DETERMINANT FORMULAE AND UNITARITY FOR THE $N = 2$ SUPERCONFORMAL ALGEBRAS IN TWO DIMENSIONS OR EXACT RESULTS ON STRING COMPACTIFICATION

Wayne BOUCHER [1], Daniel FRIEDAN [2,3] and Adrian KENT [4,3]

Enrico Fermi and James Franck Institutes, and Department of Physics, University of Chicago, Chicago, IL 60637, USA

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Determinant formulae are presented for the periodic, antiperiodic and twisted $N = 2$ superconformal algebras in two dimensions, and a classification is derived of the unitary highest weight representations. Physical realisations of several of these representations are discussed. In particular, it is noted that the unitarity constraints apply to string compactification, giving results which are nonperturbative in the compactification radius.

The infinite-dimensional conformal algebra of the two-dimensional cylinder, the Virasoro algebra, is the gauge algebra of the world surface of the covariantly quantized bosonic string [1]. The Virasoro algebra also acts in the operator representation of two-dimensional critical phenomena [2], where its unitary representation theory gives constraints on the possible values of critical indices [3]. In the investigation of unitarity, the basic technical tool is the determinant formula conjectured by Kac [4] and proved by Feigin and Fuchs [5].

The gauge algebras of the supersymmetric string, the Ramond [6] and Neveu–Schwarz [7] algebras, are the $N = 1$ superconformal algebras on the cylinder. The $N = 1$ algebras are realized in supersymmetric critical phenomena, and unitarity again restricts the possible values of critical indices, permitting the identification of physical systems with supersymmetric critical behavior [3,8]. The Neveu–Schwarz determinant formula [4] and the Ramond formula [8,9] were proved by Meurman and Rocha-Caridi [10] and by Thorn [9].

In this paper we give the determinant formulae and unitarity constraints for the $N = 2$ extensions of the $N = 1$ superconformal algebras. The proofs will be given elsewhere [11]. The $N = 2$ algebras first appeared as gauge algebras of the U(1) fermionic string [12]. Preliminary calculations towards determinant formulae were performed by Di Vecchia, Petersen and Zheng [13], Qiu and Shenker [14], and Thorn [15]. While this paper was being completed, we received preprints by Di Vecchia, Petersen and Yu [16] and Nam [17], which describe partial results overlapping with some of our work.

$N = 2$ superconformal invariance has applications to supersymmetric critical phenomena in two dimensions [18]. It also arises in the classical compactifications of the supersymmetric string. A priori, string compactifications are given by $N = 1$ superconformal nonlinear models [19,20]. But all known nontrivial examples actually have $N = 2$ superconformal symmetry [21,22]. The $N = 2$ supersymmetry of the nonlinear model is associated with spacetime supersymmetry in the string ground state after compactification [20,23]. This spacetime supersymmetry should persist to all orders in the string coupling [24].

To date, all results on the nonlinear model, including $N = 2$ superconformal symmetry, are known at best to all orders in the inverse radius of the compact dimensions. The present work is, inter alia, the first step in a program to study classical compactifications

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nonperturbatively in the compactification scale, assuming world sheet $N = 2$ superconformal symmetry. Surprisingly, the unitarity constraints already have consequences for string compactification. These first results are not particularly dramatic, but they suggest that the algebraic approach to compactification is worth pursuing.

The anti-commutation relations of the $N = 2$ algebras are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{1}{4}\tilde{c}(m^3 - m)\delta_{m,-n}, \\
[L_m, G_n^i] &= (\tfrac{1}{2}m - n)G_{m+n}^i, \\
[L_m, T_n] &= -nT_{m+n}, \\
[T_m, T_n] &= \tilde{c}m\delta_{m,-n}, \\
[T_m, G_n^i] &= \mathrm{i}\epsilon^{ij}G_{m+n}^j, \\
[G_m^i, G_n^j]_+ &= 2\delta^{ij}L_{m+n} + \mathrm{i}\epsilon^{ij}(m-n)T_{m+n} \\
&\quad + \tilde{c}(m^2 - \tfrac{1}{4})\delta^{ij}\delta_{m,-n}.
\end{aligned} \tag{1}$$

The Virasoro generators L_m are the Fourier coefficients of the traceless stress–energy tensor of a conformally invariant quantum field theory on the cylinder. Equivalently (by $z = \mathrm{e}^w$) they are the Laurent coefficients of the traceless stress–energy tensor $T(z) = \Sigma L_m z^{-m-2}$ on the plane. The T_m are the coefficients of a U(1) current algebra $J(z) = \Sigma T_m z^{-m-1}$. The G_m^i are the coefficients of the fermionic partner fields $G^i(z) = \Sigma G_m^i z^{-m-3/2}$ which complete the $N = 2$ super stress–energy multiplet. All of the generators satisfy hermiticity conditions of the form $A_m^\dagger = A_{-m}$, which follow from reality of the super stress–energy tensor. The central charge $\tilde{c}$ is related to the usual Virasoro central charge c and the usual $N = 1$ charge $\hat{c}$ by $\tilde{c} = c/3 = \hat{c}/2$. The normalization is fixed so that $\tilde{c} = 1$ for the free $N = 2$ superfield, consisting of two scalars each with $c = 1$ and two Majorana fermions each with $c = \frac{1}{2}$.

The three $N = 2$ algebras are given by three modings of the generators, corresponding to three ways of choosing boundary conditions on the cylinder. The L_m are always integrally moded, because the bosonic stress–energy tensor is always periodic on the cylinder. The P (for periodic) algebra has integer modes for T_m and G_m^i. The A (for anti-periodic) algebra has integer modes for T_m, but half-integer ($m \in \mathbf{Z} + \frac{1}{2}$) modes for G_m^i. The T (for twisted [‡1]) algebra has integer modes for G_m^1, half-integer modes for T_m and G_m^2.

We consider highest weight representations of the $N = 2$ algebras. These representations are generated from a vector of lowest L_0 eigenvalue h, called the *highest weight vector* (hwv). The hwv is necessarily annihilated by all the lowering operators L_m, G_m^i, T_m ($m > 0$). The remaining generators consist of the raising operators L_{-m}, T_{-m}, G_{-m}^i ($m > 0$) together with the zero modes. The hwv must be an eigenstate of a maximal commuting set of zero modes.

The basic technique for studying a highest weight representation is to construct it as a quotient of the Verma module, which is the largest possible representation generated from the hwv. A natural basis for the Verma module is a maximal independent set of states which are given by ordered monomials of the generators acting on the hwv. The Verma modules of all three algebras can be decomposed into eigenspaces of L_0, called *levels*. Level n is the eigenspace with L_0 eigenvalue $h + n$.

For the A algebra the zero modes are L_0 and T_0, so an hwv $|h, q\rangle$ is characterized by its energy (L_0 eigenvalue) h and charge (T_0 eigenvalue) q. Each level n of the Verma module can be further decomposed into T_0 eigenspaces with eigenvalue $q + m$, when m is called the *relative charge*. The counting of states is summarized by the partition function $P_\mathrm{A}(n, m)$ defined by

$$\sum_{n,m} P_\mathrm{A}(n,m)x^n y^m = \prod_{k=1}^{\infty} \frac{(1 + x^{k-1/2}y)(1 + x^{k-1/2}y^{-1})}{(1 - x^k)^2}. \tag{2}$$

In the P algebra the zero modes are L_0, T_0, G_0^i. There are two kinds of hwv, $|h, q \mp \frac{1}{2}\rangle_\pm$, with energy $L_0 = h$ and charge $T_0 = q \mp \frac{1}{2}$. Each satisfies an additional highest weight condition with respect to the charge, $(G_0^1 \mp \mathrm{i}G_0^2)|h, q \mp \frac{1}{2}\rangle_\pm = 0$. These two representations $P^\pm$ are isomorphic under charge conjugation ($T_m \to -T_m$, $G_m^2 \to -G_m^2$). Again, each level n is decomposed by relative charge m, with partition function

$$\sum_{n,m} P_\mathrm{P}(n,m)x^n y^m = (y^{1/2} + y^{-1/2}) \times \prod_{k=1}^{\infty} \frac{(1 + x^k y)(1 + x^k y^{-1})}{(1 - x^k)^2}. \tag{3}$$

[‡1] Twisted scalar fields were first described in ref. [25].

We will also need partition functions in the presence of single charged fermions:

$$\sum_{n,m} \widetilde{P}_X(n,m;k) x^n y^m = (1 + x^{|k|} y^{\mathrm{sgn}(k)})^{-1} \sum_{n,m} P_X(n,m) x^n y^m , \qquad (4)$$

for X = A, P. We define $\mathrm{sgn}(k) = 1$ for $k > 0$, $\mathrm{sgn}(k) = -1$ for $k < 0$ and $\mathrm{sgn}(0) = \pm 1$ for the representation $\mathrm{P}^{\pm}$.

In the T algebra the zero modes are L_0 and G_0^1. An hwv $|h\rangle$ is characterized by its energy h. Each level n splits into two equal subspaces of fermion parity $(-)^F = \pm 1$, where $(-)^F$ is the operator which commutes with L_m and T_m, anticommutes with G_m^i and is 1 on $|h\rangle$. The partition function for each fermion parity is

$$\sum_n P_T(n) x^n = \prod_{k=1}^{\infty} \frac{(1+x^k)(1+x^{k-1/2})}{(1-x^k)(1-x^{k-1/2})} . \qquad (5)$$

As usual, the inner product on the Verma module is defined by the hermiticity conditions, the commutation relations and the highest weight conditions. Subspaces of different level and relative charge or fermion parity are orthogonal. A powerful tool in the representation theory of these algebras is the determinant formula. This is a polynomial expression in $\tilde{c}, h, q$ for the determinant of the matrix of inner products of a basis of ordered monomials for a given eigenspace (up to a basis-dependent positive constant). The vanishing surfaces, or curves, of the determinant formula describe the Verma modules which contain null vectors. The presence of such null vectors can be used to solve for correlation functions in conformal quantum field theories [2,26,27,8,28]. The vanishing surfaces also mark changes in the signature of the metric, so they give crucial information about unitarity [29,3].

For the P and A algebras let $M^{\mathrm{P}}_{n,m}$ and $M^{\mathrm{A}}_{n,m}$ be the inner product matrices for level n, relative charge m. The determinant formula for the A algebra is

$$\det M^{\mathrm{A}}_{n,m}(\tilde{c},h,q) = \prod_{\substack{1 \leqslant rs \leqslant 2n \\ s \text{ even}}} (f^{\mathrm{A}}_{r,s})^{P_{\mathrm{A}}(n-rs/2,m)} \times \prod_{k \in \mathbf{Z}+1/2} (g^{\mathrm{A}}_k)^{\widetilde{P}_{\mathrm{A}}(n-|k|,m-\mathrm{sgn}(k);k)} , \qquad (6)$$

where

$$f^{\mathrm{A}}_{r,s}(\tilde{c},h,q) = 2(\tilde{c}-1)h - q^2 - \tfrac{1}{4}(\tilde{c}-1)^2 + \tfrac{1}{4}[(\tilde{c}-1)r + s]^2 \quad (s \text{ even}) ,$$
$$g^{\mathrm{A}}_k(\tilde{c},h,q) = 2h - 2kq + (\tilde{c}-1)(k^2 - \tfrac{1}{4}) \quad (k \in \mathbf{Z} + \tfrac{1}{2}) . \qquad (7)$$

For the P algebra,

$$\det M^{\mathrm{P}}_{n,m}(\tilde{c},h,q) = \prod_{\substack{1 \leqslant rs \leqslant 2n \\ s \text{ even}}} (f^{\mathrm{P}}_{r,s})^{P_{\mathrm{P}}(n-rs/2,m)} \times \prod_{k \in \mathbf{Z}} (g^{\mathrm{P}}_k)^{\widetilde{P}_{\mathrm{P}}(n-|k|,m-\mathrm{sgn}(k);k)} , \qquad (8)$$

where

$$f^{\mathrm{P}}_{r,s}(\tilde{c},h,q) = 2(\tilde{c}-1)(h - \tfrac{1}{8}\tilde{c}) - q^2 + \tfrac{1}{4}[(\tilde{c}-1)r + s]^2 \quad (s \text{ even}) ,$$
$$g^{\mathrm{P}}_k(\tilde{c},h,q) = 2h - 2kq + (\tilde{c}-1)(k^2 - \tfrac{1}{4}) - \tfrac{1}{4} \quad (k \in \mathbf{Z}) . \qquad (9)$$

For the T algebra let $M^{\mathrm{T}}_{\pm,n}$ be the matrix of inner products for level n and fermion parity $(-)^F = \pm 1$. On level 0 the determinant formulae are $\det M^{\mathrm{T}}_{+,0} = 1$, $\det M^{\mathrm{T}}_{-,0} = h - \tilde{c}/8$. For $n > 0$,

$$\det M^{\mathrm{T}}_{\pm,n}(\tilde{c},h) = (h - \tfrac{1}{8}\tilde{c})^{P_{\mathrm{T}}(n)/2} \times \prod_{\substack{1 \leqslant rs \leqslant 2n \\ s \text{ odd}}} (f^{\mathrm{T}}_{r,s})^{P_{\mathrm{T}}(n-rs/2)} , \qquad (10)$$

where

$$f^{\mathrm{T}}_{r,s}(\tilde{c},h) = 2(\tilde{c}-1)(h - \tfrac{1}{8}\tilde{c}) + \tfrac{1}{4}[(\tilde{c}-1)r + s]^2 \quad (s \text{ odd}). \qquad (11)$$

A vanishing of the determinant formula signals a new hwv generating a submodule inside the Verma module. For the P algebra, along the quadratic vanishing surface $f^{\mathrm{P}}_{r,s} = 0$ there is an hwv on level $rs/2$ with relative charge $-\frac{1}{2}\mathrm{sgn}(0)$. Along the vanishing plane $g^{\mathrm{P}}_k = 0$ there is an hwv at level $|k|$ and relative charge $-\frac{1}{2}\mathrm{sgn}(0) + \mathrm{sgn}(k)$. For $k = 0$ this reflects the unbroken supersymmetry of states with $h = \tilde{c}/8$, and the possibility of a non-zero Witten index [30,8].

For the A algebra, along the quadratic surfaces $f^{\mathrm{A}}_{r,s} = 0$ there is an hwv of level $rs/2$ and relative charge 0. Along the vanishing plane $g^{\mathrm{A}}_k = 0$ there is an hwv of level $|k|$ and relative charge sgn(k). For the T algebra, along the quadratic vanishing curve $f^{\mathrm{T}}_{r,s} = 0$, there are two hwvs at level $rs/2$ with fermion parity ± 1. Along the vanishing line $h = \tilde{c}/8$, the state at level 0 and fermion parity -1 becomes an hwv, again allowing a nonzero index.

The quadratic vanishing surfaces for the P and A algebras are roughly analogous to the vanishing curves of the $N = 0$ or $N = 1$ superconformal formulae. The vanishing planes $g^{\mathrm{P,A}}_k = 0$ are something new. For the A algebra, there are precisely two at each half-integral level; for the P algebra there is one at level 0 and two at each of the higher levels. The hwv on a vanishing plane does not generate a full Verma submodule of states, because there exist *raising* operators which annihilate the hwv. (A simple example is given by the vanishing plane $g^{\mathrm{A}}_{1/2} = 0$, where the hwv $G_{-1/2}|h, q\rangle$ is annihilated by $G_{-1/2}$.) This explains the need for the modified partition functions $\tilde{P}_{\mathrm{P}}, \tilde{P}_{\mathrm{A}}$ (eq. (4)); their precise form is justified in the proof of the determinant formulae [11].

We stress that there are no further vanishings. In particular, the formulae imply that the A algebra has hwvs only for relative charge $m = -1, 0, 1$, and the P algebra only for $m = -\frac{1}{2}, \frac{1}{2}, -3\,\mathrm{sgn}(0)/2$.

Determinant formulae give a great deal of information about the representation theory of a Lie algebra. For the $N = 0$ and $N = 1$ superconformal algebras, determinant formulae were used to prove the *non*-unitarity of a large class of representations; the remaining representations were conjectured to be unitary [3,8]. These conjectures were based on explicit low-level calculations, and on a small number of examples from statistical mechanics. The unitarity conjectures were subsequently verified by manifestly unitary constructions of all the allowed representations [31].

An interesting picture emerges when the same strategy is applied to the $N = 2$ algebras. In the region $\tilde{c} < 1$, we find a discrete series of possibly unitary representations, while the non-unitarity of the remaining representations follows from the determinant formulae. (This much is familiar from the $N = 0, 1$ results.) However, for the P and A algebras, it is no longer the case that all $\tilde{c} \geqslant 1, h \geqslant 0$ representations are unitary, since the vanishing surfaces impinge on this region. Unitarity is obvious for h above the upper boundary of the vanishing surfaces, which is composed of plane segments. But the determinant formulae indicate that some representations between this boundary and the plane $h = 0$ survive the non-unitarity proof. These lie on segments of vanishing planes.

The precise results are as follows. For the A algebra, the only possible unitary representations fall into three classes:

A_3: $\tilde{c} \geqslant 1$, $(\tilde{c}, h, q)$ such that $g^{\mathrm{A}}_n \geqslant 0$,
for all $n \in \mathbf{Z} + \frac{1}{2}$.

A_2: $\tilde{c} \geqslant 1$, $(\tilde{c}, h, q)$ such that $g^{\mathrm{A}}_n = 0$,
$g^{\mathrm{A}}_{n+\mathrm{sgn}(n)} < 0$, $f^{\mathrm{A}}_{1,2} \geqslant 0$, for some $n \in \mathbf{Z} + \frac{1}{2}$.

A_0: $\tilde{c} < 1, \tilde{c} = 1 - 2/\tilde{m}, h = (jk - \frac{1}{4})/\tilde{m}$,
$q = (j - k)/\tilde{m}$, for integer $\tilde{m} \geqslant 2$,
and $j, k \in \mathbf{Z} + \frac{1}{2}, 0 < j, k, j + k \leqslant \tilde{m} - 1$.

For the P algebra, the only possible unitaries (with sgn(0) = ±1) are

$\mathrm{P}^{\pm}_3$: $\tilde{c} \geqslant 1$, $(\tilde{c}, h, q)$ such that $g^{\mathrm{P}}_n \geqslant 0$ for all $n \in \mathbf{Z}$.

$\mathrm{P}^{\pm}_2$: $\tilde{c} \geqslant 1, (\tilde{c}, h, q)$ such that $g^{\mathrm{P}}_n = 0$,
$g^{\mathrm{P}}_{n+\mathrm{sgn}(n)} < 0$, $f^{\mathrm{P}}_{1,2} \geqslant 0$, for some $n \in \mathbf{Z}$.

$\mathrm{P}^{\pm}_0$: $\tilde{c} < 1$, $\tilde{c} = 1 - 2/\tilde{m}, h = \tilde{c}/8 + jk/\tilde{m}$,
$q = \mathrm{sgn}(0)(j - k)/\tilde{m}$, for integer $\tilde{m} \geqslant 2$,
and $j, k \in \mathbf{Z}$, $0 \leqslant j - 1, k, j + k \leqslant \tilde{m} - 1$.

For the T algebra the only possible unitaries are

T_2: $\tilde{c} \geqslant 1$, $h \geqslant \tilde{c}/8$.

T_0: $\tilde{c} < 1$, $\tilde{c} = 1 - 2/\tilde{m}, h = \tilde{c}/8 + (\tilde{m} - 2r)^2/16\tilde{m}$
for integers $\tilde{m}, r$ such that $2 \leqslant \tilde{m}$
and $1 \leqslant r \leqslant \tilde{m}/2$.

(The subscripts indicate the dimension of the moduli spaces.)

Observe that P^+_3 and P^-_3 are identical, while $\mathrm{P}^+_{2,0}$ and $\mathrm{P}^-_{2,0}$ differ only at the supersymmetric value $h = \tilde{c}/8$, with the allowed charges asymmetric around $q = 0$. Note also that, for the T_0 discrete series, only even $\tilde{m}$ values allow $h = \tilde{c}/8$. Such information constrains the spontaneous breaking of supersymmetry in finite volume and in the presence of supersymmetric mass perturbations [8,18].

The representations A_3, $P_3^{\pm}$, T_2 are obviously unitary; we conjecture that all of the remaining possibly unitary representations are indeed unitary. This conjecture is supported by examination of the inner product matrices at low levels. Further support for the conjecture is provided by manifestly unitary constructions of some of the discrete series of representations, which we now describe. The $\tilde{c} = \frac{1}{3}$ representations are realised by a single periodic scalar field, at special values of the period, demonstrating $N = 2$ supersymmetry in the gaussian model [18]. A similar construction, applied to the free $N = 1$ scalar superfield, produces $\tilde{c} = \frac{1}{2}$ representations. Tensor products of these yield $\tilde{c} = \frac{2}{3}$ and $\tilde{c} = \frac{5}{6}$.

Another construction method, which has proved useful in the context of the $N = 0, 1$ superconformal algebras [31,32], uses "quark model" fermionic oscillators. For any semisimple Lie algebra g, let $H^{i,a}(z)$ $(1 \leqslant i \leqslant 2,\ 1 \leqslant a \leqslant \dim(g))$ be $2\cdot\dim(g)$ Fermi fields. Define

$$V^{i,a}(z) = \tfrac{1}{2}\,\mathrm{i} f_{abc} :H^{i,b}(z)H^{i,c}(z): \quad \text{(no sum on } i) ,$$
$$W^i(z) = \tfrac{1}{3} V^{i,a}(z)H^{i,a}(z) \quad \text{(no sum on } i) ,$$
$$S^1(z) = V^{2,a}(z)H^{1,a}(z) , \quad S^2(z) = V^{1,a}(z)H^{2,a}(z) ,$$
$$G^i(z) = (1/\sqrt{6v})\,[W^i(z) - S^i(z)] ,$$
$$J(z) = \tfrac{1}{3} H^{1,a}(z)H^{2,a}(z) ,$$
$$T(z) = (1/4v) : V^{i,a}(z)\,V^{i,a}(z):$$
$$- (1/6v) :(V^{1,a} + V^{2,a})(z)(V^{1,a} + V^{2,a})(z): . \tag{12}$$

The operators $T(z)$, $G^i(z)$, $J(z)$ define a representation of the P, A or T algebras (depending on the choice of moding). Here f_{abc} are the structure constants of g, while the constant $v = c_\psi/2$ is defined by $f_{abc} f_{bcd} = -2v\delta_{ad}$. The (manifestly unitary) representation on the fermionic Fock space has $\tilde{c} = \dim(g)/9$, giving constructions for $\tilde{c} = \frac{1}{3}, \frac{2}{3}, \frac{8}{9}$. In addition, a similar construction using two sets of Fermi fields transforming under the fundamental representation of G_2 and replacing f_{abc} by the invariant anti-symmetric three-tensor, yields $\tilde{c} = \frac{7}{9}$.

An explicitly unitary construction of the entire discrete series would be of great value. One intriguing avenue of investigation is suggested by an apparent relationship between the $N = 2$ discrete series and the unitary representations of the affine algebra $\widehat{su(2)}$ with generators T_n^i. The discrete series of $N = 2$ central charges $c = 3\tilde{c} = 3 - 6/\tilde{m}$, $\tilde{m} = 2, 3, \ldots$, are precisely the central charges c of the Virasoro algebra $L_n^{su(2)}$ associated with the unitary representations of $\widehat{su(2)}$. The h, q weights of the $N = 2$ representations are simple functions of the possible $\widehat{su(2)}$ highest weights ($L_0^{su(2)}$ and T_0^3 eigenvalues).

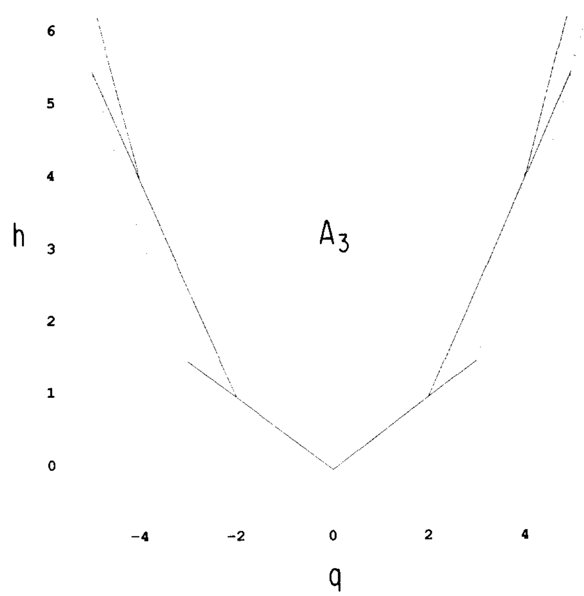

Fig. 1. Unitary representations of the A algebra at $\tilde{c} = 3$. The unitary representations in A_3 form the convex region bounded by solid lines. The unitaries in A_2 lie on the solid lines outside A_3. The parabola is $f^A_{1,2}(3, h, q) = 0$. The lines are segments of $g^A_k(3, h, q) = 0$, $|k| \leqslant 7/2$.

The unitary representations A_2 and $P_2^{\pm}$, which lie on vanishing planes in the region $\tilde{c} \geqslant 1$, are a novel feature in the representation theory of infinite-dimensional Lie algebras. They imply, for fixed $\tilde{c}$ and q, a discrete spectrum of h-values lying below the continuum of unitary representations. The unitary representations for $\tilde{c} = 3$ are pictured in figs. 1 and 2.

Representations with $\tilde{c} \geqslant 1$ arise in the compactification of supersymmetric string. $N = 2$ supersymmetric nonlinear models based on Calabi–Yau spaces are superconformally invariant, at least to all orders in perturbation theory [21]. They give representations of the A and P algebras with $\tilde{c} = d/2$, where d is the real dimension of the Calabi–Yau space. The value $\tilde{c} = 3$ corresponds to compactification from ten to four dimensions. The A representations determine the boson-

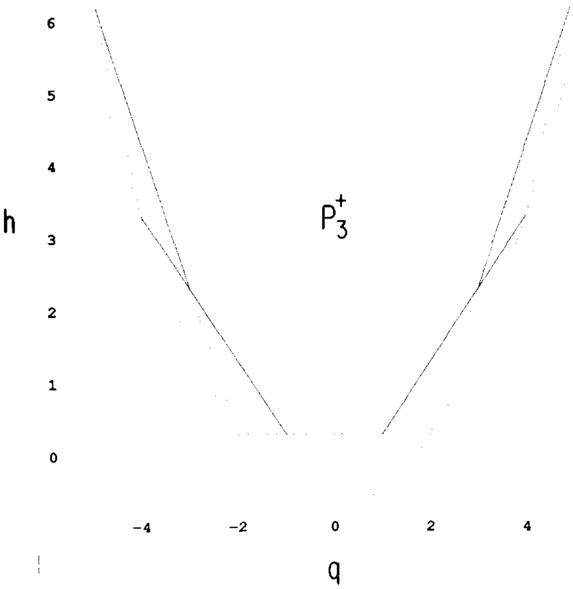

Fig. 2. Unitary representations of the P algebra at $\tilde{c} = 3$, sgn(0) = 1. The unitary representations in P_3^+ form the convex region bounded by solid lines. The unitaries in P_2^+ lie on the solid lines outside P_3^+. The parabola is $f^P_{1,2}(3, h, q) = 0$. The lines are segments of $g^P_k(3, h, q) = 0$, $|k| \leqslant 3$. The equivalent diagram for sgn(0) = −1 is given by the reflection $q \to -q$.

ic spectrum of the compactified string; the P representations determine the fermionic spectrum. Certain special operators in the nonlinear models correspond to unitary representations in the classes A_2, $P_2^\pm$. In particular, the (anti-)holomorphic ϵ-tensors correspond to the A_2 representations with $q = \pm\tilde{c}$, $h = \tilde{c}/2$, at intersections of $g^A_{\pm 1/2}$ with $f^A_{1,2}$. The covariantly constant spinors corresponds to the $P_0^\pm$ representations with $q = \mathrm{sgn}(0)\,\tilde{c}/4$, $h = \tilde{c}/8$, which are at intersections of g^P_0 with $f^P_{1,2}$. Any compactification (Calabi–Yau or not [22]) with unbroken spacetime supersymmetry must have a superconformal field with these quantum numbers. A key problem in the study of string compactification is to show non-perturbatively that these operators are present and that they satisfy the operator product relations needed for the construction of the gravitino vertex and the world sheet spacetime supersymmetry current [33].

The $N = 2$ unitary representation theory has two immediate consequences for compactification. First, if operators with the quantum numbers of the covariantly constant spinors and the (anti-)holomorphic ϵ-tensors occur in a compactification, then they must keep those quantum numbers in any compactification nearby, because of the gap in allowed h values. This assumes that q is not renormalised (even nonperturbatively). Second, the fact that the interesting (h, q) values lie at the intersection of vanishing surfaces can be used, by the methods of refs. [2,26,27,8,28], to prove the operator-product identities which imply spacetime supersymmetry [34] [‡2].

These algebraic results are interesting because they are nonperturbative in the compactification radius. They suggest that further algebraic investigation of string compactification could be fruitful.

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‡2 Note that, at special values of $\tilde{c}$, namely $\tilde{c} = 1 + 2/n$, $n = 1, 2, \ldots$, there is a discrete series of algebraically special representations, identified by a triple intersection of vanishing surfaces. It is not clear whether this condition is significant. However, for $n = 1$ and $n = 2$ it does apply to six- and four-dimensional Calabi–Yau compactifications.

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