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SUPER CHARACTERS AND CHIRAL ASYMMETRY IN SUPERCONFORMAL FIELD THEORY *

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The super partition function of two-dimensional superconformal field theory on the super torus is analyzed. Its bottom component is the integer index of the Ramond representation. The top component measures chiral asymmetry in the Ramond sector. It is a sesquilinear form in the odd "super characters" of the Ramond algebra. Formulas for the characters are derived for the irreducible unitary highest weight representations. Modular invariance introduces constraints on the chiral asymmetry which can be satisfied for all models in the discrete series.

1. Introduction

Two-dimensional conformal field theory and in particular superconformal field theory has been at the nexus of much recent activity in two-dimensional critical phenomena and string theory. The investigation of two-dimensional superconformal field theory has generally parallelled the development of ordinary field theory. The unitary highest weight representations of the superconformal algebras were classified [1–3], supersymmetric critical phenomena were found [1,2], a method for constructing correlation functions and obtaining operator product coefficients was developed [4–6], characters were calculated [7,8,3] and a method for constructing the genus one partition function and obtaining field multiplicities was developed [9–12]. There is a conjectured classification of models with superconformal central charge $\hat{c} < 1$ based on unitarity [1,2] and on modular invariance of the genus one partition function [9–12]. In this paper the genus one super partition function is constructed from super characters of the Ramond algebra. This completes the description of the field multiplicities of superconformal models [10,12].

The present paper starts from the observation [13] that $e^{-\tau H-\hat{\tau}Q}$ is the evolution operator for a supersymmetric model with hamiltonian $H = Q^2$ in euclidean "super time" $(\tau, \hat{\tau})$. The super partition function, for fermions periodic in time, would be

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$\mathrm{tr}(-1)^F e^{-\tau H - \hat{\tau} Q}$. The bottom component of this trace is the supersymmetry index. The top component vanishes, since either $Q = 0$ or the states form doublets under Q and $(-1)^F$.

The Ramond sector of a superconformal field theory has an analogous structure, except that there are two anticommuting supersymmetry generators, G_0 and $\overline{G}_0$, with the hamiltonian given by $H = G_0^2 + \overline{G}_0^2$. In complex euclidean "super time" $(\tau, \hat{\tau})$ the super partition function is

$$Z_{\mathrm{sup}} = \mathrm{tr}_{\mathrm{R}}(-1)^F \exp 2\pi i\left(\tau G_0^2 + \hat{\tau} G_0 - \bar{\tau}\overline{G}_0^2 - \hat{\bar{\tau}}\overline{G}_0\right)$$

$$= Z_{++}(\tau, \bar{\tau}) + 4\pi^2 i \hat{\tau}\hat{\bar{\tau}} Z_{\mathrm{top}}(\tau, \bar{\tau}), \tag{1}$$

$$Z_{\mathrm{top}} = \mathrm{tr}_{\mathrm{R}}(-1)^F i G_0 \overline{G}_0 q^{G_0^2} \bar{q}^{\overline{G}_0^2}, \qquad q = e^{2\pi i \tau}. \tag{2}$$

The trace is over the Ramond sector. The bottom component $Z_{++} = \mathrm{tr}_{\mathrm{R}}(-1)^F$ is the integer supersymmetry index. The top component Z_{top} now can be nonzero because there are two supersymmetry generators.

The euclidean time τ might be written as $i\beta + \theta$ with β interpreted as inverse temperature and θ as a translation in circular space, in a $1+1$ dimensional system. More abstractly, τ is a "modulus" for the two-dimensional box or torus made by identifying points in the complex plane under the lattice of translations $z \to z + m\tau + n$, for all integer m, n. The super time $\tau, \hat{\tau}$ is the supermodulus of the super torus $z \to z + m(\tau + \theta\hat{\tau}) + n$, $\theta \to \theta + m\hat{\tau}$. These are the super tori with odd spin structure – periodic fermion boundary conditions around both cycles.

Different values of $\tau, \hat{\tau}$ can represent superconformally equivalent surfaces. The modulus $\tau, \hat{\tau}$ describes the same super torus as all its modular transforms. The generators of the supermodular group are the fractional linear transformations $\tau \to \tau + 1$, $\hat{\tau} \to \hat{\tau}$ and $\tau \to -1/\tau$, $\hat{\tau} \to \tau^{-3/2}\hat{\tau}$ [14]. The super partition function is modular invariant,

$$Z_{\mathrm{top}}(\tau + 1, \bar{\tau} + 1) = Z(\tau, \bar{\tau}),$$

$$Z_{\mathrm{top}}(-1/\tau, -1/\bar{\tau}) = \tau^{3/2}\bar{\tau}^{3/2} Z_{\mathrm{top}}(\tau, \bar{\tau}). \tag{3}$$

The top component of the super partition function measures the chiral asymmetry of the two-dimensional fermions. It vanishes if the model contains chiral fermions, with separately conserved parities of left and right moving fermions, $(-1)^F$ and $(-1)^{\overline{F}}$. More generally, the super partition function must be an even function of the odd supermoduli $\hat{\tau}$ and of the odd supermoduli $\hat{\bar{\tau}}$ separately. This is necessary in superstring models where the GOS projection is done separately on left and right moving fermions. These theories are defined on chiral supermoduli space where $\hat{\tau} \to -\hat{\tau}$ and $\hat{\bar{\tau}} \to -\hat{\bar{\tau}}$ are both symmetry transformations.

Z_{top} will be decomposed, below, into a sesquilinear form

$$Z_{\text{top}}(\tau,\bar{\tau}) = \sum_{a,b} 2I_{a,b}\hat{\chi}^a(\tau)\overline{\hat{\chi}^b(\tau)} \tag{4}$$

in the odd super characters

$$\hat{\chi}^a(\tau) = \operatorname{tr}_{\lambda_a}\left(G_0 q^{G_0^2}\right). \tag{5}$$

The trace $\operatorname{tr}_\lambda$ is taken over the irreducible highest weight representation of the Ramond algebra with the single highest weight vector $|\lambda\rangle$, $G_0|\lambda\rangle = \lambda|\lambda\rangle$. The indices a, b range over the inequivalent irreducible representations with weights λ_a, λ_b. These representations seem not to have been considered previously. They introduce an extra subtlety into the classification problem for superconformal models since their highest weight vectors are G_0 eigenstates. The L_0 eigenvalue, used in previous analyses, determines the G_0 eigenvalue only up to a sign. The matrix element I_{ab} is an integer which measures the net chirality in the spin fields (Ramond representations) of weights λ_a, λ_b.

We proceed by reworking old representation-theoretic arguments to evaluate the super characters $\hat{\chi}^a(\tau)$. In the now standard way [9] the modular transformations act by linear matrices on the odd super characters $\hat{\chi}^a(\tau)$. The modular invariance of Z_{sup} becomes modular invariance of the sesquilinear matrix I_{ab}. The chirality matrix I_{ab} is found for each model in the conjectured complete list of the unitary discrete series. It did not seem a priori obvious that every model would have a consistent chirality matrix.

This construction completes the structural analysis of the genus one super partition function for superconformal field theories.

Sects. 2 and 3 describe the representations of one and two Ramond algebras. The super partition function is analyzed in sect. 4. The super characters are calculated in sect. 5 and the chirality matrices $I_{a,b}$ are found in sect. 6. Sect. 7 is the conclusion. Details are placed in the appendix.

2. Representations of the Ramond algebra

The commutation relations of the $N = 1$ superconformal algebras are

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{8}\hat{c}(m^3 - m)\delta_{m,-n},$$

$$[L_m, G_n] = \left(\tfrac{1}{2}m - n\right)G_{m+n},$$

$$[G_m, G_n]_+ = 2L_{m+n} + \tfrac{1}{2}\hat{c}\left(m^2 - \tfrac{1}{4}\right)\delta_{m,-n}. \tag{6}$$

The central charge $\hat{c}$ commutes with all the generators and is represented by a real number in each superconformal field theory (and in each irreducible representation). There are two inequivalent algebras. The Ramond algebra has anticommuting generators G_n indexed by n integer while the Neveu-Schwarz algebra has n half-integer [15]. In this paper we study only the Ramond algebra.

Previous treatments considered the extended Ramond algebra which includes a fermion chirality operator $(-1)^F$ satisfying

$$\left[L_m,(-1)^F\right]=\left[G_m,(-1)^F\right]_+=0. \tag{7}$$

The highest weight vectors of the extended algebra form a representation of $(-1)^F$ and G_0 and thus are two-fold degenerate, unless $G_0=0$. The states with $G_0=0$ ($\overline{G}_0=0$) are left (right) supersymmetric. The irreducible representations with supersymmetric highest weight vector, $G_0=0$ or $\overline{G}_0=0$, can be ignored since they do not contribute to Z_{top}.

The highest weight vectors $|h,\pm\rangle$ of the extended algebra satisfy

$$\begin{aligned}
&L_{+n}|h,\pm\rangle=0, && G_{+n}|h,\pm\rangle=0, \qquad n>0,\\
&L_0|h,\pm\rangle=h|h,\pm\rangle, && (-1)^F|h,\pm\rangle=\pm|h,\pm\rangle,\\
&G_0|h,+\rangle=|h,-\rangle, && G_0|h,-\rangle=\left(h-\tfrac{1}{16}\hat{c}\right)|h,+\rangle.
\end{aligned} \tag{8}$$

The irreducible highest weight representation $L_{\text{ext}}(h,\hat{c})$ of the extended Ramond algebra is generated by the raising operators L_{-n}, G_{-n} acting on the highest weight states $|h,\pm\rangle$.

Every representation $L_{\text{ext}}(h,\hat{c})$ of the extended Ramond algebra is reducible as a representation of the unextended Ramond algebra. The zero modes of the unextended Ramond algebra consist of the commuting generators L_0, G_0 so each irreducible representation has a single highest weight vector $|\lambda\rangle$ satisfying

$$\begin{aligned}
&L_n|\lambda\rangle=0, \qquad G_n|\lambda\rangle=0, \qquad n>0\\
&G_0|\lambda\rangle=\lambda|\lambda\rangle, \qquad L_0|\lambda\rangle=h(\lambda)|\lambda\rangle, \qquad h(\lambda)=\lambda^2+\tfrac{1}{16}\hat{c}.
\end{aligned} \tag{9}$$

Write $L(\lambda,\hat{c})$ for the irreducible highest weight representation generated by the raising operators acting on $|\lambda\rangle$.

For a non-supersymmetric highest weight vector, $\lambda^2=h-\frac{1}{16}\hat{c}\neq 0$, the irreducible representation of the extended algebra decomposes into a direct sum of two irreducibles of the unextended algebra:

$$L_{\text{ext}}(h,\hat{c})=L(\lambda,\hat{c})\oplus L(-\lambda,\hat{c}), \qquad (-1)^F|\pm\lambda\rangle=|\mp\lambda\rangle. \tag{10}$$

The operator R_λ is to be 1 on $L(\lambda,\hat{c})$ and -1 on $L(-\lambda,\hat{c})$. R_λ is constructed by having it act as $\lambda^{-1}G_0$ on the highest weight states $|h, \pm\rangle$, and extending to the descendants by requiring $[R_\lambda, G_n] = [R_\lambda, L_n] = 0$. The projections onto $L(\pm\lambda, \hat{c})$ are then $\frac{1}{2}(1 \pm R_\lambda)$.

The classification of irreducible unitary highest weight representations of the extended Ramond algebra was given in ref. [2, 3]. It is an immediate consequence that the unitary representations of the unextended Ramond algebra are for $\hat{c} \geq 1$, all λ real, and for $c < 1$ the discrete series

$$\hat{c} = \hat{c}(m) = 1 - \frac{8}{m(m+2)}, \qquad m = 2, 3, \ldots$$

$$\lambda = \lambda_{p,q}(m) = \frac{(m+2)p - mq}{\sqrt{8m(m+2)}}, \qquad \begin{array}{l} p = 1, 2, \ldots, m-1 \\ q = 1, 2, \ldots, m+1, \\ p - q \,\text{odd}. \end{array}$$

$$\lambda_{p,q} = -\lambda_{m-p,m+2-q}. \tag{11}$$

For odd values of m there is no state with $\lambda = 0$, so the unitary representations all pair under this symmetry. For m even, all representations again pair except the $\lambda = 0$ "supersymmetric" representation $p = \frac{1}{2}m$, $q = \frac{1}{2}m + 1$.

3. Representations of two Ramond algebras

In superconformal field theories of the type we are discussing there are two superconformal algebras $\{L_n, G_n\}$ and $\{\bar{L}_n, \bar{G}_n\}$. The odd generators of these two algebras anticommute, so the irreducible representations are not necessarily tensor products of irreducible representations of the individual algebras. Every irreducible representation of the two Neveu-Schwarz algebras is a tensor product. But for the two Ramond algebras the tensor product representation decomposes into the direct sum of two representations of opposite chirality.

The zero modes of the two Ramond algebras are generated by G_0 and $\bar{G}_0$,

$$[G_0, \bar{G}_0]_+ = 0. \tag{12}$$

The highest weight states, as irreducible representations of the zero mode algebra, are doublets, excepting the representations with $G_0 = 0$ or $\bar{G}_0 = 0$. The zero mode algebra is realized on the highest weight states by Pauli matrices

$$G_0 = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \bar{G}_0 = \bar{\lambda} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{13}$$

Given G_0 and $\overline{G}_0$ it is always possible to make an operator $(-1)^F$. In fact, on the highest weight states there are two possibilities

$$(-1)^F = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varepsilon = \pm 1. \tag{14}$$

The action on the descendants is then fixed by the (anti)commutation relations (7). The corresponding highest weight representations are called $L^{\varepsilon}(\lambda, \overline{\lambda}, \hat{c})$.

Acting on the highest weight states,

$$(-1)^F i G_0 \overline{G}_0 = \varepsilon \lambda \overline{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

The invariants characterizing these representations are λ^2, $\overline{\lambda}^2$ and $\varepsilon\lambda\overline{\lambda}$. That is,

$$L^{\pm}(\lambda, \overline{\lambda}, \hat{c}) = L^{\pm}(-\lambda, -\overline{\lambda}, \hat{c}) = L^{\mp}(-\lambda, \overline{\lambda}, \hat{c}). \tag{16}$$

Every representation of two Ramond algebras automatically has an operator $(-1)^{\overline{F}}$, and there are two such irreducible representations $L^{\pm}(\lambda, \overline{\lambda}, \hat{c})$ for every pair of weights $h, \overline{h}$. If λ is nonzero and real, which is implied by unitarity (and $G_0 \neq 0$), the invariant of the representation is the chirality $\varepsilon\lambda\overline{\lambda}/|\lambda\overline{\lambda}| = \pm 1$. This is the sign of that representation's contribution to Z_{top}.

In a chiral theory there are two extended Ramond algebras, $(-1)^F$, G_n and $(-1)^{\overline{F}}, \overline{G}_n$. It is useful to regard the irreducible representations of two extended Ramond algebras as the tensor product representations

$$L_{\text{ext}}(h, \overline{h}, \hat{c}) = L_{\text{ext}}(h, \hat{c}) \otimes L_{\text{ext}}(\overline{h}, \hat{c}) \tag{17}$$

of two *commuting* extended Ramond algebras $(-1)^{F_{\rm L}}, G_n^{\rm L}$ and $(-1)^{F_{\rm R}}, G_n^{\rm R}$. The representation on the tensor product is

$$G_n = G_n^{\rm L} \otimes \mathbf{1}, \qquad (-1)^F = (-1)^{F_{\rm L}} \otimes \mathbf{1},$$

$$\overline{G}_n = (-1)^{F_{\rm L}} \otimes G_n^{\rm R}, \qquad (-1)^{\overline{F}} = \mathbf{1} \otimes (-1)^{F_{\rm R}}. \tag{18}$$

The total fermion parity is $(-1)^{F_{\rm tot}} = (-1)^F(-1)^{\overline{F}}$.

$L_{\text{ext}}(h, \overline{h}, \hat{c})$ splits under the two unextended algebras into the direct sum of two irreducible representations of opposite chirality,

$$L_{\text{ext}}(h, \overline{h}, \hat{c}) = L^{+}(\lambda, \overline{\lambda}, \hat{c}) \oplus L^{-}(\lambda, \overline{\lambda}, \hat{c}), \tag{19}$$

where $\lambda^2 = h - \frac{1}{16}\hat{c}$ and $\overline{\lambda}^2 = \overline{h} - \frac{1}{16}\hat{c}$. This can be seen by writing an operator R which is 1 on $L^{+}(\lambda, \overline{\lambda}, \hat{c})$ and -1 on $L^{-}(\lambda, \overline{\lambda}, \hat{c})$. In the tensor product representation

$$R = R^{\mathrm{L}} \otimes i^{-1}(-1)^{F_{\mathrm{R}}} R^{\mathrm{R}}, \tag{20}$$

satisfying

$$R^2 = 1, \qquad \left[R, (-1)^{F_{\mathrm{tot}}}\right]_+ = 0, \qquad [R, G_n] = [R, \overline{G}_n] = 0, \tag{21}$$

and the projections are

$$L^{\pm}(\lambda, \overline{\lambda}, \hat{c}) = \tfrac{1}{2}(1 \pm R) L_{\mathrm{ext}}(h, \hat{c}) \otimes L_{\mathrm{ext}}(\overline{h}, \hat{c}). \tag{22}$$

This can be used to calculate the contribution of $L^{\pm}(\lambda, \overline{\lambda}, \hat{c})$ to Z_{top}:

$$\begin{aligned}
\mathrm{tr}_{L^{\pm}}(-1)^F iG_0\overline{G}_0 q^{G_0^2}\overline{q}^{\overline{G}_0^2} &= \mathrm{tr}_{L_{\mathrm{ext}}}\tfrac{1}{2}(1 \pm R)(-1)^F iG_0\overline{G}_0 q^{G_0^2}\overline{q}^{\overline{G}_0^2} \\
&= \pm\tfrac{1}{2}\mathrm{tr}_{L_{\mathrm{ext}}} R(-1)^F iG_0\overline{G}_0 q^{G_0^2}\overline{q}^{\overline{G}_0^2} \\
&= \pm\tfrac{1}{2}\mathrm{tr}_{L_{\mathrm{ext}}} R^{\mathrm{L}} G_0^{\mathrm{L}} q^{(G_0^{\mathrm{L}})^2} \otimes G_0^{\mathrm{R}} R^{\mathrm{R}} \overline{q}^{(G_0^{\mathrm{R}})^2} \\
&= \pm\tfrac{1}{2}\mathrm{tr}_{L_{\mathrm{ext}}} R^{\mathrm{L}} G_0^{\mathrm{L}} q^{(G_0^{\mathrm{L}})^2} \mathrm{tr}_{L_{\mathrm{ext}}} R^{\mathrm{R}} G_0^{\mathrm{R}} \overline{q}^{(G_0^{\mathrm{R}})^2}.
\end{aligned} \tag{23}$$

Each factor in this product is a super character,

$$\mathrm{tr}_{L_{\mathrm{ext}}} R^{\mathrm{L}} G_0^{\mathrm{L}} q^{(G_0^{\mathrm{L}})^2} = 2\,\mathrm{tr}_{L_{\mathrm{ext}}}\tfrac{1}{2}(1 + R^{\mathrm{R}}) G_0^{\mathrm{R}} \overline{q}^{(G_0^{\mathrm{R}})^2} = 2\,\mathrm{tr}_{\lambda} G_0 q^{G_0^2}, \tag{24}$$

so the contribution of Z_{top} is

$$\mathrm{tr}_{L^{\pm}}(-1)^F iG_0\overline{G}_0 q^{G_0^2}\overline{q}^{\overline{G}_0^2} = \pm 2\,\mathrm{tr}_{\lambda} G_0 q^{G_0^2} \mathrm{tr}_{\overline{\lambda}} G_0 \overline{q}^{G_0^2}. \tag{25}$$

This uses the fact that $\mathrm{tr}((-1)^F G_0 q^{G_0^2}) = 0$ on $L_{\mathrm{ext}}(h, \hat{c})$. The tr_{λ} is the trace on the representation $L(\lambda, \hat{c})$ of one unextended Ramond algebra.

4. The super partition function

The trace of the super evolution operator, eq. (1), is the genus one super partition function on the odd piece of genus one supermoduli space. The super partition function is a function on the supermoduli space of super-Riemann surfaces. In genus one this is the space of super tori. The supermoduli space for each genus consists of two pieces, characterized by the parity of the number of zero modes of the Dirac operator on the surface. For genus one, the even piece is an ordinary space, without odd supermoduli. It is a three-sheeted covering of the moduli space of ordinary genus one Riemann surfaces. The odd piece of genus one supermoduli space is a superspace, with one even and one odd supermodulus, τ, $\hat{\tau}$. The ordinary part of this superspace, where $\hat{\tau}=0$, is exactly the ordinary genus one moduli space. The super partition function on the odd piece of genus one supermoduli space is the trace of the super evolution operator, eq. (1).

The odd piece of supermoduli space is described by the analytic euclidean super time $\tau, \hat{\tau}$ with τ describing an ordinary torus and $\hat{\tau}$ being an odd analytic coordinate of weight $\frac{3}{2}$. The modular transformations are fractional linear transformations of the τ, $\hat{\tau}$, such that the original and transformed coordinates describe the same super torus. The generators of the modular transformations are $\tau, \hat{\tau} \to \tau+1, \hat{\tau}$ and $\tau, \hat{\tau} \to -1/\tau, \tau^{-3/2}\hat{\tau}$. The super partition function of a superconformal field theory is invariant under these transformations, as expressed in eq. (3).

Now decompose the trace (2) into a sum over irreducible representations of the two Ramond algebras. Let the irreducible representations $L^{\pm}(\lambda, \bar{\lambda}, \hat{c})$ occur with multiplicity $N^{\pm}(\lambda, \bar{\lambda})$. The top component of the super partition function can be expanded in the sum over irreducible representations

$$Z_{\mathrm{top}} = \sum_{a,b} 2 I_{a,b} \hat{\chi}^{a}(\tau) \overline{\hat{\chi}^{b}(\tau)}, \tag{26}$$

where

$$\chi^{a}(\tau) = \chi_{\lambda_a}(\tau) = \mathrm{tr}_{\lambda_a} G_0 q^{G_0^2} \tag{27}$$

are the odd super characters and

$$I_{a,b} = N^{+}(\lambda_a, \lambda_b) - N^{-}(\lambda_a, \lambda_b) \tag{28}$$

is a matrix of integers measuring the chiral asymmetry of the representations. The indices a, b range over the inequivalent irreducible representations. In the discrete series a ranges over the allowed pairs p, q with $\lambda_{p,q} > 0$, and b likewise.

This expression is the decomposition of the top component of the super partition function into a sesquilinear form in the odd super characters. In a trivial way the constant bottom component can be regarded as a sesquilinear pairing in the unique nonzero even super character, the constant. The description of the super partition function as a sesquilinear form in even and odd objects generalizes [16] to the supermoduli space of all finite genus super-Riemann surfaces [17, 18].

The absolute multiplicity of $L^{\pm}(\lambda_a, \bar{\lambda}_b, \hat{c})$ disregarding chirality is

$$N_{a,b} = N^{+}(\lambda_a, \lambda_b,) + N^{-}(\lambda_a, \lambda_b). \tag{29}$$

The chirality matrix $I_{a,b}$ satisfies the constraints

$$|I_{a,b}| \leq N_{a,b}, \qquad I_{a,b} \equiv N_{a,b} \qquad (\mathrm{mod}\, 2). \tag{30}$$

For any index a write $-a$ for the index such that $\lambda_{-a} = -\lambda_a$. In the discrete series, if a is p, q then $-a$ is $m-p, m+2-q$. The symmetries are

$$N_{a,b} = N_{-a,-b} = N_{-a,b}, \qquad I_{a,b} = I_{-a,-b} = -I_{-a,b}. \tag{31}$$

The supercharacters $\hat{\chi}^a$ and the chirality matrices $I_{a,b}$ of the discrete series models are considered in the next two sections.

The analogue of Z_{top} for higher loops (surfaces with handles) is the part of the superpartition function which is odd in the left supermoduli $\hat{\tau}_\alpha$ and in the right supermoduli $\bar{\hat{\tau}}_\alpha$ separately. This must vanish in a theory with chiral fermions, since right $(-1)^F$ and left $(-1)^{\bar{F}}$ are separately conserved.

The existence of Z_{top} was due solely to a property of Clifford algebras: an odd number of generators γ^i can form an irreducible representation in two different chiralities, while an even number d of generators can always be multiplied together to form a nontrivial operator γ^{d+1} which anticommutes with all the γ^i, a $(-1)^F$ operator. In Z_{top} there is an odd number (1) of $\hat{\tau}$ supermoduli and an odd number (1) of $\bar{\hat{\tau}}$. The left and right representations give an even number of generators $G_0, \bar{G}_0$ which can always be extended to include $(-1)^F$, but with a choice of chiralities, which is encoded in the chirality matrix $I_{a,b}$. The problem is to check that this choice of chiralities can be made consistently. If the absolute Ramond multiplicities $N_{a,b}$ are all even the zero chirality matrix $I_{a,b} = 0$ is trivially modular invariant. Then there exist chiral $(-1)^F, (-1)^{\bar{F}}$ operators and the representations are all tensor products or representations of the left and right extended Ramond algebras. However, this does not guarantee that the operator product coefficients will have the same chiral symmetry or that a modular invariant mixing of left Ramond and right Neveu-Schwarz algebras exists.

The Ramond sector representations $L^{\pm}(\lambda, \bar{\lambda}, \hat{c})$ also give one branch (of three) of the partition function on the even piece of supermoduli space. This is

$$Z_{+-}(\tau, \bar{\tau}) = \mathrm{tr}_{\mathrm{R}}\, q^{G_0^2} \bar{q}^{\bar{G}_0^2} = \sum_{a,b} 2N_{a,b} \chi^a(q) \overline{\chi^b(q)} \tag{32}$$

a sesquilinear form in the even characters

$$\chi^a(q) = \chi^{-a}(q) = \mathrm{tr}_{L(\lambda_a)} q^{G_0^2} \underset{q \sim 0}{\sim} q^{\lambda_a^2}. \tag{33}$$

The factors of two in Z_{top} and Z_{+-} come from the two-fold degeneracy of the highest weight states of the irreducible representations $L^{\pm}(\lambda, \bar{\lambda}, \hat{c})$.

The branch $Z_{+-}(\tau, \bar{\tau})$ corresponds to periodic boundary conditions in space, antiperiodic in time. It produces under modular transformations the other two branches, Z_{-+} and Z_{--}, corresponding to the other two even boundary conditions. These branches are also given by traces over the representations of the two Neveu-Schwarz algebras (see appendix B.2).

For every superconformal field theory there is a corresponding ordinary conformal field theory – its nonchiral GOS projection – the spin model. The spin model partition function is the sum over both sectors, projecting on $(-1)^F = 1$ in each sector.

$$Z_{\text{spin}} = \tfrac{1}{2}(Z_{++} + Z_{+-}) + \tfrac{1}{2}(Z_{-+} + Z_{--}). \tag{34}$$

Z_{++} is the integer index $\operatorname{tr}_{\mathrm{R}}(-1)^F$.

The partition function on the even piece of supermoduli space has been analyzed in refs. [10, 12] where the absolute multiplicities $N_{a,b}$ for the two Ramond algebras are found.

The various pieces of the super partition function in genus one are all given by the Ramond representations – on the odd part $Z_{\text{sup}}(\tau, \hat{\tau}, \bar{\tau}, \hat{\bar{\tau}})$ is given directly and on the even part analytic continuation of $Z_{+-}(\tau, \bar{\tau})$ gives the rest.

The present paper analyzes the top component of the super partition function on the odd piece of supermoduli space. It gives the chirally asymmetric part $I_{a,b}$ of the multiplicities for the representations of $\lambda \neq 0$, $\bar{\lambda} \neq 0$. It does not deal with the bottom component, the index.

A certain amount is known about the index. If m is odd then the index Z_{++} is zero by unitarity [2]. If m is even then the index is known only to the extent that the absolute multiplicity of the representation $\lambda = \bar{\lambda} = 0$ is known [10, 12] (see appendix B). This multiplicity is congruent to the index modulo two.

5. The super characters

The basic building blocks of the irreducible highest weight representations $L(\lambda, \hat{c})$ of the unextended Ramond algebra are the Verma modules $V(\lambda, \hat{c})$. The Verma module $V(\lambda, \hat{c})$ is the universal representation generated by a highest weight state $|\lambda\rangle, G_0|\lambda\rangle = \lambda|\lambda\rangle$. A basis of linearly independent descendants on level n, the eigenspace $G_0^2 = \lambda^2 + n$, is given by the states

$$G_{-1}^{m_1} \dots G_{-r}^{m_r} L_{-1}^{l_1} \dots L_{-s}^{l_s} |\lambda\rangle, \qquad m_r = 0, 1;\ l_s = 0, 1, 2, \dots, \tag{35}$$

with $n = \Sigma_r r m_r + \Sigma_s s l_s$.

The dimension of level n of $V(\lambda, \hat{c})$ is written $P_{\mathrm{R}}(n)$. The character of the Verma module is

$$\chi_{V(\lambda)}(\tau) = \sum_{n=0}^{\infty} q^{\lambda^2 + n} P_{\mathrm{R}}(n) = q^{\lambda^2} \prod_{k=1}^{\infty} \frac{1 + q^k}{1 - q^k}. \tag{36}$$

The highest weight space is one-dimensional, $P_R(0)=1$, but for $n>0$ level n is even dimensional, $P_R(n)$ is even.

The super character of a Verma module is trivial to calculate. G_0 is linear in the free variable λ as a matrix on the states (35). So if $n>0$ the G_0 eigenvalues must occur in pairs $\pm\sqrt{\lambda^2+n}$ of opposite sign and must cancel in the trace (5). The super character of the nondegenerate representation is then given entirely by the contribution of the highest weight state,

$$\hat{\chi}_{V(\lambda)}(\tau)=\lambda q^{\lambda^2}. \tag{37}$$

For $\hat{c}\geq 1$ the unitary representations $L(\lambda,\hat{c})$ are those with λ real. All of the states 35 are then linearly independent, the Verma module $V(\lambda,\hat{c})$ is itself irreducible and $L(\lambda,\hat{c})=V(\lambda,\hat{c})$. The super partition function then takes the form

$$Z_{\text{top}}(\tau,\bar{\tau})=\sum_{a,b}2I_{a,b}\lambda_a q^{\lambda_a^2}\overline{\lambda_b q^{\lambda_b^2}}. \tag{38}$$

If the result is to be a modular invariant, the number of irreducibles a,b will have to be infinite. It should be enough, however, to consider only the "rational" case – when $N_{a,b}$ and $I_{a,b}$ have finite rank [19,20] – as is always true if $\hat{c}<1$.

For $\hat{c}<1$ the unitary representations are degenerate, meaning that some of the states (35) are linearly dependent $L(\lambda,\hat{c})$ is the quotient of $V(\lambda,\hat{c})$ by the subspace of the null states. The null states are generated by two highest weight vectors in the Verma module. Each of these generates a sub-Verma module inside the original Verma module, which together span the null states. The intersection of the two sub-Verma modules is generated by two highest weight vectors which are *not* on vanishing curves of the original λ. But the intersection of *their* sub-Verma modules is generated by two highest weight vectors on vanishing curves of the original λ. The pattern repeats *ad infinitum*.

To calculate the super character, start with the super character of the Verma module, subtract the super characters of the first two sub-Verma modules, correct for the double subtraction by adding the super characters of the second pair of sub-Verma modules, and continue in an infinite alternating sum.

This nesting picture of alternating pairs of sub-Verma modules is exhibited in the ordinary character formula [18,21]

$$\chi^a(\tau)=\sum_{n=-\infty}^{\infty}q^{\lambda_{p,q,n}^2}-\sum_{n=-\infty}^{\infty}q^{\tilde{\lambda}_{p,q,n}^2} \tag{39}$$

where the weights – G_0 eigenvalues – are given up to sign by

$$\lambda_{p,q,n}=\pm\frac{(m+2)p-mq+nN}{2\sqrt{N}},\qquad \tilde{\lambda}_{p,q,n}=\pm\frac{(m+2)p+mq+nN}{2\sqrt{N}} \tag{40}$$

with $N = 2m(m+2)$. The lowest order term in the q-expansion for the character χ^a comes from the first sum in eq. (39), and corresponds to the original highest weight state. The next two terms come from the second sum and are subtracted, corresponding to the first two highest weight null vectors; the next two terms come from the first sum and so on.

To calculate the super characters it is necessary to determine which branch $\pm\sqrt{\lambda^2 + n}$ is taken by G_0 on each highest weight vector. The arguments which give these choices are summarized in appendix A. The result is

$$\hat{\chi}^a(\tau) = \hat{\chi}^{p,q} = \sum_{n=-\infty}^{\infty} \lambda_{p,q,n} q^{\lambda^2_{p,q,n}} - \sum_{n=-\infty}^{\infty} \tilde{\lambda}_{p,q,n} q^{\tilde{\lambda}^2_{p,q,n}}, \tag{41}$$

with the weights

$$\lambda_{p,q,n} = (-1)^{nm} \frac{(m+2)p - mq + nN}{2\sqrt{N}},$$

$$\tilde{\lambda}_{p,q,n} = (-1)^{q+nm} \frac{(m+2)p + mq + nN}{2\sqrt{N}}. \tag{42}$$

Symmetries of the super characters are then

$$\hat{\chi}_{p,q} = -\hat{\chi}_{m-p,m+2-q} = (-1)^m \hat{\chi}_{p+m,q-m-2} = (-1)^{q+1} \hat{\chi}_{p,-q} = -\hat{\chi}_{-p,-q}. \tag{43}$$

6. The modular invariance condition

In this section the modular transformation properties of the super characters are described as those of higher level, weight $\frac{3}{2}$ modular forms. The absolute multiplicities $N_{a,b}$ were calculated by imposing the modular invariance of Z_{spin} [10, 12] and are summarized in appendix B. Here the modular invariance of Z_{top} is imposed as a constraint on the chiral asymmetry $I_{a,b}$. Solutions were found for all of the known sets of absolute multiplicities $N_{a,b}$. These are presented in appendix B.

It is only possible to have chiral symmetry, $I_{a,b} = 0$, if all the absolute multiplicities N_{ab} are even. This happens in only two examples on the $\hat{c} < 1$ discrete series list (appendix B). These are at $m = 10$ and 12 (corresponding to the E_6 lattice).

Two extended Ramond algebras act on the Hilbert space of a model if and only if $I_{a,b} = 0$. This is because the tensor product representation is exactly two irreducibles of opposite chirality, eq. (10). But $I_{a,b} = 0$ is not sufficient to guarantee a chiral theory since the operator product coefficients must also respect both symmetries, $(-1)^F$ and $(-1)^{\bar{F}}$. The chirality condition on the operator product coefficients amounts to the condition that no odd "generalized characters" participate in the partition function at genus > 1. And in genus one every chiral theory has a chiral

GOS projection and must have a modular invariant partition function composed as a sesquilinear forms in the Ramond and Neveu-Schwarz characters $\chi_\lambda(\tau)$ and $\chi_h^{\pm}(\tau)$.

The modular transformations of the characters $\hat{\chi}^a(\tau)$, $a = p, q$ are

$$\hat{\chi}^a(\tau+1) = e^{2\pi i\lambda_a^2}\hat{\chi}^a(\tau),$$

$$\hat{\chi}^a(-1/\tau) = (-i\tau)^{3/2}\frac{2}{\sqrt{m(m+2)}}\sum_{b,-b} e^{4\pi i\lambda_a\lambda_b}\hat{\chi}^b(\tau). \tag{44}$$

The sum over b, $-b$ is meant to be over two of each irreducible, i.e. over all $p-q$ odd, $1 \le p < m$, $1 \le q < m+2$, $\lambda_{p,q} > 0$ and $\lambda_{p,q} < 0$. These super characters are explicitly modular forms of weight $\frac{3}{2}$.

The symmetries of the $\hat{\chi}^a$ allow the modular transformation equation (44) to be rewritten

$$\hat{\chi}^a(-1/\tau) = (-i\tau)^{-3/2}\sum_{a',-a'} S^a_{a'}\hat{\chi}^{a'}(\tau), \tag{45}$$

with transformation matrix

$$S^a_{a'} = \frac{2i}{\sqrt{m(m+2)}}(-1)^{((p-q)(p'-q')-1)/2}\sin\left(\frac{\pi pp'}{m}\right)\sin\left(\frac{\pi qq'}{m+2}\right). \tag{46}$$

Symmetries of $S^a_{a'}$ are

$$S^a_{a'} = S^{-a}_{-a'} = -S^{-a}_{a'}. \tag{47}$$

Modular invariance of the super partition function as decomposed into super characters in eq. (26) becomes modular invariance of the sesquilinear form $I_{a,b}$

$$I_{a',b'} = \sum_{a,-a,b,-b} S^a_{a'} I_{a,b}\overline{S^b_{b'}}. \tag{48}$$

This is the only new genus one modular invariance constraint because $\tau \to \tau+1$ invariance follows from the conditions (30) and invariance of the ordinary partition function Z_{spin}.

For each of the previously known absolute multiplicity matrices $N_{a,b}$ in the discrete series (appendix B) exactly one modular invariant chiral asymmetry matrix $I_{a,b}$ satisfying the conditions (30) has been found. It seems plausible that no other possibilities exist. The chiral asymmetries $I_{a,b}$ for the various infinite series and exceptional models are given in appendix B.

7. Conclusion

We have seen that representations of two Ramond algebras, as occur in superconformal field theories in two dimensions, are labelled by a chirality ± 1 which is the sign of $(-1)^F iG_0\bar{G}_0$ on the highest weight vectors. The top component of the super partition function in genus one, $Z_{\text{top}}(\tau, \bar{\tau})$, detects the net chiral asymmetry of the model in each representation. It can be written as a sesquilinear form $I_{a,b}$ in the odd super characters of the Ramond algebra. The modular transformation law of the super characters puts constraints on the chiral asymmetry matrix. It is possible to find a solution of the modular invariance constraint for each of the models in the discrete series.

Chiral asymmetry is an obstruction to writing separate left and right chiral fermion parity operators $(-1)^F$, $(-1)^{\bar{F}}$. Thus no superconformal model with chiral asymmetry can be a string compactification. Additional obstructions arise in higher genus. A superconformal model has a chiral symmetry only if no odd super characters contribute to its partition function for all values of the genus.

The matrix $I_{a,b}$ provides detailed information on the chiral asymmetry of a superconformal field theory. It would be interesting to find applications for this information.

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Appendix A

CALCULATION OF SUPER CHARACTERS

The ordinary character and the odd super character of the irreducible representation $L(\lambda, \hat{c})$ can be calculated as the character of the Verma module $V(\lambda, \hat{c})$ with the contributions from the null states subtracted away. The submodule of null states is generated by highest weight states, themselves generating degenerate representations. The super character requires knowledge of the G_0 eigenvalues on this series of highest weight vectors in Verma modules. Fortunately, most of this work has been done elsewhere, in that the determinant formula [1, 8, 21, 22, 23] and character formulas [3, 22, 8] for the extended algebra have been proved following the treatment [7] of the Virasoro algebra. The construction of the null states and their G_0 eigenvalues can be obtained with the Feigin Fuchs [7] representation of the superconformal null vectors in terms of vertex operators [8, 21]. The supersymmetric anomalous GL_1 current is used to construct null states which saturate the bounds for the total number of null states. The same construction can be used to calculate the null states for the unextended algebra. Here we give the numerology of the construction. The arguments are all to be found in the treatments of the unextended algebra [8,21].

Define $M_n(\lambda, \hat{c})$ to be the inner product matrix for the $P_R(n)$ states (35) of level n in the Verma module $V(\lambda, \hat{c})$. The determinant formula for the unextended algebra is simply the square root of the determinant formula for the extended algebra:

$$\det M_n(\lambda, \hat{c}) = \prod_{pq/2 \le n} \left(\lambda^2 - \lambda_{p,q}(\hat{c})^2\right)^{P_R(n-pq/2)}. \tag{A.1}$$

This is seen by calculating the inner product matrix for the Verma module of the extended algebra starting at level zero with a basis of eigenstates of G_0. The matrix of inner products is block diagonal in this basis, and each block is the inner product matrix for the unextended algebra.

The determinant and character formulas are derived from the theory of an anomalous super current. This is the theory of one free super scalar $\Phi = \phi + \theta\psi$ with current $J(z,\theta) = D\Phi = \psi(z) + \theta\partial\phi(z)$ and super stress-energy tensor $T = \frac{1}{4}JDJ - N^{-1/2}DJ$. The central charge is $\hat{c} = 1 - 16/N$, agreeing with the notation $N = 2m(m+2)$ previously used for the discrete series. In previous applications the super current algebra was extended by $(-1)^F$ but here it is left unextended.

The weight of the superconformal vertex operator of charge k, $e^{k\Phi} = e^{k\phi} + \theta\psi e^{k\phi}$, is $k(k + 2N^{-1/2})$. In Ramond representations the highest weight states $|\lambda\rangle$ of the unextended super current algebra are characterized by G_0 eigenvalue λ and by charge $\lambda - N^{-1/2}$. The charge zero highest weight state $|N^{-1/2}\rangle$ has weight $h = \frac{1}{16}$, which is just the zero point energy of the weight $\frac{1}{2}$ fermionic component of the super current. The charged state $|\lambda\rangle$ is created by acting with the charge $k = \lambda - N^{-1/2}$ vertex operator on $|N^{-1/2}\rangle$. The formulas for the conformal weight of $|\lambda\rangle$ agree: $k(k + 2N^{-1/2}) + \frac{1}{16} = \lambda^2 + \frac{1}{16}\hat{c}$.

The screening operators $Y_\pm$ are the super contour integrals of the two superconformal vertex operators of weight $\frac{1}{2}$. Their charges are $k_+ = N^{-1/2}m$, $k_- = -N^{-1/2}(m+2)$. The super contour integral of a weight $\frac{1}{2}$ superconformal field is a superconformal invariant. Explicitly it is the ordinary contour integral of the top component $\psi e^{k_\pm\phi}$. This is fermionic so the screening operators $Y_\pm$ anticommute with the G_n.

The current algebra acting on $|\lambda\rangle$ generates a representation which is identical to the Verma module $V(\lambda, \hat{c})$ of the Ramond algebra. States in the Verma module can be represented by screening operators acting on the highest weight states λ. In particular, let

$$|p,q,n\rangle = Y_+^{r(m+2)} Y_-^{sm} |\lambda_{p+nm,\, q-n(m+2)}\rangle, \tag{A.2}$$

with $n = r - s$. The charge of $|r,s,p,q\rangle$ is the same as the charge of $|\lambda_{p,q}\rangle$.

Because G_0 anticommutes with the screening charges G_0 has eigenvalue $\lambda_{p,q,n}$ given in eq. (42). The states $|n, p, q\rangle$ are exactly the highest weight states which contribute to the first sum in the super character formula (41).

Similarly, the state

$$|p,q,n\rangle' = Y_+^{q+r(m+2)} Y_-^{sm} |\tilde{\lambda}_{p+nm,-q-n(m+2)}\rangle$$
$$= Y_+^{r(m+2)} Y_-^{p+sm} |\tilde{\lambda}_{-p+nm,q-n(m+2)}\rangle, \tag{A.3}$$

with $n = r - s$ has the same charge as $|\lambda_{p,q}\rangle$ and has G_0 eigenvalue $\tilde{\lambda}_{p,q,n}$ given in formula (42). The states $|p, q, n\rangle'$ are exactly the highest weight states whose contributions are subtracted out in the super character formula (41). The relative levels of these states are given by the formula

$$\tilde{\lambda}^2_{p,q,n'} - \lambda^2_{p,q,n} = \lambda^2_{p+rm,-q-s(m+2)} - \lambda^2_{p+rm,q+s(m+2)} = \tfrac{1}{2}(p+rm)(q+s(m+2)), \tag{A.4}$$

where $r = n' + n$ and $s = n' - n$.

Appendix B

MULTIPLICITIES AND CHIRAL ASYMMETRY

This appendix summarizes conventions, give the absolute multiplicities $N_{a,b}$ for the conjectured complete list of discrete series models [10,12] and gives a modular invariant chiral asymmetry matrix $I_{a,b}$ for each model. There is no uniqueness result – these are not necessarily the only possible modular invariant $I_{a,b}$.

B.1. NOTATION

The characters and super characters are

$$\chi^a_{--}(\tau) = \mathrm{tr}_{\mathrm{NS}} q^{L_0 - \hat{c}/16} \underset{q\sim 0}{\sim} q^{h_a - \hat{c}/16},$$
$$\chi^a_{-+}(\tau) = \mathrm{tr}_{\mathrm{NS}} (-1)^F q^{L_0 - \hat{c}/16} \underset{q\sim 0}{\sim} q^{h_a - \hat{c}/16},$$
$$\chi^a_{+-}(\tau) = \mathrm{tr}_{\mathrm{R}} q^{G_0^2} \underset{q\sim 0}{\sim} q^{\lambda_a^2},$$
$$\hat{\chi}^a(\tau) = \mathrm{tr}_{\mathrm{R}} G_0 q^{G_0^2} \underset{q\sim 0}{\sim} \lambda_a q^{\lambda_a^2}. \tag{B.1}$$

The index a labelling the representations stands for a pair p, q of positive integers,

$p<m$, $q<m+2$ and $p-q$ is even for Neveu-Schwarz representations, odd for Ramond representations. The index $-a$ stands for $m-p$, $m+2-q$. The highest weights are

$$\lambda_a=\lambda_{p,q}(m)=\frac{(m+2)p-mq}{\sqrt{8m(m+2)}} \tag{B.2}$$

for the Ramond algebra and

$$h_a=h_{p,q}(m)=\frac{\frac{1}{4}((m+2)p-mq)^2-1}{2m(m+2)}, \tag{B.3}$$

for the Neveu-Schwarz algebra.

The partition function of the spin model is

$$Z_{\text{spin}}=\tfrac{1}{2}(Z_{++}+Z_{+-})+\tfrac{1}{2}(Z_{-+}+Z_{--}). \tag{B.4}$$

The super partition function on the odd part of genus one supermoduli space is

$$Z_{\text{sup}}=Z_{++}(\tau,\bar{\tau})+4\pi^2 i\hat{\tau}\hat{\bar{\tau}}Z_{\text{top}}(\tau,\bar{\tau}), \tag{B.5}$$

where the bottom component Z_{++} is the index and the top component is

$$Z_{\text{top}}(\tau,\bar{\tau})=\sum_{a,\bar{a}}2I_{a,\bar{a}}\hat{\chi}^a(\tau)\overline{^{\bar{a}}\hat{\chi}^a(\tau)}. \tag{B.6}$$

The indices $a,\bar{a}$ stand for p,q and $\bar{p},\bar{q}$.

The superpartition function on the even piece of supermoduli space has three branches:

$$Z_{+-}(\tau,\bar{\tau})=\sum_{a,\bar{a}}2N_{a,\bar{a}}\chi^a_{+-}(\tau)\overline{\chi^{\bar{a}}_{+-}(\tau)},$$

$$Z_{--}(\tau,\bar{\tau})=\sum_{a,\bar{a}}N^{\text{NS}}_{a,\bar{a}}\chi^a_{--}(\tau)\overline{\chi^{\bar{a}}_{--}(\tau)},$$

$$Z_{-+}(\tau,\bar{\tau})=\sum_{a,\bar{a}}N^{\text{NS}}_{a,\bar{a}}\chi^a_{-+}(\tau)\overline{\chi^{\bar{a}}_{-+}(\tau)}. \tag{B.7}$$

$N^{\text{NS}}_{a,\bar{a}}$ is the multiplicity of the Neveu-Schwarz representation with $h=h_a$, $\bar{h}=h_{\bar{a}}$. The sums in the partition function are over the irreducible representations, each counted once. Only one of a and $-a$ is included in each sum. If the sum is expanded to cover both a and $-a$ then a factor of $\frac{1}{2}$ should be included for each index.

B.2 MULTIPLICITIES

In the discrete series there are three infinite series of models [10] and eight exceptional models [12], corresponding to pairs of root diagrams of simply laced Lie algebras [12]. The *scalar* series corresponds to the diagrams (A_{m-1}, A_{m+1}), the *alternate* series to $(A_{m-1}, D_{\frac{1}{2}m+2})$ and the spin series to $(D_{\frac{1}{2}m+1}, A_{m+1})$, and the same naming for the series where $m \to m+2$. The eight exceptional models correspond to

$$
\begin{aligned}
&\left(A_{m \mp 1}, E_6\right), && m = 10, 12, \\
&\left(D_{\frac{1}{2}m+1,2}, E_6\right), && m = 10, 12, \\
&\left(A_{m \mp 1}, E_7\right), && m = 16, 18, \\
&\left(A_{m \mp 1}, E_8\right), && m = 28, 30.
\end{aligned}
\tag{B.8}
$$

Except for the models $(E_6, D_{\frac{1}{2}m+1,2})$ at $m = 10, 12$ the Neveu-Schwarz multiplicity $N^{\mathrm{NS}}_{a,\bar{a}}$ is given by the same expression in $p, q, \bar{p}, \bar{q}$ as the Ramond multiplicity $N_{a,\bar{a}}$.

The scalar series is

$$
m \geq 3, \qquad N_{a,\bar{a}} = \delta_{p,\bar{p}}\delta_{q,\bar{q}} + \delta_{p+\bar{p},m}\delta_{q+\bar{q},m+2}\,. \tag{B.9}
$$

The alternate series is

$$
\begin{aligned}
&m = 4k \geq 4, && N_{a,\bar{a}} = \left(\delta_{p,\bar{p}} + \delta_{p+\bar{p},m}\right)\left(\delta_{q,\bar{q}} + \delta_{q+\bar{q},m+2}\right), && q \text{ odd}, \\
&m = 4k+2 \geq 6, && N_{a,\bar{a}} = \left(\delta_{p,\bar{p}} + \delta_{p+\bar{p},m}\right)\left(\delta_{q,\bar{q}} + \delta_{q+\bar{q},m+2}\right), && p \text{ odd}.
\end{aligned}
\tag{B.10}
$$

The spin series is

$$
\begin{aligned}
&m = 4k \geq 8, && N_{a,\bar{a}} = \begin{cases} \delta_{p,\bar{p}}\delta_{q,\bar{q}} + \delta_{p+\bar{p},m}\delta_{q+\bar{q},m+2}\,, & p \text{ odd} \\ \delta_{p+\bar{p},m}\delta_{q,\bar{q}} + \delta_{p,\bar{p}}\delta_{q+\bar{q},m+2}\,, & p \text{ even} \end{cases} \\
&m = 4k+2 \geq 6, && N_{a,\bar{a}} = \begin{cases} \delta_{p,\bar{p}}\delta_{q,\bar{q}} + \delta_{p+\bar{p},m}\delta_{q+\bar{q},m+2}\,, & q \text{ odd} \\ \delta_{p+\bar{p},m}\delta_{q,\bar{q}} + \delta_{p,\bar{p}}\delta_{q+\bar{q},m+2}\,, & q \text{ even}. \end{cases}
\end{aligned}
\tag{B.11}
$$

The four pairs of exceptional models have a combination of delta functions in either p or q related to the exceptional groups $E_{6,7,8}$. In each case there is a second model obtained by $p, \bar{p} \leftrightarrow q, \bar{q}$, $m \to m+2$.

For $(A_{m \mp 1}, E_6)$, the $m = 10$ model is

$$
\begin{aligned}
N_{a,\bar{a}} = {} & \delta_{p+\bar{p},m}\left[\left(\delta_{q,5} + \delta_{q,11}\right)\left(\delta_{\bar{q},1} + \delta_{\bar{q},7}\right) + \left(\delta_{q,1} + \delta_{q,7}\right)\left(\delta_{\bar{q},5} + \delta_{\bar{q},11}\right)\right] \\
& + \delta_{p,\bar{p}}\left[\left(\delta_{q,1} + \delta_{q,7}\right)\left(\delta_{\bar{q},1} + \delta_{\bar{q},7}\right) + \left(\delta_{q,5} + \delta_{q,11}\right)\left(\delta_{\bar{q},5} + \delta_{\bar{q},11}\right)\right] \\
& + \left(\delta_{p,\bar{p}} + \delta_{p+\bar{p},m}\right)\left(\delta_{q,4} + \delta_{q,8}\right)\left(\delta_{\bar{q},4} + \delta_{\bar{q},8}\right)
\end{aligned}
\tag{B.12}
$$

and the $m=12$ model is obtained by exchanging $p, \bar{p} \leftrightarrow q, \bar{q}$.

In the $(\mathrm{D}_{\frac{1}{2}m+1,2}\,\mathrm{E}_6)$, models, $N_{a,\bar{a}}$ and $N^{\mathrm{NS}}_{a,\bar{a}}$ are different. For $m=10$,

$$N_{a,\bar{a}} = 2(\delta_{p,\bar{p}} + \delta_{p+\bar{p},m})(\delta_{q,4} + \delta_{q,8})(\delta_{\bar{q},4} + \delta_{\bar{q},8}), \tag{B.13}$$

$$N^{\mathrm{NS}}_{a,\bar{a}} = (\delta_{p,\bar{p}} + \delta_{p+\bar{p},m})(\delta_{q,1} + \delta_{q,7} + \delta_{q,5} + \delta_{q,11})(\delta_{\bar{q},1} + \delta_{\bar{q},7} + \delta_{\bar{q},5} + \delta_{\bar{q},11}). \tag{B.14}$$

The $m=12$ model is given by exchanging $p, \bar{p} \leftrightarrow q, \bar{q}$. These two are the only models with even multiplicities in the Ramond sector, so the only ones where $I_{a,\bar{a}} = 0$ is possible.

For $(A_{m\mp1}, \mathrm{E}_7)$, the $m=16$ model is

$$N_{a,\bar{a}} = (\delta_{p,\bar{p}} + \delta_{p+\bar{p},m})\begin{pmatrix} (\delta_{q,1} + \delta_{q,17})(\delta_{\bar{q},1} + \delta_{\bar{q},17}) + (\delta_{q,3} + \delta_{q,15})\delta_{\bar{q},9} \\ \quad + (\delta_{q,7} + \delta_{q,11})(\delta_{\bar{q},7} + \delta_{\bar{q},11}) + (\delta_{q,9}\delta_{\bar{q},9}) \\ \quad + (\delta_{q,5} + \delta_{q,13})(\delta_{\bar{q},5} + \delta_{\bar{q},13}) + (\delta_{\bar{q},3} + \delta_{\bar{q},15})\delta_{q,9} \end{pmatrix} \tag{B.15}$$

and the $m=18$ model is obtained by exchanging $p, \bar{p} \leftrightarrow q, \bar{q}$.

The (A_{m-1}, E_8), $m=28$ model is

$$\begin{aligned} N_{a,\bar{a}} = (\delta_{p,\bar{p}} + \delta_{p+\bar{p},m})\big[&(\delta_{q,1} + \delta_{q,11} + \delta_{q,19} + \delta_{q,29})(\delta_{\bar{q},1} + \delta_{\bar{q},11} + \delta_{\bar{q},19} + \delta_{\bar{q},29}) \\ &+ (\delta_{q,7} + \delta_{q,13} + \delta_{q,17} + \delta_{q,23})(\delta_{\bar{q},7} + \delta_{\bar{q},13} + \delta_{\bar{q},17} + \delta_{\bar{q},23})\big] \end{aligned} \tag{B.16}$$

and the $m=30$ model is obtained by exchanging $p, \bar{p} \leftrightarrow q, \bar{q}$ and taking $m \to m+2$.

B.3 CHIRALITIES

Chiral asymmetry matrices $I_{a,\bar{a}}$ are given here for the discrete series models whose absolute multiplicities were given in the previous section.

The scalar series has

$$m \geq 3, \qquad I_{a,\bar{a}} = \delta_{p,\bar{p}}\delta_{q,\bar{q}} - \delta_{p+\bar{p},m}\delta_{q+\bar{q},m+2}. \tag{B.17}$$

The alternate series has

$$\begin{aligned} m = 4k \geq 4, \qquad & I_{a,\bar{a}} = (\delta_{p,\bar{p}} - \delta_{p+\bar{p},m})(\delta_{q,\bar{q}} + \delta_{q+\bar{q},m+2}), \qquad q \text{ odd} \\ m = 4k+2 \geq 6, \qquad & I_{a,\bar{a}} = (\delta_{p,\bar{p}} + \delta_{p+\bar{p},m})(\delta_{q,\bar{q}} - \delta_{q+\bar{q},m+2}), \qquad p \text{ odd}. \end{aligned} \tag{B.18}$$

The spin series has

$$m=4k\geq 8, \qquad I_{a,\bar a}=\begin{cases}\delta_{p,\bar p}\delta_{q,\bar q}-\delta_{p+\bar p,m}\delta_{q+\bar q,m+2}, & p\text{ odd}\\ \delta_{p+\bar p,m}\delta_{q,\bar q}-\delta_{p,\bar p}\delta_{q+\bar q,m+2}, & p\text{ even}\end{cases}$$

$$m=4k+2\geq 6, \qquad I_{a,\bar a}=\begin{cases}\delta_{p,\bar p}\delta_{q,\bar q}-\delta_{p+\bar p,m}\delta_{q+\bar q,m+2}, & q\text{ odd}\\ -\delta_{p+\bar p,m}\delta_{q,\bar q}+\delta_{p,\bar p}\delta_{q+\bar q,m+2}, & q\text{ even}.\end{cases} \tag{B.19}$$

For $(\mathrm{A}_{m\mp1},\mathrm{E}_6)$, the $m=10$ model has

$$\begin{aligned}I_{a,\bar a}=\delta_{p+\bar p,m}&\left[\left(\delta_{q,5}-\delta_{q,11}\right)\left(\delta_{\bar q,1}-\delta_{\bar q,7}\right)+\left(\delta_{q,1}-\delta_{q,7}\right)\left(\delta_{\bar q,5}-\delta_{\bar q,11}\right)\right]\\ &+\delta_{p,\bar p}\left[\left(\delta_{q,1}-\delta_{q,7}\right)\left(\delta_{\bar q,1}-\delta_{\bar q,7}\right)+\left(\delta_{q,5}-\delta_{q,11}\right)\left(\delta_{\bar q,5}-\delta_{\bar q,11}\right)\right]\\ &+\left(\delta_{p,\bar p}-\delta_{p+\bar p,m}\right)\left(\delta_{q,4}+\delta_{q,8}\right)\left(\delta_{\bar q,4}+\delta_{\bar q,8}\right)\end{aligned} \tag{B.20}$$

and for $m=12$ exchange $p,\bar p\leftrightarrow q,\bar q$.

The $(D_{\frac12 m+1,2},\mathrm{E}_6)$, $m=10,12$ models have

$$I_{a,\bar a}=0. \tag{B.21}$$

For $(\mathrm{A}_{m\mp1},\mathrm{E}_7)$, the $m=16$ model has

$$I_{a,\bar a}=\left(\delta_{p,\bar p}-\delta_{p+\bar p,m}\right)\left(\begin{aligned}&\left(\delta_{q,1}+\delta_{q,17}\right)\left(\delta_{\bar q,1}+\delta_{\bar q,17}\right)-\left(\theta_{q,3}+\delta_{q,15}\right)\delta_{\bar q,9}\\ &\quad+\left(\delta_{q,7}+\delta_{q,11}\right)\left(\delta_{\bar q,7}+\delta_{\bar q,11}\right)+\left(\delta_{q,9}\delta_{\bar q,9}\right)\\ &\quad+\left(\delta_{q,5}+\delta_{q,13}\right)\left(\delta_{\bar q,5}+\delta_{\bar q,13}\right)-\left(\delta_{\bar q,3}+\delta_{\bar q,15}\right)\delta_{q,9}\end{aligned}\right) \tag{B.22}$$

and for the $m=18$ model exchange $p,\bar p\leftrightarrow q,\bar q$.

For $(\mathrm{A}_{m\mp1},\mathrm{E}_8)$, the $m=28$ model has

$$\begin{aligned}I_{a,b}=\left(\delta_{p,\bar p}-\delta_{p+\bar p,m}\right)&\left[\left(\delta_{q,1}-\delta_{q,11}-\delta_{q,19}+\delta_{q,29}\right)\left(\delta_{\bar q,1}-\delta_{\bar q,11}-\delta_{\bar q,19}+\delta_{\bar q,29}\right)\right.\\ &\left.+\left(\delta_{q,7}-\delta_{q,13}-\delta_{q,17}+\delta_{q,23}\right)\left(\delta_{\bar q,7}-\delta_{\bar q,13}-\delta_{\bar q,17}+\delta_{\bar q,23}\right)\right]\end{aligned} \tag{B.23}$$

and for the $m=30$ model exchange $p,\bar p\leftrightarrow q,\bar q$.

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