The augmentation of the top complex is the usual

$$\mathcal{D}_{-1}^{\text{int}}(Q) = \mathbb{Z}\,, \qquad \partial\colon \delta_\xi \in \mathcal{D}_0^{\text{int}}(Q) \mapsto 1 \in \mathcal{D}_{-1}^{\text{int}}(Q) \tag{7.26}$$

Write the augmented space of integral currents in Q

$$\mathcal{D}^{\text{int}}(Q) = \bigoplus_{j=-1} \mathcal{D}_j^{\text{int}}(Q)\,. \tag{7.27}$$

The morphism Π

The morphism maps Π_j are

$$\Pi_{-1} = m \mapsto m\partial\xi_0\,, \quad m \in \mathbb{Z} \tag{7.28}$$

$$\Pi_j = \begin{cases} \Pi_{j,n-1}, & j = 0,1 \\ \pi_j \circ \Pi_{j,n-1}, & j = 2,3 \end{cases} \tag{7.29}$$

$$\Pi_j = 0\,, \quad j \geq 4\,. \tag{7.30}$$

where

$$\Pi_{j,n-1}\colon \mathcal{D}_j^{\text{int}}(\mathcal{D}_{n-1}^{\text{int}}(M)) \to \mathcal{D}_{j+n-1}^{\text{int}}(M) \tag{7.31}$$

is the map of equation (3.2), and π_j is the projection on the quotient space

$$\pi_j\colon \mathcal{D}_{j+n-1}^{\text{int}}(M) \to \mathcal{Q}_j = \mathcal{D}_{j+n-1}^{\text{int}}(M)/\mathcal{Q}_{2-j}^{\perp}\,, \quad j = 2,3\,. \tag{7.32}$$

In particular,

$$\Pi_0\colon \delta_\xi \mapsto \xi\,, \quad \xi \in Q\,. \tag{7.33}$$

To see that the Π_j form a morphism of chain complexes, first calculate explicitly

$$\partial\Pi_0 = \Pi_{-1}\partial\,. \tag{7.34}$$

Then note that

$$\partial\Pi_j = \Pi_{j-1}\partial\,, \quad j \geq 1, \tag{7.35}$$

because the operator $\partial_{*j,n-1}$ in equation (3.10) vanishes on $\mathcal{D}_j^{\text{int}}(Q)$ for $j \geq 1$, because ∂ takes Q to the single point $\{\partial\xi_0\}$ and there are no j-currents in a single point if $j \geq 1$.

Translation invariance

The fiber $Q = \mathcal{D}^{\text{int}}_{n-1}(M)_{\partial\xi_0}$ is an affine space for the metric abelian group $\mathcal{G} = \mathcal{D}^{\text{int}}_{n-1}(M)_0$, which acts by translations T^ξ

$$Q = \xi_0 + \mathcal{G}\,, \qquad T^\xi \colon \xi_0 \mapsto \xi_0 + \xi\,, \quad \xi \in \mathcal{G}\,, \quad \xi_0 \in Q\,. \tag{7.36}$$

Let T^ξ act trivially on the Q_j, $j \neq 0$.

The translations act on integral currents in Q by pushing forward

$$T^\xi_* \colon \mathcal{D}^{\text{int}}_j(Q) \to \mathcal{D}^{\text{int}}_j(Q) \tag{7.37}$$

where T^ξ_* acts trivially on $\mathcal{D}^{\text{int}}_{-1}(Q) = \mathbb{Z}$.

The morphism Π is translation-invariant,

$$\Pi_j T^\xi_* = T^\xi \Pi_j\,. \tag{7.38}$$

The isomorphism $\Pi_{1*}\colon T_{\xi_0}Q \to \mathcal{D}_1$

Since Q is an affine space for the group $\mathcal{G}$, the tangent spaces $T_{\xi_0}Q$ are all the same, and all equal to $T_0\mathcal{G}$. The map Π_1 induces isomorphisms (of real vector spaces)

$$\Pi_{1*}\colon T_{\xi_0}Q \to \mathcal{D}_1\,, \qquad \Pi_1^*\colon \Omega_1 \to T^*_{\xi_0}Q \tag{7.39}$$

where, as in section 5,

$$T^*_{\xi_0}Q = \Omega_1(Q)_{inv}\,, \qquad T_{\xi_0}Q = (\Omega_1(Q)_{inv})^*\,. \tag{7.40}$$

The J operator and the skew form pulled up to $\mathcal{Q}$

Transport the J operator from $\mathcal{D}_1$ and Ω_1 to $T_{\xi_0}Q$ and $T^*_{\xi_0}Q$, via the isomorphisms (7.39),

$$J\Pi_{1*} = \Pi_{1*}J\,, \qquad J\Pi_1^* = \Pi_1^* J\,. \tag{7.41}$$

The J operator acting on the cotangent space $T^*_{\xi_0}Q$ then determines a J operator on the space $\Omega_1(Q)$ of 1-forms on Q.

Pull back the skew form $I\langle\,\xi_1,\xi_2\,\rangle$ to $\mathcal{D}^{\text{int}}(Q)$ along the morphism Π,

$$I_Q\langle\,\eta_1,\eta_2\,\rangle = \Pi^* I\langle\,\eta_1,\eta_2\,\rangle = I\langle\,\Pi\eta_1,\Pi\eta_2\,\rangle\,, \qquad \eta_1,\eta_2 \in \mathcal{D}^{\text{int}}(Q)\,. \tag{7.42}$$

$$I_Q\langle\,\xi_1,\xi_2\,\rangle \in \operatorname{Hom}(\mathcal{D}^{\text{int}}(Q)\times\mathcal{D}^{\text{int}}(Q),\mathbb{Z}) \tag{7.43}$$

$$I_Q\langle\,\eta_1,\eta_2\,\rangle = 0 \quad \text{if } \deg(\eta_1)+\deg(\eta_2) \neq 2 \tag{7.44}$$

$$I_Q\langle\,\eta_1,\eta_2\,\rangle = -I_Q\langle\,\eta_2,\eta_1\,\rangle \tag{7.45}$$

$$I_Q\langle\,\partial\eta_1,\eta_2\,\rangle = -I_Q\langle\,\eta_1,\partial\eta_2\,\rangle \tag{7.46}$$

$$I_Q\langle\,\eta_1,T^\xi_*\eta_2\,\rangle = I_Q\langle\,\eta_1,\eta_2\,\rangle\,, \quad \deg(\eta_2) \geq 1\,, \quad \xi \in \mathcal{G} \tag{7.47}$$

$$J^2 = -1 \quad \text{on } T_{\xi_0}Q \tag{7.48}$$

$$I_Q\langle\,J\eta_1,J\eta_2\,\rangle = I_Q\langle\,\eta_1,\eta_2\,\rangle\,, \quad \eta_1,\eta_2 \in T_{\xi_0}Q \tag{7.49}$$

$$I_Q\langle\,\eta,J\eta\,\rangle > 0\,, \quad \eta \neq 0 \in T_{\xi_0}Q \tag{7.50}$$

where $I_Q\langle\,\eta_1,\eta_2\,\rangle$ is extended to $T_{\xi_0}Q$ by linearity.

7.6 Cauchy-Riemann equations on $\mathcal{Q}$

Now we have the ingredients to write Cauchy-Riemann equations on Q analogous to the Cauchy-Riemann equations for a Riemann surface Σ, in the form of equations (6.1-6.3),

$$F_Q(\eta_0, \partial\eta_2) = 2\pi i I_Q\langle\, \eta_0, \eta_2 \,\rangle\,, \qquad \eta_0 \in \mathcal{D}_0^{\mathrm{int}}(Q)\,, \quad \eta_2 \in \mathcal{D}_2^{\mathrm{int}}(Q)\,. \tag{7.51}$$

where the fundamental solution $F_Q(\eta_0, \eta_1)$ is a homomorphism (almost everywhere defined)

$$F_Q\colon \mathcal{D}_0^{\mathrm{int}}(Q) \times \mathcal{D}_1^{\mathrm{int}}(Q) \to \mathbb{C} \tag{7.52}$$

whose extension to $\mathcal{D}_0(Q) \times \mathcal{D}_1(Q)$ satisfies

$$F_Q(\eta_0, (J-i)\eta_1) = 0\,. \tag{7.53}$$

On Q, we can impose the additional condition that the fundamental solution $F_Q(\eta_0, \eta_1)$ is the pullback of a form $F(\xi_0, \xi_1)$ on $Q_0 \times Q_1$,

$$F_Q = \Pi^* F\,, \qquad F_Q(\eta_0, \eta_1) = F(\Pi_*\eta_0, \Pi_*\eta_1) \tag{7.54}$$

$$F \in \mathrm{Hom}(\mathcal{Q}_0 \times \mathcal{Q}_1, \mathbb{C}) \tag{7.55}$$

$$F(\xi_0, \partial\xi_2) = 2\pi i I\langle\, \xi_0, \xi_2 \,\rangle \tag{7.56}$$

$$F(\xi_0, (J-i)\xi_1) = 0\,, \quad \xi_0 \in \mathcal{D}_0,\ \xi_1 \in \mathcal{D}_1\,. \tag{7.57}$$

7.7 $n = 1$

For $n = 1$, for M a Riemann surface Σ, the bundle of quasi Riemann surfaces

$$0 \longrightarrow \mathcal{G}(\Sigma) \xrightarrow{\partial} \mathcal{Q}(\Sigma) \xrightarrow{\partial} \mathcal{B}(\Sigma) \longrightarrow 0 \tag{7.58}$$

is

$$0 \longrightarrow \mathcal{D}_0^{\mathrm{int}}(\Sigma)_0 \xrightarrow{\partial} \mathcal{D}_0^{\mathrm{int}}(\Sigma) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0 \tag{7.59}$$

since $\partial\mathcal{D}_0^{\mathrm{int}}(\Sigma) = \mathcal{D}_{-1}^{\mathrm{int}}(\Sigma) = \mathbb{Z}$. The fiber $\mathcal{Q}(\Sigma)_{\partial\xi_0} = \mathcal{D}_0^{\mathrm{int}}(\Sigma)_m$, $\partial\xi_0 = m$, is the space of integral 0-currents of total mass m (called the space of integral 0-cycles of degree m in the usual terminology). The chain complex (7.6) of the Q_j is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathcal{D}_2^{\mathrm{int}}(\Sigma) \xrightarrow{\partial} \mathcal{D}_1^{\mathrm{int}}(\Sigma) \xrightarrow{\partial} \mathcal{D}_0^{\mathrm{int}}(\Sigma)_{m\mathbb{Z}} \xrightarrow{\partial} m\mathbb{Z} \longrightarrow 0\,. \tag{7.60}$$

The skew form $I\langle\, \xi_1, \xi_2 \,\rangle$ is the skew intersection form $I_\Sigma\langle\, \xi_1, \xi_2 \,\rangle$ (which extends uniquely to the augmented chain complex of Σ).

7.8 Surjectivity of Π_3

For $n > 1$, the morphism maps Π_j are all surjective. This allows the Q_j to be constructed as the quotients of the $\mathcal{D}_j^{\text{int}}(Q)$ by the null spaces of the skew form $I_Q\langle \eta_1, \eta_2 \rangle$ on Q. All the structure of the quasi Riemann surface is encoded in the structure of Q, in the skew form $I_Q\langle \eta_1, \eta_2 \rangle$ and the J operator on $T_{\xi_0}Q$.

However, when $n = 1$, when M is a Riemann surface Σ, the Π_j, $j \neq 3$, are surjective, but Π_3 is *not* surjective. The map Π_3 is the restriction of $\Pi_{3,0}$, which takes integral 3-currents in $\mathcal{D}_0^{\text{int}}(\Sigma)$ to integral 3-currents in Σ. But there are no 3-currents in the 2-manifold Σ. So $\Pi_3 = 0$. The augmentation on the left by $\mathcal{D}_3^{\text{int}}(\Sigma) = \mathbb{Z}$ is artificial. The augmented chain complex makes sense, but there are in fact no 3-currents in a Riemann surface.

The failure of Π_3 to be surjective is aggravating. It blocks the possibility of there being morphisms of quasi Riemann surfaces from the $\mathcal{Q}(M)_{\partial\xi_0}$ for $n > 1$ to the quasi Riemann surfaces associated to a Riemann surface. Such morphisms might be useful for tranporting two dimensional quantum field theories from Riemann surfaces to the $\mathcal{Q}(M)_{\partial\xi_0}$. It would be useful if there were an *actual* augmentation $\Sigma' \supset \Sigma$ of a Riemann surface Σ such that $\mathcal{D}_3^{\text{int}}(\Sigma')$ was actually $\mathbb{Z}$.

8 The complex examples $\mathcal{Q}(M)$ for any n

Now let n be any positive integer. When n is even, the operator $J = \epsilon_n *$ and the skew-hermitian intersection form $I_M\langle \bar{\xi}_1, \xi_2 \rangle$ are imaginary. The construction of the quasi Riemann surface $\mathcal{Q}(M)_{\partial\xi_0}$ has to be modified so that J and $I_M\langle \bar{\xi}_1, \xi_2 \rangle$ will act on the tangent space.

The bundle $\mathcal{Q}(M) \to \mathcal{B}(M)$ of complex quasi Riemann surfaces associated to M is given by the exact sequence

$$0 \longrightarrow \mathcal{G}(M) \xrightarrow{\partial} \mathcal{Q}(M) \xrightarrow{\partial} \mathcal{B}(M) \longrightarrow 0 \tag{8.1}$$

$$\mathcal{B}(M) = \partial\mathcal{D}_{n-1}^{\text{int}}(M), \quad \text{as before,} \tag{8.2}$$

$$\mathcal{Q}(M) = \mathcal{D}_{n-1}^{\text{int}}(M) \oplus i\partial\mathcal{D}_n^{\text{int}}(M) \tag{8.3}$$

$$\mathcal{G}(M) = \mathcal{D}_{n-1}^{\text{int}}(M)_0 \oplus i\partial\mathcal{D}_n^{\text{int}}(M)\,. \tag{8.4}$$

The fibers of $\mathcal{Q}(M) \to \mathcal{B}(M)$ are

$$\mathcal{Q}(M)_{\partial\xi_0} = \mathcal{D}_{n-1}^{\text{int}}(M)_{\partial\xi_0} \oplus i\partial\mathcal{D}_n^{\text{int}}(M)\,, \qquad \partial\xi_0 \in \partial\mathcal{D}_{n-1}^{\text{int}}(M)\,. \tag{8.5}$$

Fix $\partial\xi_0 \in \partial\mathcal{D}_{n-1}^{\text{int}}(M)$ and write the fiber as $Q = \mathcal{Q}(M)_{\partial\xi_0}$ and the gauge group as $\mathcal{G} = \mathcal{G}(M)$. Again, Q is an affine space for the abelian group $\mathcal{G}$.

As in the real case, there is a morphism of chain complexes of metric abelian groups

$$\begin{array}{ccccccccccccc}
\mathcal{D}_4^{\text{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_3^{\text{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_2^{\text{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_1^{\text{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_0^{\text{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_{-1}^{\text{int}}(Q) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Pi_4} & & \downarrow{\scriptstyle \Pi_3} & & \downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_1} & & \downarrow{\scriptstyle \Pi_0} & & \downarrow{\scriptstyle \Pi_{-1}} & & \\
0 & \longrightarrow & Q_3 & \xrightarrow{\partial} & Q_2 & \xrightarrow{\partial} & Q_1 & \xrightarrow{\partial} & Q_0 & \xrightarrow{\partial} & Q_{-1} & \longrightarrow & 0
\end{array} \tag{8.6}$$

with the usual augmentation $\mathcal{D}^{\mathrm{int}}_{-1}(Q) = \mathbb{Z}$, and with

$$Q_{-1} = \mathbb{Z}\partial\xi_0 \tag{8.7}$$

$$Q_0 = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}^{\mathrm{int}}_{n}(M) \tag{8.8}$$

$$Q_1 = \mathcal{D}^{\mathrm{int}}_{n}(M) \oplus i\mathcal{D}^{\mathrm{int}}_{n}(M) \tag{8.9}$$

$$Q_2 = \left[\mathcal{D}^{\mathrm{int}}_{n+1}(M) \oplus i\mathcal{D}^{\mathrm{int}}_{n+1}(M)\right] / {Q_0}^{\perp} \tag{8.10}$$

$$Q_3 = \left[\mathcal{D}^{\mathrm{int}}_{n+2}(M) \oplus i\mathcal{D}^{\mathrm{int}}_{n+2}(M)\right] / {Q_{-1}}^{\perp} \tag{8.11}$$

where now ${Q_0}^{\perp}$ and ${Q_{-1}}^{\perp}$ are the orthogonal complements in the skew-hermitian intersection form $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ of M defined in equation (4.3). Now Q_3 is isomorphic to $\mathbb{Z}\oplus i\mathbb{Z}$. Again, the morphism maps Π_j are given by the $\Pi_{j,n-1}$. Again, Π_0 is the canonical map $\delta_\xi \mapsto \xi$, $\xi \in Q$, and Π_{-1} is $1 \mapsto \partial\xi_0$. Again, the Π_j are invariant under translation by $\mathcal{G}$.

At this point, the only manifestation of the imaginary unit i that is used in the construction of the metric abelian groups in (8.6) is the complex conjugation involution $\xi \mapsto \bar{\xi}$ that acts as an automorphism of (8.6).

By construction, $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ gives a skew-hermitian form $I\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ on $\oplus_j Q_j$,

$$I\langle\, \bar{\xi}_1, \xi_2 \,\rangle \in \mathrm{Hom}(\bigoplus_{j_1} \bar{Q}_{j_1} \times \bigoplus_{j_2} Q_{j_2}, \mathbb{Z}\oplus i\mathbb{Z}) \tag{8.12}$$

satisfying (where defined)

$$I\langle\, \bar{\xi}_1, \xi_2 \,\rangle = 0 \qquad \text{unless } \deg'(\xi_1) + \deg'(\xi_2) = 2, \tag{8.13}$$
$$\text{where } \deg'(\xi) = j \text{ for } \xi \in Q_j$$

$$I\langle\, \bar{\xi}_1, \xi_2 \,\rangle = -\,\overline{I\langle\, \bar{\xi}_2, \xi_1 \,\rangle} \tag{8.14}$$

$$I\langle\, \partial\bar{\xi}_1, \xi_2 \,\rangle = -I\langle\, \bar{\xi}_1, \partial\xi_2 \,\rangle \tag{8.15}$$

$$I\langle\, \bar{\xi}_1, \xi_2 \,\rangle \text{ is nondegenerate.} \tag{8.16}$$

The vector spaces

$$\Omega_1 = \mathrm{Hom}(Q_1, \mathbb{R}) = \mathrm{Hom}(\mathcal{D}^{\mathrm{int}}_{n}(M), \mathbb{R}) \oplus i\,\mathrm{Hom}(\mathcal{D}^{\mathrm{int}}_{n}(M), \mathbb{R})\,, \qquad \mathcal{D}_1 = \Omega_1^* \tag{8.17}$$

are *complex* vector spaces. Given that $*$ acts on $\mathrm{Hom}(\mathcal{D}^{\mathrm{int}}_{n}(M), \mathbb{R})$, the operator $J = \epsilon_n *$ acts on Ω_1 and $\mathcal{D}_1$. The skew-hermitian form $I\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ extends by linearity to a skew-hermitian complex form on $\mathcal{D}_1$. They satisfy on $\mathcal{D}_1$ (where defined)

$$J^2 = -1 \tag{8.18}$$

$$I\langle\, \bar{\xi}_1, \xi_2 \,\rangle = -\,\overline{I\langle\, \bar{\xi}_2, \xi_1 \,\rangle} \tag{8.19}$$

$$I\langle\, \overline{J\xi_1}, J\xi_2 \,\rangle = I\langle\, \bar{\xi}_1, \xi_2 \,\rangle, \tag{8.20}$$

$$I\langle\, \bar{\xi}, J\xi \,\rangle > 0, \quad \xi \neq 0\,. \tag{8.21}$$

The skew-hermitian form $I\langle \bar{\xi}_1, \xi_2 \rangle$ pulls back along the maps Π_j to give a skew-hermitian form on $\mathcal{D}^{\rm int}(Q)$,

$$I_Q\langle \bar{\eta}_1, \eta_2 \rangle \in \mathrm{Hom}(\overline{\mathcal{D}^{\rm int}(Q)} \times \mathcal{D}^{\rm int}(Q), \mathbb{Z} \oplus i\mathbb{Z}) \tag{8.22}$$

which inherits the properties of $I\langle \bar{\xi}_1, \xi_2 \rangle$. Again, $I\langle \bar{\xi}_1, \xi_2 \rangle$ is invariant under the translations acting on the $\mathcal{D}^{\rm int}_j(Q)$, $j \geq 1$.

The map Π_1 induces isomorphisms of vector spaces

$$\Pi_{1*} \colon T_{\xi_0}Q \to \mathcal{D}_1\,, \qquad \Pi_1^* \colon \Omega_1 \to T^*_{\xi_0}Q \tag{8.23}$$

so $T_{\xi_0}Q$ and $T^*_{\xi_0}Q$ are complex vector spaces, and thus also $\Omega_1(Q)$. Again, J is transported from Ω_1 and $\mathcal{D}_1$ to $T_{\xi_0}Q$ and to $T^*_{\xi_0}Q$. and thereby determines a J operator on $\Omega_1(Q)$. On the tangent space $T_{\xi_0}Q$, J and $I_Q\langle \bar{\eta}_1, \eta_2 \rangle$ inherit the properties (8.18-8.21). This might be called an affine Kähler structure on Q.

9 Definition of quasi Riemann surface

The aim now is to define quasi Riemann surface and bundle of quasi Riemann surfaces in a way that encompasses all the examples $\mathcal{Q}(M)$ and $Q(M)$ as narrowly as possible. At a minimum, the structure should be sufficient to express Cauchy-Riemann equations.

9.1 A quasi Riemann surface

A *quasi Riemann surface* is a complete metric space Q with the following properties.

1. Q is an affine space of a metric abelian group $\mathcal{G}$. $\mathcal{G}$ acts on Q by translations

$$Q = \xi_0 + \mathcal{G}\,, \qquad T^\xi \colon \xi_0 \mapsto \xi + \xi_0\,, \quad \xi \in \mathcal{G}\,, \quad \xi_0 \in Q\,. \tag{9.1}$$

2. There is a morphism of chain complexes of metric abelian groups

$$\begin{tikzcd}
\mathcal{D}^{\rm int}_4(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_4"] & \mathcal{D}^{\rm int}_3(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_3"] & \mathcal{D}^{\rm int}_2(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_2"] & \mathcal{D}^{\rm int}_1(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_1"] & \mathcal{D}^{\rm int}_0(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_0"] & \mathcal{D}^{\rm int}_{-1}(Q) \arrow[r] \arrow[d, "\Pi_{-1}"] & 0 \\
0 \arrow[r] & Q_3 \arrow[r, "\partial"] & Q_2 \arrow[r, "\partial"] & Q_1 \arrow[r, "\partial"] & Q_0 \arrow[r, "\partial"] & Q_{-1} \arrow[r] & 0
\end{tikzcd} \tag{9.2}$$

$$\mathcal{D}^{\rm int}_{-1}(Q) = \mathbb{Z}\,, \quad \partial \colon \delta_\xi \mapsto 1\,, \quad \xi \in Q\,. \tag{9.3}$$

3. Q_{-1} is a cyclic group with generator $\partial\xi_0$, which is possibly 0,

$$Q_{-1} = \mathbb{Z}\partial\xi_0\,, \qquad \Pi_{-1} \colon 1 \mapsto \partial\xi_0\,. \tag{9.4}$$

4. Q_0 is the smallest metric abelian group such that

$$\mathcal{G} = \partial^{-1}\{0\} \subset Q_0\,, \quad Q = \partial^{-1}\{\partial\xi_0\} \subset Q_0\,, \text{ and } \xi + \xi_0 = T^\xi\xi_0\,, \ \xi \in \mathcal{G}\,, \ \xi_0 \in Q\,. \tag{9.5}$$

So $Q = Q_0 = \mathcal{G}$ if $\partial\xi_0 = 0$, otherwise Q_0 is isomorphic to $\mathcal{G} \times \mathbb{Z}\partial\xi_0$, the ismormorphism depending on a choice of $\xi_0 \in \mathcal{Q}$.

The morphism map Π_0 is

$$\Pi_0 \colon \delta_\xi \mapsto \xi\,, \quad \xi \in Q\,. \tag{9.6}$$

5. All the Π_j are surjective except possibly Π_3.

6. The morphism Π is translation-invariant under $\mathcal{G}$, where $\mathcal{G}$ acts by T^ξ on the Q_j, trivially for $j \neq 0$, and by T^ξ_* on the $\mathcal{D}^{\mathrm{int}}_j(Q)$, trivially for $j = -1$.

7. Π_1 induces isomorphisms

$$T_{\xi_0}Q = \mathcal{D}_1\,, \quad T^*_{\xi_0}Q = \Omega_1\,, \quad \text{where } \; \Omega_1 = \mathrm{Hom}(Q_1, \mathbb{R})\,, \quad \mathcal{D}_1 = \Omega_1^*\,. \tag{9.7}$$

8. There is an involution $\xi \to \bar{\xi}$, called "complex conjugation", which acts on Q and on the Q_j, acting trivially on Q_{-1} and on $\mathcal{G}$, and which acts on the $\mathcal{D}^{\mathrm{int}}_j(Q)$ via its action on Q. The complex conjugation is an automorphism of the diagram (9.2).

The quasi Riemann surface is *real* when complex conjugation acts trivially, *complex* when complex conjugation is not trivial.

9. In the complex case, $\mathcal{D}_1$ and Ω_1 are complex vector spaces. That is, the two eigenspaces of complex conjugation are isomorphic, so that there is a linear action of i on $\mathcal{D}_1$ satisfying $\overline{i\xi} = -i\bar{\xi}$.

10. There is a skew-hermition form $I\langle\,\bar{\xi}_1, \xi_2\,\rangle$ defined almost everywhere on $\oplus_j Q_j$,

$$I\langle\,\bar{\xi}_1, \xi_2\,\rangle \in \mathrm{Hom}(\bigoplus_{j_1} \bar{Q}_{j_1} \times \bigoplus_{j_2} Q_{j_2}, \mathbb{Z} \oplus i\mathbb{Z}) \tag{9.8}$$

satisfying (where defined)

$$I\langle\,\bar{\xi}_1, \xi_2\,\rangle = 0 \qquad \begin{array}{l}\text{unless } \deg'(\xi_1) + \deg'(\xi_2) = 2, \\ \text{where } \deg'(\xi) = j \text{ for } \xi \in Q_j\end{array} \tag{9.9}$$

$$I\langle\,\bar{\xi}_1, \xi_2\,\rangle = -\,\overline{I\langle\,\bar{\xi}_2, \xi_1\,\rangle} \tag{9.10}$$

$$I\langle\,\partial\bar{\xi}_1, \xi_2\,\rangle = -I\langle\,\bar{\xi}_1, \partial\xi_2\,\rangle \tag{9.11}$$

$$I\langle\,\bar{\xi}_1, \xi_2\,\rangle \text{ is nondegenerate.} \tag{9.12}$$

11. There is a linear operator J acting on $\mathcal{D}_1$ (and on Ω_1) satisfying (where defined)

$$J^2 = -1 \quad \text{on } \mathcal{D}_1 \tag{9.13}$$

$$I\langle\,\overline{J\xi_1}, J\xi_2\,\rangle = I\langle\,\bar{\xi}_1, \xi_2\,\rangle, \quad \xi_1, \xi_2 \in \mathcal{D}_1 \tag{9.14}$$

$$I\langle\,\bar{\xi}, J\xi\,\rangle > 0, \quad \xi \neq 0 \in \mathcal{D}_1 \tag{9.15}$$

where $I\langle\,\bar{\xi}_1,\xi_2\,\rangle$ is extended by linearity to a skew-hermitian complex form on $\mathcal{D}_1$.

The real quasi Riemann surfaces can be identified with the complex quasi Rieman surfaces where J and $I\langle\,\bar{\xi}_1,\xi_2\,\rangle$ are both real.

9.2 A bundle of quasi Riemann surfaces

A *bundle of quasi Riemann surfaces* is a an exact sequence of metric abelian groups

$$0 \to \mathcal{G} \xrightarrow{\partial} \mathcal{Q} \xrightarrow{\partial} \mathcal{B} \to 0 \tag{9.16}$$

regarded as a fiber bundle $\mathcal{Q} \to \mathcal{B}$. the fibers $\mathcal{Q}_{\partial\xi_0}$ are quasi Riemann surfaces, all of which share the structure of the degenerate fiber $\mathcal{Q}_0 = \mathcal{G}$,

$$\begin{tikzcd}
\mathcal{D}_3^{\mathrm{int}}(\mathcal{G}) \arrow[r, "\partial"] \arrow[d, "\Pi_3"] & \mathcal{D}_2^{\mathrm{int}}(\mathcal{G}) \arrow[r, "\partial"] \arrow[d, "\Pi_2"] & \mathcal{D}_1^{\mathrm{int}}(\mathcal{G}) \arrow[r, "\partial"] \arrow[d, "\Pi_1"] & \mathcal{D}_0^{\mathrm{int}}(\mathcal{G}) \arrow[r] \arrow[d, "\Pi_0"] & 0 \\
0 \arrow[r] & \mathcal{G}_2 \arrow[r, "\partial"] & \mathcal{G}_1 \arrow[r, "\partial"] & \mathcal{G} \arrow[r] & 0
\end{tikzcd} \tag{9.17}$$

Every fiber $\mathcal{Q}_{\partial\xi_0}$ has the same $Q_1 = \mathcal{G}_1$, the same morphism map Π_1, the same skew-hermitian form $I\langle\,\bar{\xi}_1,\xi_2\,\rangle$ on Q_1, and the same J operator. The nondegenerate fibers $\mathcal{Q}_{\partial\xi_0}$, $\partial\xi_0 \neq 0$, are then characterized by the extensions

$$0 \to \mathcal{G} \to Q_0 \xrightarrow{\partial} \mathbb{Z}\partial\xi_0 \to 0 \tag{9.18}$$

$$0 \to Q_3 \xrightarrow{\partial} Q_2 \to \mathcal{G}_2 \to 0 \tag{9.19}$$

and by the skew-hermitian form on $\mathbb{Z}\partial\xi_0 \times Q_3$

Finally, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{D}_3^{\mathrm{int}}(\mathcal{B}) \arrow[r, "\partial"] \arrow[d] & \mathcal{D}_2^{\mathrm{int}}(\mathcal{B}) \arrow[r, "\partial"] \arrow[d, "D_2"] & \mathcal{D}_1^{\mathrm{int}}(\mathcal{B}) \arrow[r, "\partial"] \arrow[d, "D_1"] & \mathcal{D}_0^{\mathrm{int}}(\mathcal{B}) \arrow[r] \arrow[d, "D_0"] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r, "\partial"] & \mathcal{Q} \arrow[r, "\partial"] & \mathcal{B} \arrow[r] & 0\,.
\end{tikzcd} \tag{9.20}$$

In the examples $\mathcal{Q}(M)$, where $\mathcal{B} = \partial\mathcal{D}_{n-1}^{\mathrm{int}}(M)$, the map D_j is given by $\Pi_{j,n-2}$. D_1 can be interpreted as a translation-invariant connection in the bundle $\mathcal{Q} \to \mathcal{B}$. For $\sigma\colon [0,1] \to \mathcal{B}$ a 1-simplex in $\mathcal{B}$, $[\sigma]$ is in $\mathcal{D}_1^{\mathrm{int}}(\mathcal{B})$ so $D_1[\sigma]$ is in $\mathcal{Q}$ with $\partial D_1[\sigma] = \sigma(1) - \sigma(0)$. Parallel transport along σ takes $\xi \in \mathcal{Q}_{\sigma(0)}$ to $\xi + D_1[\sigma] \in \mathcal{Q}_{\sigma(1)}$. If σ_2 is a 2-simplex in $\mathcal{B}$, then parallel transport around the boundary $\partial\sigma_2$ takes ξ to $\xi + D_2[\sigma_2]$. So D_2 is the curvature of the connection.

10 Questions

Some questions and speculations about quasi Riemann surfaces are listed here. A companion note [3] contains more questions, comments, and speculations.

1. Define a *quasi holomorphic curve* in a quasi Riemann surface Q to be a function $C\colon \Sigma \to Q$ from a Riemann surface Σ to Q that preserves the J operators and the skew-hermitian forms on integral currents, so a solution of the Cauchy-Riemann equations on Q pulls back to a solution of the Cauchy-Riemann equations on Σ.

   Do quasi holomorphic curves exist in a general Q? in the $\mathcal{Q}(M)_{\partial\xi_0}$? In particular, do *local* quasi holomorphic curves exist, where $\Sigma = \mathbb{D}$ is the unit complex disk? Are there "enough" quasi holomorphic curves to distinguish solutions of the Cauchy Riemann equations on Q? Can explicit quasi holomorphic curves be constructed in the basic case $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$, $\Sigma = \mathbb{D}$ or $\mathbb{CP}^1$?

2. Can quasi holomorphic curves Q be classified? In particular, does there exist a unique two-dimensional space Σ' such that $Q = \mathcal{Q}(\Sigma')_m$ for some $m \in \mathbb{Z}$? Σ' would have to be some kind of generalization of ordinary Riemann surface. It would have integral currents. Its intersection form would be nonzero on currents of degrees adding to 2. And it would have a J operator in the middle dimension. It would have to have 3-currents, $\mathcal{D}_3^{\mathrm{int}}(\Sigma') = \mathbb{Z}$. For this point, Σ' cannot be exactly a Riemann surface. The jacobian of Σ' would have to equal the Jacobian of Q, where the jacobian is the homology in the middle dimension as a lattice in a complex Hilbert space. On this point also, Σ' cannot be exactly a Riemann surface, sincejacobians can presumably occur in the $\mathcal{Q}(M)$ that are not the jacobians of Riemann surfaces. Finally, the topology of Q would need to be weakened to make such an equivalence possible, since the quasi Riemann surfaces $\mathcal{Q}(M)$ with the metric topology induced from M, detects all the homology groups of M.

   If such two-dimensional spaces Σ' exist, are they close enough to Riemann surfaces to support some form of complex analysis in one complex variable?

   If there is such a classification of quasi Riemann surfaces, what are their automorphism groups? These would contain all the conformal symmetry groups of manifolds M.

3. Can an intrinsic conformal tensor calculus be developed on quasi Riemann surfaces analogous to the conformal tensor calculus on Riemann surfaces (perhaps along the lines suggested at the end of section 5 above)? Is there an analog of the local conformal transformations in one complex variable? Such a conformal tensor calculus would have to be consistent with any quasi holomorphic curves, and the speculative equivalence to some $\mathcal{Q}(\Sigma')_m$.

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