Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions

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Conformal invariance and unitarity severely limit the possible values of critical exponents in two-dimensional systems.

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One of the most intriguing features of statistical mechanical systems is the existence of scale-invariant critical points. Polyakov[1] has shown that *local* scale invariance, i.e., conformal invariance, can be used to construct critical theories. In two dimensions the group of conformal transformations is exceptionally large (infinite dimensional) since any analytic function mapping the complex plane to itself is conformal. Belavin, Polyakov, and Zamolodchikov (BPZ)[2] have shown how the rich structure of the conformal group in two dimensions can be used to analyze conformally invariant field theories.

Many two-dimensional statistical mechanical systems can also be interpreted as $(1+1)$-dimensional quantum field theories. The distinguishing feature of the quantum theories is unitarity, equivalent to reflection positivity in the statistical systems. We point out here that unitarity, in the presence of the large conformal transformation group, puts a powerful constraint on the allowed physical systems.

Conformally invariant systems are described by correlation functions of a collection of conformal fields $\phi(z,\bar{z})$. The fields can be interpreted as operators and the correlation functions as vacuum expectation values. The infinitesimal conformal transformations $z \to z+\epsilon z^{n+1}$, $\bar{z} \to \bar{z}+\bar{\epsilon}\bar{z}^{n+1}$ are generated by operators L_n, $\bar{L}_n$:

$$[L_n,\phi] = z^{n+1}\partial_z\phi + h(n+1)z^n\phi,$$
$$[\bar{L}_n,\phi] = \bar{z}^{n+1}\partial_{\bar{z}}\phi + \bar{h}(n+1)\bar{z}^n\phi. \tag{1}$$

The correlation functions are invariant under the global conformal transformations generated by L_n, $\bar{L}_n$, for $n=-1,0,1$. In particular,

$$\langle\phi(re^{i\theta})\phi(0)\rangle = r^{-2(h+\bar{h})}e^{-2i\theta(h-\bar{h})}, \tag{2}$$

and so ϕ has scaling dimension $x = h+\bar{h}$ and spin $h-\bar{h}$.

We are using here a nonstandard operator interpretation in which radial ordering takes the place of time ordering. Reflection positivity in the radial and time directions are equivalent by global conformal invariance. In terms of coordinates $z = e^{\tau+i\theta}$ the radial quantization gives quantum field theory in imaginary "time" τ and periodic one-dimensional "space" θ.

Local scale invariance is equivalent to the existence of a conserved traceless stress energy tensor $T(z)dz^2 + \bar{T}(\bar{z})d\bar{z}^2$ which acts as the generator of conformal transformations:

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2}L_n,$$
$$\bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{z}^{-n-2}\bar{L}_n. \tag{3}$$

It satisfies the self-adjointness condition $[T(1/\bar{z}) \times d(1/\bar{z})^2]^\dagger = T(z)dz^2$. Equivalently,

$$L_n^\dagger = L_{-n}. \tag{4}$$

The L_n obey the commutation relations

$$[L_m,L_n] = (m-n)L_{m+n} + c[(m^3-m)/12]\delta_{m,-n}. \tag{5}$$

The $\bar{L}_n$ commute with all the L's and satisfy the same Hermiticity and commutation relations (with the same c, under the assumption that the theory is $z \leftrightarrow \bar{z}$ invariant). The algebra (5) is the Virasoro algebra.[3] The central term with coefficient c describes the particular realization of conformal symmetry. It appears in a quantum theory because the composition law for transformations need only be satisfied by the operator representation up to a phase. The central term also measures the response of the ground-state energy to curving the underlying space—the trace anomaly.[4]

Our result is that unitarity restricts the allowed values of c, for $c<1$, to

$$c = 1 - 6/m(m+1), \tag{6a}$$

$$m = 2,3,4,\ldots . \tag{6b}$$

For a model in which c takes one of these values, the scaling dimensions $x = h + \bar{h}$ are rational numbers with h and $\bar{h}$ limited to the values

$$h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{7a}$$

$$p = 1, 2, \ldots, m-1, \quad q = 1, 2, \ldots, p. \tag{7b}$$

This follows from a study of the representations of the Virasoro algebra.

The lowering operators for L_0 ($\bar{L}_0$) are the L_n ($\bar{L}_n$), $n > 0$. A state annihilated by all the lowering operators is called a highest-weight vector (HWV). The vacuum $|0\rangle$ is a HWV because it has the lowest eigenvalue of the "Hamiltonian" $L_0 + \bar{L}_0$. More generally, there is a one-to-one correspondence between the conformal fields and the HWV's, the field ϕ corresponding to $\phi(0)|0\rangle$, which is a HWV by Eq. (1). The invariance of the correlation functions under global conformal transformations implies that the vacuum is also annihilated by L_0, $\bar{L}_0$, L_{-1}, and $\bar{L}_{-1}$. It follows that $\phi(0)|0\rangle$ is an eigenstate of L_0 ($\bar{L}_0$) with eigenvalue h ($\bar{h}$). The space of states is a sum of irreducible representations of the product algebra (of L's and $\bar{L}$'s), each generated from one of the HWV's.

The L's and $\bar{L}$'s form identical commuting Virasoro algebras and so we can restrict our attention to representations of the L's. A representation of the Virasoro algebra is built from a HWV by applying the L_{-n}, $n \geq 1$. A state is in the nth level if its L_0 eigenvalue is $h + n$. The nth level is spanned by the vectors

$$L_{-k_1} \cdots L_{-k_m} \phi(0)|0\rangle, \tag{8}$$

for $k_1 \geq \ldots \geq k_m > 0$ and $\sum k_i = n$. There are $P(n)$ such states, where $P(n)$ is the number of ways of writing n as a sum of positive integers. The higher-level states correspond to operators of higher scaling dimension obtained by applying products of stress-energy tensors to ϕ.

Unitarity means that the inner product in the space of states is positive definite. The inner product of any two of the states (8) can be computed from Eqs. (4) and (5). A state $|\psi\rangle$ with $\langle\psi|\psi\rangle$ negative is called a "ghost." If a ghost is found on any level the representation cannot occur in a unitary theory.

Kac[5] has given a formula for the determinant of the matrix of inner products of the states (8):

$$\det M_{(n)}(c,h) - C \prod_{pq \leq n} [h - h_{p,q}(c)]^{P(n-pq)}, \tag{9}$$

where p,q range over the positive integers, $h_{p,q}(c)$ is given by Eqs. (6a) and (7a), and C is a positive constant. The matrix of inner products is manifestly positive definite for $1 < c$, $0 < h$.

We immediately eliminate all regions where the determinant is negative because they necessarily contain an odd number of ghosts. A straightforward examination of the determinant formula shows that any point c,h in the half-plane $c < 1$ which is not on an $h_{p,q}$ curve will have a negative determinant at some level.

In fact the unitary theories are far more limited. All points on vanishing curves have ghosts except possibly the "first intersections." A first intersection is an intersection of vanishing curves that on some level is the intersection closest to $c = 1$ on a given curve $h_{p,q}$. These are exactly the points listed in Eqs. (6a)–(7b). The proof[6] that there is a ghost in every open interval bounded by first intersections on a vanishing curve $h_{p,q}$ is based on the following observations: (1) When $h = h_{p,q}(c)$ there is a null HWV at level pq which generates a subrepresentation, and (2) whenever a first intersection appears on $h_{p,q}$ (at level n) the determinant $\det M_{(n-pq)}(c, h + pq)$ for the subrepresentation is nonzero, so that there can be no HWV inside the subrepresentation at level $n - pq$.

All unitary representations for $c < 1$ are contained in the list, Eqs. (6a)–(7b), but we have not proved that all representations on the list are in fact unitary. We have verified numerically that they are through level 12 and we have a heuristic argument that they remain ghost-free to all levels. The argument is based on the structure of subrepresentations on the $c = 1$ line[7] and the assumed existence of an analytic diagonalization of the matrix in the whole region of interest. Such a diagonalization might be provided by techniques[8] which give analytic deformations of correlation functions away from $c = 1$.

Models with Hermitian transfer matrices provide concrete examples of reflection-positive systems. Systems with continuously variable critical exponents like the Gaussian model have $c \geq 1$ where unitarity allows all $h \geq 0$. In the range $c < 1$ we find by matching scaling dimensions that the $m = 3$ representations describe the Ising model, $m = 4$ the tricritical Ising, $m = 5$ the three-state Potts, and $m = 6$ the tricritical three-state Potts.[9] Table I compares the known scaling dimensions for these models[10] with the allowed values of h. All of the known exponents are accounted for. We know of no models with $m \geq 7$. The fact that $h = 1$ is never unitary for $c < 1$ rules out marginal operators as well as continuous internal symmetries generated

TABLE I. Comparison with known scaling dimensions. The $h_{p,q}$ from Eqs. (6a)–(7b) are listed with p running horizontally from 1 to $m-1$ and q running vertically from 1 to p. The spins $h-\bar{h}$ are consistent with what is known. Some operators have alternative interpretations as derivatives.

model	x	h	$\bar{h}$	$h_{p,q}(c)$
Ising	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$m=3$: $\begin{array}{cc} & \frac{1}{16} \\ 0 & \frac{1}{2} \end{array}$
	1	$\frac{1}{2}$	$\frac{1}{2}$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	
tricritical 3-state Potts	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$m=6$: $\begin{array}{ccccc} & & & & \frac{1}{7} \\ & & & \frac{5}{56} & \frac{5}{7} \\ & & \frac{1}{21} & \frac{33}{56} & \frac{12}{7} \\ & \frac{1}{56} & \frac{10}{21} & \frac{85}{56} & \frac{22}{7} \\ 0 & \frac{3}{8} & \frac{4}{3} & \frac{23}{8} & 5 \end{array}$
	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	
	$\frac{20}{21}$	$\frac{10}{21}$	$\frac{10}{21}$	
	$\frac{10}{7}$	$\frac{5}{7}$	$\frac{5}{7}$	
	$\frac{17}{7}$	$\frac{12}{7}$	$\frac{5}{7}$	
	$\frac{8}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	
	$\frac{23}{7}$	$\frac{22}{7}$	$\frac{1}{7}$	
	5	5	0	
tricritical Ising	$\frac{3}{40}$	$\frac{3}{80}$	$\frac{3}{80}$	$m=4$: $\begin{array}{ccc} & & \frac{1}{10} \\ & \frac{3}{80} & \frac{3}{5} \\ 0 & \frac{7}{16} & \frac{3}{2} \end{array}$
	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	
	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{16}$	
	$\frac{6}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	
	3	$\frac{3}{2}$	$\frac{3}{2}$	
3-state Potts	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$m=5$: $\begin{array}{cccc} & & & \frac{1}{8} \\ & & \frac{1}{15} & \frac{2}{3} \\ & \frac{1}{40} & \frac{21}{40} & \frac{13}{8} \\ 0 & \frac{2}{5} & \frac{7}{5} & 3 \end{array}$
	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	
	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	
	$\frac{9}{5}$	$\frac{7}{5}$	$\frac{2}{5}$	
	$\frac{14}{5}$	$\frac{7}{5}$	$\frac{7}{5}$	
	3	3	0	

by local currents. The possibility of large discrete-symmetry groups seems worth exploring.

Although unitarity provides strong constraints on possible representations of conformal invariance, a sensible theory also requires closure of the operator-product expansion and crossing symmetry of correlation functions.[1,2] BPZ[2] have shown how to implement crossing symmetry when the space of states is made up of representations with null states, using the fact that such states give rise to linear differential equations on the correlation functions. It follows from our result that the differential equation technique applies to all unitary models with $c<1$. BPZ[11] have found finite sets of conformal fields that must close under the operator-product expansion for the special values of c corresponding to m rational, $m=r/(s-r)$, $r<s$, $3r \geqslant 2s$. The scaling dimensions are given by $h=h_{p,q}(c)$, $1 \leqslant p < r$, $1 \leqslant q < s$. Note that the unitary representations correspond to $s=r+1$. Dotsenko[12] has used the differential equation technique to find a closed operator algebra for the three-state Potts model and to construct some of its correlation functions. The representations which actually appear in the tricritical Potts model form a similar closed algebra by Dotsenko's argument.

Kac[5] has also written a determinant formula for the supersymmetric extension of the Virasoro algebra, the Ramond-Neveu-Schwarz algebra.[13] Only first intersections can be ghost-free. The allowed representations, for $2c/3<1$, are

$$2c/3 = 1 - 8/m(m+2),$$
$$h = \frac{[(p-q)m+2p]^2-4}{8m(m+2)}, \qquad (10)$$

for $m=2,3,4,\ldots$ and p,q integers, both even or both odd, $0<p<m$, $0<q\leqslant p$. Note that $h=\frac{1}{2}$ does not occur, so that there are no internal supercurrents. The representations $c=\frac{7}{10}$, $h=0,\frac{1}{10}$ are composed of the Virasoro representations $c=\frac{7}{10}$, $h=0,\frac{3}{2},\frac{1}{10},\frac{3}{5}$. These are the representations occurring in the Z_2 invariant subalgebra of the tricritical Ising model. It seems that this subalgebra is supersymmetric.[14] Supersymmetric models at $m=4$, $c=1$ using $h=0,\frac{1}{6},1$ or $h=0,\frac{1}{16},1$ can be identified with special points in the Gaussian model.[14] The representations of the superalgebra might also be of interest in superstring theory.[15]

It should now be straightforward to sort out all possible unitary conformally invariant models with $c<1$ ($2c/3<1$ in the super case). The differential equation techniques of BPZ always apply because the only allowed representations have null states, and the problem is finite because only a finite number of representations are allowed for each possible m. Such a systematic construction of conformally invariant models would partially realize the bootstrap program initiated by Kadanoff[16] and Polyakov.[17] The extreme rigidity of the conformal bootstrap in two dimensions is a striking feature of the present result. It seems that unitarity is the crucial constraint for $c < 1$. For $c \geqslant 1$ crossing symmetry would have to limit the possible realizations of conformal invariance.

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Note added.—D. A. Huse has informed us that G. E. Andrews, R. J. Baxter, and P. J. Forrester have solved two new infinite sets of two-dimensional models. The critical exponents in one of the sets suggest that the corresponding models provide examples for each value of c in Eqs. (6a)–(6b). The supersymmetry which we have noted at $c = 0.7$ has also been described by A. M. Zamolodchikov, Yad Fiz. (to be published).

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