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THE CONFORMAL FIELD THEORY OF ORBIFOLDS

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A prescription for the calculation of any correlation function in orbifold conformal field theory is given. The method is applied to the scattering of four twisted string states, which allows the extraction of operator product coefficients of conformal twist fields. We derive Yukawa couplings in the effective field theory for fermionic strings on orbifolds.

1. Introduction

The classical equation of motion of string [1] is equivalent to the condition that the two-dimensional quantum field theory (QFT) describing spacetime be conformally invariant [2, 3]. The search for solutions becomes the classification of all possible 2d conformal QFT's [4]. Handicapped by our lack to date of effective

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methods for solving the bootstrap equations, we know very few conformal field theories explicitly [4–10], although we know of the existence of large classes [11] (nonlinear models on Calabi-Yau manifolds). Those we can solve exactly – realizations of affine Lie algebras [8, 9, 12], the unitary discrete series [7], and orbifolds [13] – all have special simplifying features. Until effective methods are developed to solve the general conformal field theory, it is worthwhile to explore the possibilities in those we already know. Of these, the class of orbifold conformal field theories seems to offer the greatest potential for producing a phenomenologically acceptable string background. An orbifold is the quotient of a manifold $\mathscr{M}$ (typically a torus) by the action of a discrete group P whose action leaves fixed a number of points on $\mathscr{M}$. At the fixed points the manifold has conical singularities, but this makes no difference to the string. Here it is our aim to describe the orbifold conformal field theory – its spectrum of fields and their anomalous dimensions, operator products, and correlation functions. The string functional integral is a sum over maps from 2d parameter space into the orbifold. Over most of the orbifold, string propagation is essentially like that in flat spacetime. If the string world-sheet passes through one of the fixed points, the map is locally branched over the covering space $\mathscr{M}$. These "twisted" strings have interesting properties, and we will be especially interested in seeing how to work with them.

The plan is as follows: in sect. 2, we present a general discussion of free field theory and the *twist fields* that create twisted states of string in both the bosonic and fermionic string theories, and we outline a method of solving for their correlations in terms of the Green function of the string coordinate fields in the presence of twists. The local behavior of the string coordinates in the neighborhood of these twist fields is worked out. In sect. 3 the global properties of the Green functions are related to the structure of the group of identifications which defines the orbifold. Bosonic twist correlation functions are then computed in sect. 4 and factorized to exhibit the operator product coefficients of the product of two twists. In sect. 5 these results are extended to the fermionic string; here the operator product coefficients can be identified with the Yukawa couplings of certain massless states in the compactified string theory. Certain recent results on the violation of conformal invariance of chiral sigma models by instantons [14] are shown to agree with exact calculations in the orbifold limit. Finally, in sect. 6 we present our conclusions.

2. Spin fields and twist fields

2.1. CONFORMAL FIELD THEORY

The general conformal field theory is made up of a set of conformal fields [15, 4] (creating highest weight vectors of the 2d conformal algebra), whose operator products form a closed algebra. For the application to string theory, one is interested in those subalgebras of the theory which are *local*, i.e. such that the operator products are single valued. In this case one can construct well-defined correlation functions on any 2d surface, barring global anomalies. On non-simply-connected world-sheets, additional restrictions come from the requirement of single-valuedness of correlations as a function of the moduli (shapes) of surfaces. The local fields create asymptotic states of string; the correlation functions yield string scattering amplitudes. The conformal fields ϕ are distinguished by their operator product with the stress-energy tensor $T(z)$, the generator of 2d conformal transformations, which has the form

$$T(z)\phi(w,\bar{w}) = \frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\bar{w})}{z-w} + \cdots, \tag{2.1}$$

where h is the conformal weight (scaling dimension) of ϕ. Non-highest weight fields have higher order singularities in their operator product with T. The field $\phi(w,\bar{w})$ has a similar operator product with the antianalytic stress tensor $\bar{T}(\bar{z})$, defining the conformal weight $\bar{h}$.

The simplest conformal field theories are those of free bosons and fermions. For a boson, one typically considers the set of conformal fields consisting of the identity operator, the field $\partial X(z)$, and its exponentials e^{ipX}. We take the field $X(z,\bar{z})$ to parametrize a circle of radius R; then the set of allowed exponentials is discrete. X itself is not a conformal field because its correlation functions contain logarithms [16]. For a fermion, one again has the identity operator and the field $\psi(z)$. These fields describe fermion correlations which are well-defined in the local coordinate z; however, we know that fermions are allowed to be double-valued as representations of the 2d Lorentz group, $\psi(e^{2\pi i}z) = -\psi(z)$. The point $z=0$ in the local conformal coordinate is then the location of a branch singularity in the fermion field, which is a source of stress-energy: $\langle T(z)\rangle = \langle \frac{1}{2}\psi\,\partial\psi\rangle \sim h/z^2$ as $z \to 0$. Independence of the position of the branch cut ensures that this stress-energy is generated by a local (conformal) field S. There are in general fields S and $\tilde{S}$ obeying

$$\psi(z)S(0) \sim z^{-1/2}\tilde{S}(0), \tag{2.2}$$

so that ψ changes sign when carried once around the origin. The fields S and $\tilde{S}$ are known as *spin* fields [17]. As (2.2) shows, the full set of conformal fields I, ψ, S and $\tilde{S}$ is not local; one can however construct a local field theory via an appropriate projection on the allowed fields [19,17]. This is just the pattern found in the fermionic string: The fermionic vertex [18] is a spin field; locality is given by the GOS [19] projection. Several copies of this basic system can be used to form a free-fermion current algebra; then there can be an infinite number of conformal fields (one for each highest weight representation of the algebra). The reason is that the constraint of being a highest weight vector of the conformal algebra is restrictive for a single fermion, but is only one constraint on several degrees of freedom for several fermions. These considerations apply as well to the free boson systems we consider. We will relax the condition for locality temporarily, to see what sorts of conformal fields there are in free field theory; later we will try to find mutually local subsets of these fields.

It is often useful to consider the Hilbert space interpretation of conformal field theory. The vacuum $|0\rangle$ is the unique $SL_2(C)$ invariant state. Each conformal field creates a highest weight state of the conformal algebra when acting on the vacuum. Acting further with components of the stress tensor fills out the full highest weight representation, or *Verma module*. Often there is a discrete quantum number which classifies the states. In the fermionic string theory, spacetime fermion number splits the states between the Neveu-Schwarz sector – consisting of the vacuum $|0\rangle$, the world sheet spinor state $\psi|0\rangle$, and their descendants under the conformal algebra; and the Ramond sector – the states $S|0\rangle$, $\tilde{S}|0\rangle$, and their descendants. These *sectors* of the Hilbert space are distinguished in the free fermion case by the boundary condition on the field ψ. The fermion ψ acts within a given sector, whereas the spin fields S and $\tilde{S}$ interchange sectors.

Locally on the surface, the Hilbert space is the set of wave functions along a contour in z. Correlation functions on the sphere can be evaluated as operator expectation values in the Hilbert space. Particularly simple are fermion correlation functions in either sector; by Wick's theorem, all correlation functions are simply related to the two-point function of free fermions,

$$G_{\mathrm{NS}}(z,w)=\langle\psi(z)\psi(w)\rangle_{\mathrm{NS}}=\frac{-1}{z-w},$$

$$G_{\mathrm{R}}(z,w)=\langle\psi(z)\psi(w)\rangle_{\mathrm{R}}=\frac{-1}{z-w}\frac{1}{2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right).$$

In the Ramond sector, the "in" and "out" states in the Hilbert space are created by conjugate spin fields at $z=0$ and ∞; the operator product (2.2) is responsible for the additional factor in parentheses. The Ramond sector Green function enables us to determine the conformal weight of the spin field S; the expectation value of the stress tensor is

$$\begin{aligned}\langle T(z)\rangle&=\tfrac{1}{2}\left(\partial_w G_{\mathrm{R}}(z,w)-\frac{1}{(z-w)^2}\right)_{z=w}\\&=\tfrac{1}{16}z^{-2}.\end{aligned}\tag{2.3}$$

We recognize the leading term in the operator product expansion of T with the conformal field S and see that the dimension of S is $\frac{1}{16}$.

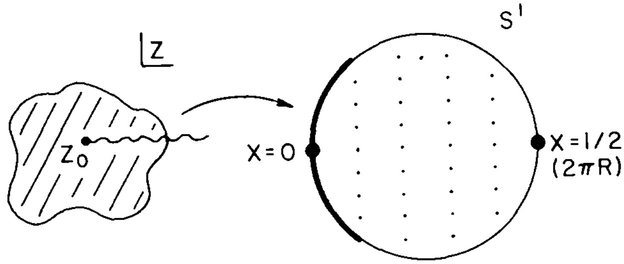

Fig. 1. The map from the world-sheet (the z-plane) to a circle in spacetime. Points of the circle have been identified under $X \to -X$, via the dotted lines. The hatched portion of the world-sheet near the twist field $\sigma(z_0)$ maps to the neighborhood of the fixed point $X=0$ (denoted by a heavy line).

Given the freedom to consider sets of fields which are not relatively local, we might try to construct the analogue of the spin field for bosonic fields X. The simplest example would be a *twist field* $\sigma(z)$ about which X is antiperiodic [20]. The operator product of the conformal field ∂X with σ is then

$$\partial X(z)\,\sigma(w,\bar{w}) \sim (z-w)^{-1/2}\tau(w,\bar{w}) + \cdots .$$

The field τ is an *excited* twist field, also double-valued with respect to X. Note that the conformal weight h of τ is $\frac{1}{2}$ greater than that of σ, whereas the two values for $\bar{h}$ are the same. Unlike the fermion case, making X antiperiodic has consequences for the geometry of spacetime. As the point z on the world-sheet approaches w, the map $X(z)$ must equal minus itself. $X=-X$ means that spacetime (the circle parametrized by $X \in [0, 2\pi R]$) has been identified under the action of a discrete Z_2 symmetry. This is the simplest example of an orbifold. The condition of symmetry under $X \leftrightarrow -X$ is satisfied for either $X=0$ or $X=\frac{1}{2}(2\pi R)$. Tracing X around an infinitesimal contour about z shows that the string wraps once around one of these *fixed points* (fig. 1). There is a separate twist field for each such fixed point. The term fixed point will always refer to the location of the twisted string in spacetime. The Hilbert space now has two sectors: the *untwisted* sector, containing the vacuum $|0\rangle$ and its descendants, and the Verma modules of combinations of the highest weight fields ∂X and e^{ipX}; and the *twisted* sector, with the states created by twist operators together with various untwisted fields acting on them. The simplest twist field σ turns out to play a role in 2d critical phenomena. It completes the correlation between the operators of the gaussian model and the Ashkin-Teller model [21].

The above Z_2 twist field can be generalized to any finite group. In particular, we will be interested in Z_N twists. In the neighborhood of a twist field located at the origin the coordinate field X undergoes a phase rotation

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{2\pi i k/N} X(z, \bar{z}) \tag{2.4}$$

(k integer) which is called the *monodromy* of the field X. For this definition of the twist to make sense the coordinate X must be complex. We adopt the convention $X = X_1 + iX_2$, $\bar{X} = X_1 - iX_2$. The basic twist field σ_+ twists the field X by $e^{+2\pi i k/N}$, and its antitwist σ_- twists X by $e^{-2\pi i k/N}$. In terms of operator products, we have

$$\begin{aligned}
&\partial_z X \sigma_+(w, \bar{w}) \sim (z-w)^{-(1-k/N)} \tau_+(w, \bar{w}) + \cdots, \\
&\partial_z \bar{X} \sigma_-(w, \bar{w}) \sim (z-w)^{-k/N} \tau'_+(w, \bar{w}) + \cdots, \\
&\partial_{\bar{z}} X \sigma_+(w, \bar{w}) \sim (\bar{z}-\bar{w})^{-k/N} \tilde{\tau}'_+(w, \bar{w}) + \cdots, \\
&\partial_{\bar{z}} \bar{X} \sigma_+(w, \bar{w}) \sim (\bar{z}-\bar{w})^{-(1-k/N)} \tilde{\tau}_+(w, \bar{w}) + \cdots,
\end{aligned} \tag{2.5}$$

with similar equations for σ_-. The above relations define four different excited twist fields τ_+ (and four τ_-'s). The primes distinguish between two different types of excited twist fields, of different conformal weights h but the same $\bar{h}$ (or vice-versa). The tildes denote fields related by complex conjugation on the world-sheet, $z \leftrightarrow \bar{z}$, $h \leftrightarrow \bar{h}$. The non-integer power of the singularity in (2.5) is determined by the phase condition (2.4), the integer power by the fact that ∂X acting on the ground state in the twisted sector of the Hilbert space creates an excited state (and also by the requirement that σ be a highest weight field), as we will see shortly. We will discuss the relation of the conformal fields to the orbifold geometry in the next section.

Just as Z_2 twist fields have their partners in the ordinary spin field S, Z_N twist fields for X have their counterparts in the theory of free fermions. Again we must group the fermions into complex pairs $\psi = \psi_1 + i\psi_2$, $\bar{\psi} = \psi_1 - i\psi_2$. We might begin the discussion of Z_N spin fields with a description of their operator products with $\psi, \bar{\psi}$, in analogy to (2.2), (2.5); but it is simplest just to bosonize. Write

$$\begin{aligned}
&\bar{\psi}\psi = 2i\, \partial H, \\
&\sqrt{\tfrac{1}{2}}\, i\psi = e^{+iH}, \\
&\sqrt{\tfrac{1}{2}}\, i\bar{\psi} = e^{-iH};
\end{aligned} \tag{2.6}$$

then the Z_N spin fields are simply $s_\pm = e^{\pm ikH/N}$ with dimension $h = \frac{1}{2}(k/N)^2$. Properly these relations include a cocycle for the lattice of charges in the exponentials of the bosonized representation, which will also depend on the fields σ as well; we leave the determination of this cocycle to future work. The correlation functions, and in particular the operator product expansions, are then simply those of free field exponentials:

$$\psi(z)s_+(w) \sim (z-w)^{+k/N} t'_+(w),$$

$$\bar{\psi}(z)s_+(w) \sim (z-w)^{-k/N} t_+(w) + \cdots, \tag{2.7}$$

where $t'_+ = -i\sqrt{2}\,e^{i(k/N+1)H}$, $t_+ = -i\sqrt{2}\,e^{i(k/N-1)H}$, and so on. The nonsingularity on the r.h.s. of the first line means that ψ annihilates the ground state in the Z_N spin sector created by s_+ ($\partial\psi$ will not, because the z-derivative of (2.7) is singular). The fields $t_\pm$ and $\tau_\pm$ will be important in what follows, for they are used to construct the lowest dimension conformal superfields in the twisted sector; the fields $t'_\pm$, $\tau'_\pm$ will appear only in the twist superfields for massive fermionic string states.

One way to obtain the operator product expansions (2.5) is to look at the Hilbert space interpretation of the twisted sectors. Think of a twist field σ placed at the origin; then the field ∂X must pick up a phase $e^{2\pi ik/N}$ when taken around the origin. The Laurent expansions of ∂X, $\partial\bar{X}$ must have the form

$$\partial_z X = \sum_{m=-\infty}^{\infty} \alpha_{m-k/N} z^{-m-1+k/N},$$

$$\partial_z \bar{X} = \sum_{m=-\infty}^{\infty} \bar{\alpha}_{m+k/N} z^{-m-1-k/N}, \tag{2.8}$$

with m integer. The mode operators have the canonical commutation reations $[\bar{\alpha}_{m+k/N}, \alpha_{n-k/N}] = (m+k/N)\delta_{m,-n}$. The twisted ground state $|\sigma\rangle = \sigma(0)|0\rangle$ is annihilated by all the positive frequency mode operators

$$\alpha_{m-k/N}|\sigma\rangle = 0, \qquad m>0,$$

$$\bar{\alpha}_{m+k/N}|\sigma\rangle = 0, \qquad m \geqslant 0.$$

When $\partial X(z)$, $\partial\bar{X}(z)$ act on the twisted ground state, one finds precisely the operator product expansion (2.5); the antitwist operator products follow similarly, as do those for twisted fermions.

2.2. CORRELATION FUNCTIONS

The operator product relations (2.5) describe the behavior of the string coordinates in the neighborhood of a twist field. More generally, we would like to study the global surface properties of the twists – to calculate their correlation functions. One might hope to find operator expressions for the σ's and τ's analogous to those for X and $\bar{X}$. Then one could compute amplitudes using operator matrix elements in the Hilbert space. This was actually the method first used to construct the spin and twist fields [18, 20]. The reason this is difficult to carry out in practice is that the twist and spin fields are nonlocal with respect to the fields X and ψ; they implement a change in boundary conditions. The procedure used was to invent an operator which annihilated any state in the untwisted sector and replaced it by a collection of states in the twisted sector, and vice versa. Any of the sectors of the Hilbert space contains a complete set of states, so a state in one sector can be re-expanded in terms of another. The operator so constructed is rather unwieldy; computation of its correlation functions involves the evaluation of exponentials of infinite sums over modes in the different sectors. It was a remarkable feat to calculate the four-spin operator correlation function by this method [22]. Now the spin operator correlations can be obtained more simply through bosonization of the free-fermion current algebra [23]. Nevertheless, the twist operator correlations cannot be found so simply; the generic twist field has no apparent local realization in terms of free fields. Such a realization would be difficult because the twist fields can be defined for spacetime manifolds of any characteristic radius, whereas the equivalence of different QFT's typically works only for special values of the couplings. Therefore, we are forced to proceed in a rather indirect manner.

What does a twist correlation look like? The location of vertices are punctures on the world-sheet; since the twist fields put branch cuts in the coordinates X, $\bar{X}$, the twist correlations will be branched about the punctures for the twists, with branch cuts running between them. Note that it is not the z-plane which is cut; rather, the fields are multivalued about points in z. For example, in the Z_N case described above the cuts in X and $\bar{X}$ are not identical. Now the path integral in the presence of a collection of operators $\mathcal{O}_i(z_i)$ (including the twist fields)

$$Z(z_i) = \mathrm{e}^{-S_{\mathrm{eff}}} = \int \mathscr{D}X \exp\left(-\frac{1}{2\pi}\int \partial X \bar{\partial} X\right) \prod_{i=1}^{M} \mathcal{O}_i(z_i)$$

is some function of the z_i which we wish to determine. Choose one of the $\mathcal{O}$'s, say $\mathcal{O}_1$, to be the stress-energy tensor $T(z_1)$. As we bring z_1 near one of the other z_i, the operator product expansion (2.1) gives the dimension of $\mathcal{O}_i$ as the leading term; more important, the residue of the subleading singularity is the derivative of the correlation function with respect to z_i. Integrating this residue, one obtains the twist correlation functions.

The expectation value of the stress tensor $T = -\frac{1}{2}\partial X \partial \bar{X}$ in the presence of twist fields can be determined from the properties of the gaussian fields X and $\bar{X}$. The classical stress-energy is obtained from the solution(s) to the equation of motion

$$\bar{\partial}\partial X_{\mathrm{cl}}(z, \bar{z}) = \bar{\partial}\partial \bar{X}_{\mathrm{cl}}(z, \bar{z}) = 0, \tag{2.9}$$

with the correct monodromy about the points z_i; then $T_{\rm cl} = -\frac{1}{2}\,\partial X_{\rm cl}\,\partial \bar{X}_{\rm cl}$. The quantum contribution to the stress tensor $T_{\rm qu}$ is determined from the connected Green function of the ∂X's at coincident points (cf. eq. (2.3)) in the presence of twists:

$$g(z,w) \equiv \frac{-\frac{1}{2}\langle \partial_z X(z)\,\partial_w \bar{X}(w)\,\sigma_1(z_1)\dots\sigma_M(z_M)\rangle}{\langle \sigma_1(z_1)\dots\sigma_M(z_M)\rangle},$$

$$\frac{\langle T(z)\sigma_1(z_1)\dots\sigma_M(z_M)\rangle}{\langle \sigma_1(z_1)\dots\sigma_M(z_M)\rangle} = \left[g(z,w) - \frac{1}{(z-w)^2} \right]_{z=w}. \tag{2.10}$$

The classical term is just the disconnected part of this correlation function $-\frac{1}{2}\langle \partial X\rangle\langle \partial \bar{X}\rangle$. Eq. (2.10) is of course singular, but subtracting the leading singularity amounts to normal ordering the composite operator T. The connected Green function is in turn uniquely specified by the fact that it has a double pole at $z=w$, must change by the appropriate phase when taken around any z_i, and is otherwise single-valued. Such a Green function can be constructed on surfaces of arbitrary genus [24]; here we restrict attention to correlation functions on the sphere.

The Green function $g(z,w)$ in the case of two twists is completely determined by the operator product relations (2.5). $SL_2(C)$ invariance allows us to put one twist σ_+ at $z=0$ and its antitwist σ_- at $z=\infty$. Then the behavior as $z, w \to 0, \infty$ and the property that $g(z,w) \sim 1/(z-w)^2 +$ finite as $z \to w$ force g to be

$$g_{M=2}(z,w) = z^{-(1-k/N)} w^{-k/N} \left[\frac{(1-k/N)z + kw/N}{(z-w)^2} \right]. \tag{2.11}$$

This is determined as follows: the factors outside the brackets are the singularities of the operator products (2.5) of ∂X, $\partial \bar{X}$ with $\sigma_+(0)$; then one puts in the double pole; then the numerator in brackets must leave untouched the residue of the double pole while cancelling any single pole terms. Since the arrangement is precisely that of a correlation of the basic fields X, $\bar{X}$ in the twisted sector, one can also calculate (2.11) via operator methods. The expectation value of the stress tensor is found in the coincidence limit $z \to w$ upon subtraction of the leading singularity. We find

$$\langle \sigma_- | T(z) | \sigma_+ \rangle = \frac{1}{z^2} \cdot \frac{1}{2}\frac{k}{N}\left(1 - \frac{k}{N}\right),$$

so that the twist field σ has dimension $h = \bar{h} = \frac{1}{2}(k/N)(1-k/N)$.

One thing we would also like to determine are the coefficients in the operator product expansion of products of various combinations of conformal fields

$$\mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2) \sim \sum_k C_{ij}^k \mathcal{O}_k(z_1,\bar{z}_1)(z_2-z_1)^{h_k - h_j - h_i}(\bar{z}_2-\bar{z}_1)^{\bar{h}_k-\bar{h}_j-\bar{h}_i}. \tag{2.12}$$

For instance, the C's are the three-particle couplings in the low-energy effective lagrangian of the string theory. We might try to determine them directly from the three-point correlation functions. If one of the $\mathcal{O}_i$ on the l.h.s. is not a twist field, then (2.12) can be found from the two-twist Green function or by operator methods. For products of twist fields only, we might try to use the stress tensor procedure outlined above. The problem is that the z dependence of a three-point function is completely determined by $\mathrm{SL}_2(\mathrm{C})$ invariance [4], so the single pole term in $\langle T\sigma_1\sigma_2\sigma_3\rangle$ contains essentially no information. However, the information can be determined from the four-point function in the limits where it factorizes on three-point functions. Then the correlation function has nontrivial dependence on the $\mathrm{SL}_2(\mathrm{C})$ invariant ratio $x=(z_1-z_2)(z_3-z_4)/(z_1-z_3)(z_2-z_4)$ of the four points, so that the derivative with respect to x is nontrivial. Also, factorization of the four-point function on a known three-point function serves to normalize the amplitude for factorization in other channels. For instance, we can calculate

$$Z(x)=\langle\sigma_+(\infty)\sigma_-(1)\sigma_+(x)\sigma_-(0)\rangle\,.$$

The limit $x\to 0$ factorizes the amplitude on states in the untwisted sector,

$$\lim_{x\to 0} Z(x)=|C_{\mathcal{O}[X]\sigma_+\sigma_-}|^2\cdot|x^{h_{\mathcal{O}}-h_+-h_-}|^2\,, \tag{2.13}$$

because the two twists cancel. The resulting operator product coefficients,

$$C_{\mathcal{O}[X]\sigma_+\sigma_-}=\langle\sigma_-|\mathcal{O}[X]|\sigma_+\rangle\,,$$

may be calculated using operator methods in Hilbert space or from the Green function (2.11). The simplest way to normalize the four-point function is to choose $\mathcal{O}$ to be the identity and use the known normalization of the two-point function. The limit $x\to\infty$ factorizes the amplitude on "doubly twisted" fields σ_{--} consistent with conserved quantum numbers,

$$\lim_{x\to\infty} Z(x)=|C_{\sigma_{-}\ \ \sigma_+\sigma_+}|^2\cdot|x^{h\ \ \ -2h_+}|^2\,,$$

since the twists "add" rather than cancel. To recapitulate, the strategy is as follows: from the Green function of X in the presence of twists, the logarithmic derivative of the four-point function is evaluated. One then integrates the amplitude, determining the overall normalization by factorization on a known channel. The desired three-twist operator product coefficients are then determined by factorization in the crossed channel. Of course the normalized amplitude contains much more information than this, in particular four-particle couplings in the low-energy effective action as well as dynamical scattering amplitudes.

A variation on the above procedure is somewhat simpler to carry out in practice. As alluded to above, the field X can be split into a classical piece X_{cl} and a quantum fluctuation X_{qu}. The evaluation of the gaussian functional integral naturally divides into a sum over classical solutions, times the quantum effective action, both evaluated in the presence of twist fields. The classical equation of motion (2.9) is solved by holomorphic and antiholomorphic fields $\partial X_{cl}(z)$ and $\bar{\partial} X_{cl}(\bar{z})$ which have the appropriate monodromy about the locations of the twist fields. One normalizes these solutions as described in the next section, and then sums $e^{-S_{cl}}$ over these solutions to obtain the classical contribution to the amplitude. The quantum contribution is determined as before via the stress tensor and is independent of the particular classical solution. The combined result is

$$Z(z_i) = \sum_{\langle X_{cl} \rangle} e^{-S_{cl}} \cdot Z_{qu}(z_i). \tag{2.14}$$

This method has the advantage of determining the relative normalization of different classical solution sectors in (2.14) automatically; the normalization is somewhat harder to dig out in the previous method since each classical solution sector has a separate integration constant upon integrating $T_{cl} + T_{qu}$.

So far we have discussed Z_N twist and spin fields without regard to making a local conformal field theory out of them. To do this we should construct mutually local combinations of twist and spin fields. For example, consider the case of a closed bosonic string compactified on a circle twisted by $X \leftrightarrow -X$. Physical vertices are conformal fields of dimension $h = \bar{h} = 1$. The lowest dimension field in the twisted sector is σ, with $h = \bar{h} = \frac{1}{16}$. We can make a physical vertex by combining σ with a plane wave in the uncompactified dimensions (parametrized by $\mathscr{X}$)

$$V_\sigma[\mathscr{X}] = \sigma \, e^{ik_\mu \mathscr{X}^\mu},$$

with $k^2 = 2 - \frac{1}{8}$ so that V_σ has dimension one. Other fields in the twisted sector consist of products of this with even numbers of X fields, e.g. $\sigma(e^{ipX} + e^{-ipX})e^{ik_\mu \mathscr{X}^\mu}$, $\tau(e^{ipX} - e^{-ipX})e^{ik_\mu \mathscr{X}^\mu}$, even numbers of derivatives of X applied to any of these (again with appropriate k^2), etc. There are also fields which create states in the untwisted sector when acting on the vacuum. In order to be local with respect to the twist fields, these must again contain even powers of X, for instance $(e^{ipX} + e^{-ipX})e^{ik_\mu \mathscr{X}^\mu}$, $\partial X \partial^2 X e^{ik_\mu \mathscr{X}^\mu}$, and so on. The locality constraint is much like the fermion parity projection in the superstring. The general condition is that only untwisted fields which are singlets under the twist group are allowed, together with the twist fields (which only factorize on the singlet untwisted fields).

2.3. FERMIONIC STRINGS

For fermionic string theories (type II or heterotic), world-sheet supersymmetry constrains the possible fields; whatever twist is imposed on the coordinate field X ($\bar{X}$), a compensating twist should be put in $\bar{\psi}$ (ψ), so that the superpartner of the stress tensor

$$T_{\mathrm{F}} = -\tfrac{1}{4}(\partial X\bar{\psi} + \partial \bar{X}\psi)$$

is single-valued. The OPE's (2.5), (2.6) show that the lowest dimension field with this property is $\mathscr{T}_0 = s_+\sigma_+$ of dimension $\frac{1}{2}k/N$. The fact that the term quadratic in k/N cancels in the conformal weight greatly simplifies the construction of local fields. World-sheet supersymmetry implies that this field must have a superpartner; since it is the field of lowest possible dimension, it must be the lowest component of a 2d superfield. The upper component may be found by applying the superstress tensor T_{F}, and noting the general form of a superfield OPE

$$T_{\mathrm{F}}(z)\phi_0(w) \sim \frac{\frac{1}{2}}{z-w}\phi_1(w) + \cdots,$$

$$T_{\mathrm{F}}(z)\phi_1(w) \sim \frac{h}{(z-w)^2}\phi_0(w) + \frac{\frac{1}{2}}{z-w}\partial_w\phi_0 + \cdots.$$

Here the superfield is $\Phi = \phi_0 + \theta\phi_1$. The OPE's (2.5), (2.6) imply that the superpartner of $\mathscr{T}_0$ is $\mathscr{T}_1 = -\frac{1}{2}t_+\tau_+$.

However, neither $\mathscr{T}_0$ nor $\mathscr{T}_1$ is an acceptable conformal field for us; they violate 2d Bose (Fermi) spin-statistics because their dimensions h and $\bar{h}$ don't differ by (half) integers. To remedy the situation we must consider the addition of another field $\tilde{s}$ or $\tilde{t}$ to make up the fractional difference between h and $\bar{h}$ caused by s and t. Obvious candidates are spin fields for world-sheet fermions $\lambda(\bar{z})$ of opposite chirality. In the superstring, these fermions are superpartners of X under an antisuperconformal algebra; for the heterotic string they are gauge fermions of $\mathrm{Spin}(32)/\mathrm{Z}_2$ or $\mathrm{E}_8 \otimes \mathrm{E}_8$. It is again somewhat simpler to pass to the bosonized representation of the fields; we write $\bar{\lambda}\lambda = 2i\,\partial\tilde{H}$. The parts of twist fields that act on this system are then simply fractional charge exponentials, just as for ψ. The heterotic twist superfield is therefore

$$\mathscr{T}_+(z,\bar{z},\theta) = \tilde{s}\left(s_+\sigma_+ - \tfrac{1}{2}\theta t_+\tau_+\right). \tag{2.15}$$

For the superstring we are instructed to complete a multiplet of the antianalytic supersymmetry, for instance

$$\mathscr{T}_+(z,\bar{z},\theta,\bar{\theta}) = \tilde{s}_+s_+\sigma_+ - \tfrac{1}{2}\theta\tilde{s}_+t_+\tau_+ - \tfrac{1}{2}\bar{\theta}\tilde{t}_+s_+\tilde{\tau}_+ + \tfrac{1}{4}\bar{\theta}\theta\tilde{t}_+t_+\upsilon_+, \tag{2.16}$$

with υ_+ obtained from $\partial X\bar{\partial}\bar{X}\cdot\sigma_+$.

In the standard presentation, vertex operators for physical states are θ-integrals of dimension-$\frac{1}{2}$ conformal superfields. This conformal weight can be arranged for spacetimes of n complex dimensions by taking combinations of k for each dimension so that the total $\sum_{i=1}^{n} k_i = N$. This choice for the k_i is of interest because it leaves some of the spacetime supersymmetry charges unbroken. For example, let's consider the case of a Z_N twist on a spacetime of three complex dimensions. The twist need not act effectively on all three coordinates; also the k's for each dimension need not be the same. The dimension-$\frac{1}{2}$ heterotic twist superfield is

$$\hat{\mathscr{T}}_{\text{het}} = \prod_{i=1}^{3} \mathscr{T}_0^{(i)} + \theta \left(\sum_{i=1}^{3} \mathscr{T}_1^{(i)} \prod_{j \neq i} \mathscr{T}_0^{(j)} \right); \tag{2.17}$$

its superstring counterpart is

$$\hat{\mathscr{T}}_{\text{ss}} = \prod_{i=1}^{3} \mathscr{T}^{(i)}(z, \bar{z}, \theta, \bar{\theta}) \tag{2.18}$$

and contains many terms. The theta integral picks out the highest component of each twist superfield in turn. This field combines with fields of the uncompactified directions $\mathscr{X}(z, \bar{z}, \theta) = \mathscr{X}(z, \bar{z}) + \theta \Psi(z)$ to make a vertex

$$\begin{aligned} V_0(z, \bar{z}) &= \int \mathrm{d}\theta\, \hat{\mathscr{T}}_{\text{het}} \mathrm{e}^{ik_\mu \mathscr{X}^\mu}(z, \bar{z}, \theta) \\ &= \sum_{i=1}^{3} \mathscr{T}_1^{(i)} \prod_{j \neq i} \mathscr{T}_0^{(j)} \mathrm{e}^{ik_\mu \mathscr{X}^\mu}(z, \bar{z}) + \prod_{i=1}^{3} \mathscr{T}_0^{(i)} ik_\mu \Psi^\mu \mathrm{e}^{ik_\mu \mathscr{X}^\mu}(z, \bar{z}). \end{aligned} \tag{2.19}$$

Other fermionic string twist vertices are similarly fashioned.

Actually, in the fermionic string theory there are many equivalent ways of representing the same physical state – different "pictures" of the Hilbert space [25], which correspond to different choices of the vacuum of the superconformal ghost system. The different pictures are labelled by an integer-valued charge, the "Bose sea level" of the superconformal ghosts β, γ. These ghost fields are part of a chiral superconformal field theory made out of the superfields

$$B = \beta + \theta b, \qquad C = c + \theta \gamma,$$

which can be thought of as "differential forms" on the superconformal algebra [26]. Their role is to ensure the consistency of string propagation generated by the stress energy tensor. Eq. (2.19) is not the only vertex for the creation of physical twisted string states, rather it is one canonical choice. The alternate representations are related by a "picture-changing" operation [25]. To describe it, we bosonize the superconformal ghosts [25].

$$\gamma = \eta\, \mathrm{e}^{\phi}, \qquad \beta = \partial \xi\, \mathrm{e}^{-\phi},$$

$$\langle \phi(z) \phi(w) \rangle = -\ln(z - w), \qquad \langle \xi(z) \eta(w) \rangle = \frac{1}{z - w}.$$

The zero mode of the field ξ does not appear in the bosonization formulae, so its inclusion would cause an unwanted degeneracy in the representation of the algebra of fields. Therefore we can restrict our attention to those fields which contain no undifferentiated ξ's. Nevertheless, in some instances it is convenient to use ξ itself in the representation of fields in this smaller algebra. The picture-changing field is one example; it can be represented as

$$\{Q_{\mathrm{BRST}}, \xi\} = \mathrm{e}^{\phi} T_{\mathrm{F}}[X, \psi] + \cdots . \tag{2.20}$$

The BRST charge Q_{BRST} is the exterior derivative on the Virasoro algebra [26], valued in the representation given by the spacetime conformal field theory:

$$Q_{\mathrm{BRST}} = \oint j_{\mathrm{BRST}} = \oint C\left(T[X, \psi] + \tfrac{1}{2} T[B, C]\right).$$

Physical states are created by vertices or conformal fields V which are representatives of the cohomology classes of Q_{BRST}. This means that they satisfy $\oint_z j_{\mathrm{BRST}} V(z) = 0$, but V itself cannot be written as a contour integral of the BRST current around some other field. Each Bose sea level contains an equivalent copy of these representatives, related by the application of the field (2.20). That is, given a physical vertex V_0, then $V_1 = \{Q_{\mathrm{BRST}}, \xi\} V_0$ represents the same state of string, but in the next Bose sea up. conversely, every V_0 can be written as $\{Q_{\mathrm{BRST}}, \xi\} V_{-1}$. The canonical set of V_0's is the collection of upper components of highest weight, dimension-$\frac{1}{2}$ superfields. The corresponding V_{-1}'s are the lower components of the superfields, times $\mathrm{e}^{-\phi}$. The form of the picture-changing field (2.20) guarantees that we obtain V_0 from V_{-1}, because that was how we constructed the higher component from the lower one by the application of T_{F} (the exponentials of the ghost boson ϕ cancel). The set of manipulations used here are formally identical to those used in the flat-space case discussed in [25]; they apply to any fermionic string background since the picture-changing procedure requires only the BRST charge and the ghosts. For example, in the heterotic twist field constructed above, the picture V_{-1} of the vertex is (compare (2.19))

$$V_{-1} = \mathrm{e}^{-\phi} \prod_{i=1}^{3} \mathscr{T}_0^{(i)} \mathrm{e}^{i k_\mu \mathscr{X}^\mu}(z, \bar{z}).$$

In the Hilbert space interpretation, this vertex at $k = 0$ creates states $|\mathscr{T}_+^0\rangle = V_{-1}(k = 0)|0\rangle$ analogous to the "usual" NS ground states $\psi^\mu{}_{1/2}|0\rangle_{-1} = \mathrm{e}^{-\phi}\psi^\mu(0)|0\rangle$. Superficially, these states bear little resemblance to the superfield vertex operators V_0 (eq. (2.19)) and $(\partial X^\mu + ik \cdot \psi\psi^\mu)\mathrm{e}^{ikX}$ they correspond to. The relationship between them is of course given by picture-changing. In the picture-changed Hilbert space corresponding to superfield vertices, these latter operators create the states

$\Sigma_{i=1}^{3}\alpha^{i}_{-k_i/N}\bar{\psi}^{i}_{-(1/2-k_i/N)}|\hat{\mathscr{T}}_{+}^{0}\rangle$ and $\alpha^{\mu}_{-1}|0\rangle$ at zero momentum. The Hilbert space description will be useful for the enumeration of fermionic string states in sect. 5.

On any given topology of world-sheet, there are ghost zero modes which are the globally defined solutions to the equations of motion $\bar{\partial}B=\bar{\partial}C=0$. In the path integral these produce a "background charge" which must be cancelled in order for the correlations not to vanish. Since the ghosts reflect the geometry of the world-sheet, the amount of charge depends on the topology of world-sheet. Note that twists *do not* change the world-sheet topology, so we don't twist the ghosts. For the sphere, there are three c zero modes and two γ zero modes corresponding to the (super)Killing vector fields of OSp(2,1), the supersymmetric extension of $SL_2(C)$. The c zero modes can be used to fix the location of any three vertices on the world-sheet [25] and cancel the background charge, their correlations building the jacobian for fixing $SL_2(C)$. That is, we remove the integral of $V(z)$ over the world-sheet and replace it by $cV(z)$ for three of the vertices. Similarly the remaining fermionic generators of OSp(2,1) can remove the theta integrals of two V_0 superfields and replace them by picture changed V_{-1}'s, having the form $e^{-\phi}$ times the lower component of a superfield. The ϕ insertions cancel the superconformal ghost background charge, and their correlations generate the jacobian for fixing the remaining fermionic symmetries of OSp(2,1). The invariance of correlation functions under the picture-changing operation [25] guarantees that answers will not depend on which two vertices are changed. We show in sect. 5 how picture-changing can simplify calculations in the fermionic string.

The superfields that twist strings are Neveu-Schwarz fields; they create asymptotic bosonic states in spacetime that sit at the fixed points of the twist. In the fermionic string theory there are also fields that create asymptotic spacetime fermion states. In the untwisted sector of the string conformal field theory, these are simply the Z_2 spin fields we discussed above. In the twisted sector, there should be analogues of the spin fields. They can be found by the application of the spacetime supersymmetry charge Q to the boson states [25]. The supersymmetry charges which are left unbroken by the twisting are the ones invariant (i.e. having nonsingular operator product) with respect to the twist. Since all the twists we consider leave an unbroken U(1) (the complex structure), the components of the ten-dimensional Q that remain are the spin fields in the SO(9,1) current algebra of the ψ's built out of this U(1) (see below). It is possible to choose Z_N twists which leave no components of the spacetime supersymmetry charge Q unbroken. There are again a variety of pictures in which to represent the fields which create twisted spacetime fermions; the most canonical one is $V_{-1/2}$:

$$V_{-1/2}=\tilde{s}\,e^{-\phi/2}S_{+}\prod_{i}\sigma_{+}^{(i)},$$

$$S_{+}=\mathscr{S}\prod_{i}\exp\left(i\left(\frac{k_i}{N}-\tfrac{1}{2}\right)H_i\right). \tag{2.21}$$

The field S_+ is a spin field in the twisted sector of the fermion system ($\mathscr{S}$ is a spin field of the uncompactified directions), and we call the entire combination in (2.21) a *spin-twist* field. We will use them below in the calculation of Yukawa couplings for orbifold compactifications. As in the ordinary flat-space string, we project the set of states onto the subset with even 2d spinor parity $(-)^F$ [19]. The fermion number is defined so that the $SL_2(C)$ invariant vacuum has zero fermion number. The states of even parity are the only states which have local operator products with the spacetime supersymmetry charge Q.

Our main objective in this paper is to give a complete description of the conformal field theory of strings in orbifold backgrounds. These are special limits of more general string compactifications on Ricci-flat Kähler manifolds (Calabi-Yau manifolds) [3] and their generalizations [11,27]. The typical Calabi-Yau manifold is not a rigid object; its shape depends on several parameters or moduli. For some topologies, particular corners of the space of moduli correspond to manifolds where all the curvature has been squeezed down to a discrete set of points. At these points the manifold looks locally like the quotient of a flat space by the action of a discrete group which has a fixed point at the singularity. For our methods to apply, the manifold must have this structure not just locally, but globally as well. The twist fields describe asymptotic states of string which "sit" at the fixed points f. They, together with the conformal fields for untwisted string states which are invariant under the action of the group, form the local algebra of fields we want. Locality is guaranteed by BRST invariance, single-valuedness with respect to the covering group of the orbifold, and the fermion parity projection. In the case of the heterotic string, for a given spacetime orbifold there are generically a number of gauge field expectation values satisfying these requirements. The simplest solution is to embed the spin connection in the gauge group. This results in a nonchiral conformal field theory describing the compactified dimensions. More recently [11,27] other, chiral solutions have been found where the gauge field is different from the spin connection, and/or other gravitational fields (such as the antisymmetric tensor) are excited. For each of these different solutions the set of mutually local conformal fields can be described. We might also remark that the "level-matching" constraint for modular invariance described in [28] implies locality; fields should have integer-power operator products when loop amplitudes are factorized on correlation functions on the sphere.

We conclude this section with a discussion of the additional global symmetries of the orbifold conformal field theory. Because the above twist construction for $N > 2$ requires that spacetime have a complex structure, there is a conserved U(1) current $J = -i\sum_i \partial H_i$ whose charge J_0 generates rotations of the phase of the complex coordinates. Conformal fields may be classified by their charge as well as their conformal weight; the superstress tensor, for instance, divides into $T_F^+ = -\frac{1}{4}\partial X\cdot\bar{\psi}$ and $T_F^- = -\frac{1}{4}\partial\bar{X}\cdot\psi$. The fields T, $T_F^{\pm}$ and J form a global $N=2$ superconformal algebra [29]

$$T(z)T(w) \sim \frac{\frac{3}{4}\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{z-w} + \cdots,$$

$$T(z)T_{\mathrm{F}}^{\pm}(w) \sim \frac{\frac{3}{2}T_{\mathrm{F}}^{\pm}(w)}{(z-w)^2} + \frac{\partial_w T_{\mathrm{F}}^{\pm}}{z-w} + \cdots,$$

$$T_{\mathrm{F}}^{+}(z)T_{\mathrm{F}}^{-}(w) \sim \frac{\frac{1}{8}\hat{c}}{(z-w)^3} + \frac{\frac{1}{4}J(w)}{(z-w)^2} + \frac{\frac{1}{4}T(w)}{z-w} + \cdots,$$

$$J(z)T_{\mathrm{F}}^{\pm}(w) \sim \frac{\pm T_{\mathrm{F}}^{\pm}(w)}{z-w} + \cdots,$$

$$J(z)J(w) \sim \frac{\frac{1}{2}\hat{c}}{(z-w)^2} + \cdots.$$

The first line is the conformal algebra; the second says that $T_{\mathrm{F}}^{\pm}$ are dimension-$\frac{3}{2}$ conformal fields; the third is the supersymmetry algebra; the fourth says that T_{F}^{+} has charge ± 1; and the last is the current algebra of the U(1) current. The central charge $\hat{c}$ equals $2n$ for a $2n$-dimensional orbifold. This $N=2$ algebra is to be distinguished from the local $N=1$ algebra generated by T and $T_{\mathrm{F}} = T_{\mathrm{F}}^{-} + T_{\mathrm{F}}^{-}$ because there are no ghosts for the extra global conformal symmetries.

One of the interesting features of this extended superconformal algebra is that the current J is basically decoupled from the rest of the conformal fields [30, 31]. By this we mean that fields in the different charge sectors of the orbifold conformal field theory are all of the form

$$\Phi_{J_0=\alpha} = \mathrm{e}^{i\alpha\hat{H}}\tilde{\Phi},$$

where the decoupled scalar field $\hat{H}$ is the integral of the U(1) current,

$$\hat{H} = i\int \mathrm{d}z\, J(z) = \sum_i H_i,$$

and $\tilde{\Phi}$ commutes with the current J. In particular, we find the form of the spacetime supersymmetry charge and holomorphic ε tensor to be

$$\mathrm{Q} = \mathrm{e}^{-\phi/2}\mathscr{S}_{\pm}\mathrm{e}^{\pm i\hat{H}/2},$$

$$\varepsilon = \mathrm{e}^{i\hat{H}}.$$

Here $\mathscr{S}_{\pm}$ are spin fields for the (four) uncompactified dimensions. Vertices containing s, the field that twists world-sheet spinors, must have $\Sigma_i(k_i/N) = 1$ in order to have local operator product relations with Q and thereby spacetime supersymmetric partners. This is part of the projection $(-)^F = (-)^{F_4+J_0} = 1$, where F_4 is the fermion number of the uncompactified dimensions plus ϕ charge.

3. Global monodromy and the orbifold space group

The string coordinate fields $X(z, \bar{z})$ must satisfy the local monodromy conditions (2.4) in the presence of twist operators. These give rise to operator products (2.5), which determine the asymptotic behavior of the Green function $g(z, w)$ when z or w approaches the location z_i of a twist operator. However, this information is not generally sufficient to uniquely determine $g(z, w)$, because of the presence of a number n_{cl} of holomorphic fields $\partial X^{(n)}(z; z_i)$ which are solutions to the equations of motion having the correct local monodromy around each of the z_i. In other words, we can add to $g(z, w; z_i)$ the n_{cl}^2 terms

$$A_{mn}(z_i; \bar{z}_i)\, \partial X^{(m)}(z; z_i)\, \partial \bar{X}^{(n)}(w; z_i) \tag{3.1}$$

(which are nonsingular as z approaches w) without affecting the local monodromy. In order to determine the constants A_{mn} we also need to impose *global* monodromy requirements; i.e. we need to specify how $X(z, \bar{z})$ changes when it is transported around closed loops $\mathscr{C}_i$ in the complex z-plane which encircle two or more of the vertex locations z_i. A vertex can be thought of as a node or puncture on the world-sheet. Thus loops which surround sets of twists are homotopically nontrivial. The changes in X are best understood viewing the string as a map from the 2d world-sheet into spacetime. The twist fields provide local boundary conditions for the map near the z_i, which are just the local monodromy conditions (2.5) discussed previously. However, we will see that they also contain the global information needed to completely determine the Green functions and to properly normalize the classical solutions X_{cl}. To get at this global information we will need to know something about the background geometry which is responsible for the existence of the twist fields, namely the geometry of orbifolds [13]. Therefore we digress here to discuss some relevant features of orbifolds.

For our purposes it is most useful to think of a d-dimensional orbifold Ω as being constructed by identifying points of d-dimensional euclidean space $\mathscr{R}^d$ under a *space group* S of rotations θ and translations v:

$$\Omega = \mathscr{R}^d/\mathrm{S}\,.$$

A typical element of S takes $X \to \theta X + v$ and will be denoted by (θ, v). The multiplication law for elements of S is then

$$(\theta_1, v_1) \cdot (\theta_2, v_2) = (\theta_1\theta_2, \theta_1 v_2 + v_1), \tag{3.2}$$

and the inverse of an element is

$$(\theta, v)^{-1} = (\theta^{-1}, -\theta^{-1}v).$$

One can also construct the orbifold Ω by starting with a d-dimensional torus T^d and identifying points of T^d under a group $\bar{\mathrm{P}}$ of rotations and translations:

$$\Omega = \mathrm{T}^d/\bar{\mathrm{P}}.$$

In the cases we will consider in this paper, $\bar{\mathrm{P}}$ will consist of rotations alone. Then $\bar{\mathrm{P}}$ may be identified with the *point group* P, which is the discrete group of rotations obtained from the space group S simply by ignoring the translations v. The subgroup of S formed by the pure translations $(1, v)$ is referred to as the lattice Λ of S. Identification of points of euclidean space under the subgroup Λ alone defines the torus T^d; points of T^d can then be identified under $\bar{\mathrm{P}}$ to form the orbifold. $\bar{\mathrm{P}}$ should be a subgroup of the isometry group of the torus, which means in particular that P will consist of rotations which are automorphisms of the lattice Λ.

The first reason for introducing the space and point groups of the orbifold is to note the correspondence of elements of these groups with the various twist fields. This correspondence can be understood from the point of view of the Hilbert space for the string propagting on the orbifold, since vertex operators are associated with each state in the Hilbert space. The Hilbert space can be decomposed into various sectors; in each sector the string field X obeys different boundary conditions. For example, there are *winding* sectors where the closed string boundary conditions are

$$X(\sigma + 2\pi) = X(\sigma) + v \tag{3.3}$$

for any lattice vector v, because we have identified points differing by v as being the same. These sectors are present for the case of a background which is either a torus or an orbifold. In the orbifold case there are also *twisted* sectors because we have identified points under rotations as well as translations. One might think that there is a separate sector of the Hilbert space for each element of the space group. This is not true, because the space group is nonabelian. The string field in the "sector twisted by g" obeys

$$X(\sigma + 2\pi) = gX(\sigma), \tag{3.4}$$

where g is an element of S. But it also obeys

$$hX(\sigma + 2\pi) = (hgh^{-1})hX(\sigma), \tag{3.5}$$

where h is any other element of S. So the sectors twisted by hgh^{-1} are in fact all the same sector; i.e. there is one sector of the Hilbert space for each *conjugacy class* of the space group S, not for each element [13].

This result can be illustrated for the one-dimensional orbifold formed by identifying the real line $\mathscr{R}^1$ under the space group S with elements $(1, n)$ (translation by n units) and $(-1, n)$ (inversion of the line followed by translation), where n runs over the integers [13]. Equivalently this "Z_2" orbifold may be formed by identifying points of the circle under an inversion $X \to -X$ as described in sect. 2 and in fig. 1. The point group is $P = Z_2$. The translation elements of S are conjugate in pairs, $(1, n_0) \sim (1, -n_0)$; the inverting elements belong to one of two classes, $\{(-1, \text{even})\}$ or $\{(-1, \text{odd})\}$. Note that the sets of even and odd translations associated with the last two conjugacy classes define *cosets* of the lattice $\Lambda = \{(1, n)\}$. The translation elements describe winding sectors, not twisted sectors, and have soliton operators rather than twist operators associated with them. The two remaining conjugacy classes imply that there are really two twisted sectors, or rather two subsectors of a single twisted sector. From the point of view in which the circle is identified under the transformation $X \to -X$ to form the orbifold, the two subsectors are due to the fact that this transformation has two fixed points on the circle, at $X = 0$ and $X = \frac{1}{2}(2\pi R)$. Each fixed point provides a possible location for the ground state of the string, when the string satisfies antiperiodic boundary conditions on the circle. Excited states in the twisted sector of the Hilbert space correspond to fluctuations of the string about each of these fixed points. So there are actually two Z_2 twist fields for this orbifold; each creates the twisted sector ground state located at one of the two fixed points, and each is associated with an entire conjugacy class of space group elements, $\{(-1, \text{even})\}$ or $\{(-1, \text{odd})\}$.

A second simple example of an orbifold is the tetrahedron. It can be constructed by starting with the torus defined by the hexagonal lattice shown in fig. 2a and identifying points under a rotation by π about the origin ($X \to -X$). The origin is taken to be at the center of the unit cell (point D in the figure); the four fixed points of this rotation on the torus (A, B, C, D) are indicated by solid dots. After this Z_2 identification of points, we can pick one of the two equilateral triangles which make up a unit cell (fundamental region) for the two-torus – say the one with corners A, A′, A″ in the figure – and use it as the fundamental region for the orbifold. Each edge of this triangle is divided into two segments by the fixed point at its midpoint, and the two segments are identified with each other under the π rotation (plus translation by a lattice vector). So we can fold up the three corners of the triangle along the dashed lines, and glue each edge to itself to create the tetrahedron depicted in fig. 2b. We see that there is a deficit angle of π at each vertex due to a curvature delta function located there. Like the tetrahedron, orbifolds in general are flat everywhere except at fixed points or fixed surfaces where they become singular. (Usually the singularity is more complicated than a curvature delta function, however.) As in the previous one-dimensional example, there is one Z_2 twist field

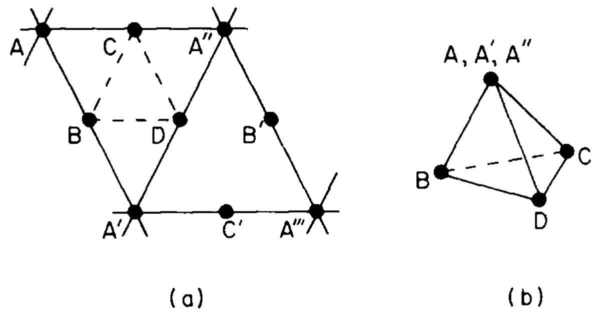

Fig. 2. The tetrahedron as a two-dimensional Z_2 orbifold. (a) The parallelogram with vertices A, A', A'', A''' is the unit cell for the hexagonal lattice. Points are identified under a rotation by π about the center of the unit cell; the fixed points are indicated by solid dots. (b) The triangle with corners A, A', A'' in (a) can be folded up along the dashed lines to create the tetrahedron with vertices A, B, C, D.

(and one conjugacy class of space group elements) associated with each of the fixed points of the Z_2 transformation, which here are just the four vertices of the tetrahedron.

We will be describing below various classical solutions which contribute to the twist correlation functions; we can use the tetrahedron example to illustrate the solutions as maps from the world-sheet S^2 onto the tetrahedron. For example, if we calculate a four-point function where the four twist fields create states located at the four different fixed points A, B, C, and D, then one classical solution (in fact the one with minimum action) is a one-to-one map from the world-sheet sphere to the tetrahedron. Since the tetrahedron is topologically equivalent to the two-sphere, this classical solution is the topologically stable solution guaranteed by the second homotopy group $\pi_2(S^2) = Z$. In fig. 2a this solution appears as a string stretching along the left edge of the unit cell for the torus (line segment ABA′) which propagates to the right until it reaches the parallel line segment CDC′). On the tetrahedron in fig. 2b the string starts wrapped around the edge AB and ends up wrapped around the edge CD. The fact that the classical solution is so simple on the unfolded version of the tetrahedron (fig. 2a) is essentially why orbifold scattering amplitudes are exactly calculable. Other classical solutions for the same set of twist fields (i.e. the same set of fixed points $\{A, B, C, D\}$) will cover the tetrahedron a number of times, either by "bouncing" off the edges AB and CD several times, or by starting off wrapped around the edge AB several times. On the unfolded version of the tetrahedron, the string either propagates further to the right or starts stretched a longer distance parallel to the left edge of the torus. Hence these solutions will have greater action (world-sheet area) and smaller contribution to the amplitude. For a different collection of twist fields, the set of solutions will be different. For example, if all the fixed points of the twist fields are taken to be the same point, then the zero-action solution, in which the string just sits at that fixed point, dominates the amplitude. In the calculation of the next section we will sum over the entire set of solutions to obtain the exact amplitude.

The description of the one-dimensional Z_2 orbifold in terms of its space group and point group generalizes easily to "Z_N" orbifolds, which give rise to the Z_N twist fields considered in this paper. The point group P for a Z_N orbifold is generated by a single rotation θ of order N in SO($2n$): $\mathrm{P} = \overline{\mathrm{P}} = Z_N$. (We take the orbifold to be even-dimensional, $d = 2n$.) The space group S for this orbifold consists of elements of the form (θ^j, v), where $j = 0, 1, 2, \dots, N-1$, θ^j denotes the jth power of θ, and v runs over the $2n$-dimensional lattice Λ. The translation elements of S belong to conjugacy classes of the form $\{(1, \theta^j v_0)\}$, with v_0 fixed and j running from 0 to $N-1$. These classes describe winding sectors; one sums over the images of the winding vector under θ to get a Z_N invariant state. There are also for *each* $j = 1, 2, \dots, N-1$ several conjugacy classes in S, having the form $\{(\theta^j, v)\}$, where v runs over some coset of the lattice Λ. These classes are in the $N-1$ twisted sectors. To determine what the cosets are, start with some element (θ^j, v_0) and conjugate it with the elements (θ^k, u):

$$(\theta^k, u) \cdot (\theta^j, v_0) \cdot (\theta^{-k}, -\theta^{-k} u) = (\theta^j, \theta^k v_0 + (1 - \theta^j) u).$$

Thus the cosets have the form

$$\{\theta^k v_0 + (1 - \theta^j) u,\ k \in Z,\ u \in \Lambda\}$$

for some fixed v_0.

We will concentrate below on the singly-twisted and singly-antitwisted sectors, $j = \pm 1$. ($j = -1$ is of course equivalent to $j = N-1$.) For these two sectors the elements of a given coset all have the same fixed point on the torus: If two points f_1 and f_2 in $\mathscr{R}^d$ are fixed points of two such elements, say

$$\begin{aligned} f_1 &= \theta f_1 + \theta^{k_1} v_0 + (1 - \theta) u_1, \\ f_2 &= \theta f_2 + \theta^{k_2} v_0 + (1 - \theta) u_2, \end{aligned} \tag{3.6}$$

then $(1-\theta)(f_1 - f_2) = (\theta^{k_1} - \theta^{k_2}) v_0 + (1-\theta)(u_1 - u_2)$, and similarly for θ^{-1}. Provided that none of the eigenvalues of θ is equal to 1, we see that f_1 and f_2 differ by lattice vectors and hence are the same fixed point on the torus. (If some of the eigenvalues of θ are equal to 1, then its fixed point set has nonzero dimension; that is, θ has fixed *tori* rather than fixed points, and the elements of a given coset for θ have the same fixed torus.) So there is a one-to-one correspondence between the conjugacy classes $\{(\theta, v)\}$ and the fixed points f (or fixed tori in some cases) of the rotation θ acting on the torus T^{2n}. That is, the "sector twisted by θ" (or by θ^{-1}) has a subsector for each fixed point of θ. The coset of Λ obtained by setting $v_0 = 0$

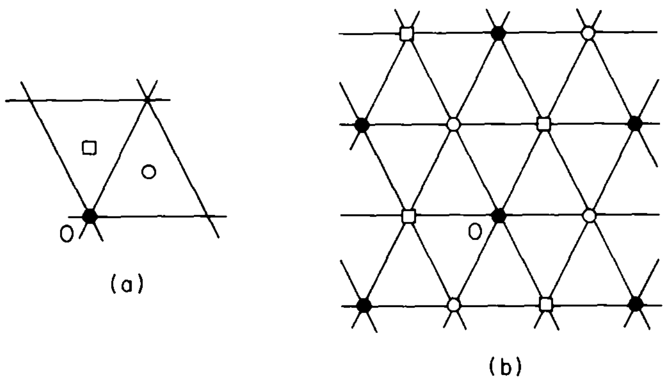

Fig. 3. A two-dimensional Z_3 orbifold. (a) The three fixed points for the $\frac{2}{3}\pi$ rotation on the hexagonal lattice Λ. (b) The same lattice Λ. The three cosets of λ are denoted by the same symbols used for the corresponding fixed points in (a).

(corresponding to the fixed point at the origin) is actually a sublattice of Λ, which can be denoted by $(1-\theta)\Lambda \equiv \{(1-\theta)u,\ u \in \Lambda\}$. The other cosets are not lattices but lattices "shifted" by some vector v_0. From (3.6) we see that v_0 is just $(1-\theta)f$, modulo $(1-\theta)\Lambda$ (similarly for θ^{-1}), and so we can denote the cosets of Λ by

$$(1-\theta^j)(f+\Lambda), \qquad f \text{ a fixed point of } \theta^j, \qquad j = \pm 1 .$$

The case of a two-dimensional Z_3 orbifold is illustrated in fig. 3. This case requires the two-dimensional lattice to be hexagonal in order for θ, the rotation by $\frac{2}{3}\pi$, to be an automorphism. This rotation has three fixed points on the torus, which are shown in fig. 3a. The three cosets of lattice vectors corresponding to each of the three fixed points are shown in fig. 3b.

The conjugacy classes for the higher-twist sectors ($j \neq \pm 1$) are *not* usually in one-to-one correspondence with the fixed points of θ^j. The reason is that some of the fixed points may not be fixed by θ; physical states are then θ-invariant linear combinations of states located at different fixed points of θ^j. So there are actually a large number of twist fields associated with a Z_N orbifold. Each may be labelled by two indices: $\sigma = \sigma_{j,\varepsilon}$. The first index $j = 1, 2, \dots, N-1$ denotes one of the $N-1$ twisted sectors of the Hilbert space – the sector twisted by θ^j. The second index $\varepsilon = 1, 2, \dots, n_f(j)$ labels a conjugacy class within that sector; for $j = \pm 1$ it indicates the fixed point f (or fixed torus) of θ at which the twist field creates a twisted state. We will also denote the $j = \pm 1$ single twist and antitwist fields by $\sigma_{\pm}$; they create states which are the antiparticles of each other.

Now that we have established the correspondence between twist fields and conjugacy classes of the space group of the orbifold, we can describe the proper global monodromy conditions on the field X. The class of space group elements associated with a given twist field $\sigma_{j,\epsilon}(0)$ determines how $X(z,\bar{z})$ is rotated and translated when it is carried around that operator in the z-plane:

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta^j X(z,\bar{z}) + v\,, \tag{3.7}$$

where v belongs to a coset of Λ which depends on the index ϵ. Eq. (3.7) is just the boundary condition (3.4) for the cylinder (τ,σ) which describes propagation of a string in the sector twisted by θ^j (with $g=(\theta^j, v)$), after mapping the cylinder to the complex z-plane with the exponential map $z = e^{\tau + i\sigma}$. Let us now split the field X into a classical piece X_{cl} and a quantum fluctuation X_{qu} as described in sect. 2, and ask how each piece separately changes when X circles the twist field. The classical field should have exactly the same behavior as the full field:

$$X_{\text{cl}}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta^j X_{\text{cl}}(z,\bar{z}) + v\,, \tag{3.8}$$

which implies that the boundary conditions for X_{qu} simply ignore the shift by v:

$$X_{\text{qu}}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta^j X_{\text{qu}}(z,\bar{z})\,. \tag{3.9}$$

The local monodromy condition (2.4) given in sect. 2 also ignored the shifts by v, so it strictly applies only to X_{qu}.

When X is transported around a collection of twist fields, it simply changes by a product of space group elements, one for each of the twist fields. But each twist field is associated with an infinite number of space group elements in the same conjugacy class, so there are an infinite number of such products. This implies that there are an infinite number of allowed global monodromy conditions on X, or equivalently on the classical field X_{cl}, since X_{cl} has the same boundary conditions as X. On the other hand, the quantum fluctuation X_{qu} ignores the translations v in the space group, so its boundary conditions are actually specified by the point group P alone. It obeys a *unique* set of global monodromy conditions, because there is only *one* element of the point group (θ^j) associated with the twist field $\sigma_{j,\epsilon}$ (at least for the abelian point groups we are considering here).

In fact, all the global information needed to determine both the quantum Green functions and the proper set of classical solutions can be obtained from the monodromy conditions for transporting X_{qu} or X_{cl} around collections of fields which have *net twist zero*. These are the combinations of twists for which the product of the point group elements is the identity rotation. Around such paths the full field X therefore is not rotated (acquires no multiplicative phase) but may be translated by some amount. Now, due to the local monodromy conditions on X, correlation functions for $\partial_z X$ in the presence of twist fields located at z_i on the world-sheet have branch cuts in z which terminate at the z_i. We will abuse terminology and refer to the world-sheet as the "cut z-plane". This does *not* mean that there are cuts in the world-sheet, but only in the fields $\partial_z X$ and $\partial_{\bar{z}} X$ living on the world-sheet. (The cuts signify that the world-sheet, which is a continuous map onto the orbifold, need not be continuous on the spacetime torus which covers the orbifold.) We need to keep track of the phase of the multivalued field X, or equivalently which sheet of the cut z-plane we are on. The paths we are interested in, around which X acquires no phase, are just the "closed loops" on this sheeted surface, which return to the same point in the z-plane *and* on the same sheet.

Around the closed loops, which we denote by $\mathscr{C}_i$, X changes by a pure translation $(1, v)$, with v running over some coset of the lattice Λ. This is also the change in X_{cl}, and X_{qu} is required to be strictly periodic around these loops, because it ignores the translations v. That is,

$$0 = \Delta_{\mathscr{C}_i} X_{\mathrm{qu}} = \oint_{\mathscr{C}_i} \mathrm{d}z\, \partial X_{\mathrm{qu}} + \oint_{\mathscr{C}_i} \mathrm{d}\bar{z}\, \bar{\partial} X_{\mathrm{qu}}, \tag{3.10}$$

for all closed loops $\mathscr{C}_i$. These global monodromy conditions, in addition to the local conditions (3.9), suffice to determine the Green functions for the quantum fluctuations, and hence to find the quantum contribution to the amplitude. We will explicitly carry out such a calculation involving four twist fields in sect. 4. Here we just note that (3.10) makes no reference to the classical solution about which X_{qu} is a fluctuation, or to which fixed points f the various twist fields correspond to, so the quantum Green functions will be independent of all such information. Also, (3.10) involves the antiholomorphic field $\bar{\partial} X$ and so it will generically introduce antiholomorphic $\bar{z}_i$ dependence into the Green function via the constants (in z and w) $A_{mn}(z_i, \bar{z}_i)$ in (3.1). Similarly the change in X_{cl} around a closed loop $\mathscr{C}$ is

$$\Delta_{\mathscr{C}} X_{\mathrm{cl}} = \oint_{\mathscr{C}} \mathrm{d}z\, \partial X_{\mathrm{cl}} + \oint_{\mathscr{C}} \mathrm{d}\bar{z}\, \bar{\partial} X_{\mathrm{cl}} = v, \tag{3.11}$$

where v runs over a coset of Λ which depends on exactly which twist fields are encircled by $\mathscr{C}$.

For example, suppose $\mathscr{C}$ encircles two operators, a twist σ_{+,ε_1} associated with the space group elements

$$\left\{ \left(\theta, (1-\theta)\left(f_{\varepsilon_1} + u_1\right)\right),\ u_1 \in \Lambda \right\},$$

and an antitwist σ_{-,ε_2} associated with the space group elements

$$\left\{ \left(\theta^{-1}, (1-\theta^{-1})\left(f_{\varepsilon_2} + u_2\right)\right),\ u_2 \in \Lambda \right\}.$$

Then the products of these elements have the form

$$\left(1, (1-\theta)\left(f_{\varepsilon_1} - f_{\varepsilon_2} + u\right)\right), \qquad u \in \Lambda,$$

and hence we would require in (3.11) that

$$v \in (1-\theta)(f_{\epsilon_1} - f_{\epsilon_2} + \Lambda). \tag{3.12}$$

Note that there will be a properly normalized classical solution associated with each element v in (3.11), and we must impose (3.11) for each independent closed loop $\mathscr{C}$ in the z-plane. These conditions will be used in sect. 4 to determine explicitly the properly normalized classical solutions for the case of two Z_N twists and two Z_N antitwists.

Taking the closed loop $\mathscr{C}$ to encircle *all* the vertex operators provides us with a "space group selection rule" for correlation functions: The set of products of space group elements for all the fields in a correlation function must include the identity element of the space group, $(1,0)$; otherwise the correlation function vanishes. This is because the loop surrounding all the fields is homotopically trivial (it can be pulled off to infinity), so it can be viewed also as enclosing no vertex operators. If we ignore the translations present in the space group elements (i.e. consider them as elements of the point group) we obtain a coarser "point group selection rule", that all nonvanishing correlation functions must have net twist zero. This selection rule is just the orbifold analog of the fact that fermionic string amplitudes involving an odd number of spacetime fermions vanish. We can apply these selection rules to the one-dimensional Z_2 orbifold discussed above. The point group selection rule tells us that correlation functions must have an even number of Z_2 twist fields; the space group selection rule tells us in addition that the numbers of twist fields associated with *each* of the two fixed points must be even, unless soliton operators are also present.

Most of the above considerations apply to more general orbifolds than Z_N orbifolds; however, the Z_N case is particularly simple because the spacetime coordinates can always be chosen to diagonalize the order-N rotation θ:

$$\theta = \exp\left(2\pi i \sum_{i=1}^{n} \frac{k_i}{N} J_{2i-1,2i}\right), \tag{3.13}$$

where the k_i's are integers between 0 and N. So θ independently rotates n two-planes in spacetime. Choosing complex coordinates $X^i = X_{2i-1} + iX_{2i}$ and $\overline{X}^i = X_{2i-1} - iX_{2i}$ for each of the two-planes, θ simply acts on the X^i by multiplication by the phase $e^{2\pi i k_i/N}$. The twist operator σ_+ associated with θ will therefore satisfy a separate local monodromy condition (2.4) with each of the n fields X^i. We can think of σ_+ as the product of n separate twist operators $\sigma_+^{(i)}$, each of which rotates a single complex coordinate (field) X^i (and similarly for σ_-), and then write the correlation functions for the $\sigma_\pm$'s as the product of n correlation functions, each of which involves only $\sigma_\pm^{(i)}$ twist operators for some i. In other words, we have effectively factorized the problem into n two-dimensional problems. In fact this factorization typically fails, because the global monodromy conditions for the classical solutions, like eq. (3.11), involve lattice vectors v with nonzero components along more than one complex direction X^i. (If the lattice Λ is a direct sum of n orthogonal two-dimensional lattices, $\Lambda = \Lambda_2 \oplus \Lambda'_2 \oplus \cdots$, and if the rotation θ takes each two-dimensional lattice into itself, then the sum over classical solutions will factor into n separate sums, however.) On the other hand, the quantum piece of the calculation can always be factored into two-dimensional pieces, because the global monodromy requirements (3.10) for the quantum fluctuations X_{qu} do not couple the different X^i's. This allows us to use Green functions $g(z, w)$ involving only a single complex field ∂X, rather than n of them.

In contrast, orbifolds can also be constructed using finite nonabelian point groups P. These orbifolds will give rise to "nonabelian" twist fields; i.e. the local monodromy conditions for different twist operators will not in general be simultaneously diagonalizable. (Since the action of any given twist field can always be diagonalized, the twist looks abelian locally. It is only the global behavior which is nonabelian.) Therefore Green functions in the presence of these twists will necessarily be matrix-valued,

$$g^{ij}(z, w; z_i) \equiv \frac{\left\langle -\frac{1}{2}\partial_z X^i \partial_w \bar{X}^j \sigma_1(z_1)\sigma_2(z_2)\dots\right\rangle}{\left\langle \sigma_1(z_1)\sigma_2(z_2)\dots\right\rangle}, \tag{3.14}$$

and will satisfy noncommuting monodromy conditions. The construction of these Green functions remains an open problem.

We conclude this section with a discussion of orbifold geometry for the heterotic string. As described in sect. 2, spacetime bosons for the twisted sector correspond to dimension-$\frac{1}{2}$ twist superfields, whereas spacetime fermions are created by dimension-1 spin-twist fields. Recall that the fields ∂X^i and ψ^i acquire the same phase around a twist superfield, so that the superpartner of the stress tensor is single-valued. From the point of view of orbifold geometry, this requirement results from the identical transformations of ∂X^i and ψ^i under the space group elements associated with the twist. For the $E_8 \otimes E_8$ heterotic string, each space group element may also be accompanied by an $E_8 \otimes E_8$ gauge transformation, subject to constraints arising from modular invariance [13]. One can represent the $E_8 \otimes E_8$ current algebra by two sets of 16 left-handed (antianalytic) fermions $\lambda(\bar{z})$, each transforming as the **16** of $O(16) \subset E_8$. Choosing appropriate complex combinations of the fields λ^i and $\bar{\lambda}^i$, one can diagonalize any given gauge transformation. For example, a Z_N space group element will be accompanied by some Z_N gauge transformation. Around a twist field $\tilde{s}$ for this combined spatial rotation and gauge transformation, the fields λ^i will have the monodromy

$$\lambda^i(e^{-2\pi i}\bar{z}) = e^{2\pi i \tilde{k}_i/N}\lambda^i(\bar{z}) \tag{3.15}$$

for some integers $\tilde{k}_i$. Like the ψ, s system, the λ, $\tilde{s}$ system can be bosonized; then $\tilde{s}$ becomes a fractional exponential,

$$\tilde{s}(\bar{z}) = \prod_i e^{i\tilde{k}_i \tilde{H}_i(\bar{z})/N}.$$

That is, $\tilde{s}$ implements a translation or "shift" on the root lattice of $E_8 \otimes E_8$ by a vector $\delta = (\tilde{k}_1/N, \tilde{k}_2/N, \ldots)$, such that $N\delta$ is a lattice vector [13]. In general, several twist fields $\tilde{s}$ will have the same dimension, since several "shifted" vectors $l + \delta$ may have the same length. Here l is an $E_8 \otimes E_8$ lattice vector, and so its components will be either all integers or all half-integers, with an even sum (corresponding to the Neveu-Schwarz and Ramond sectors for the λ^i, respectively). Thus there is an additional degeneracy of the Hilbert space, which occurs because the physical states form multiplets under the gauge group left unbroken by the twist.

There are typically a number of consistent possible gauge transformations which can accompany the elements of a given orbifold space group S. One possibility is to embed the spin connection in the gauge connection; i.e. we take $\tilde{k}_i = k_i$, $i = 1, \ldots, n$; $\tilde{k}_i = 0$, $i > n$ in (3.15). Then the left- and right-handed fermions transform in the same way under the twist, and the nonlinear model for this orbifold is nonchiral. In particular, the modes which appear in the (type II) superstring also appear as states of the heterotic string in this background, in the (0) picture. We will use this correspondence in sect. 4 to show that superpotentials vanish for these modes. The directions in the parameter space of the nonlinear model along which the orbifold singularities are resolved to form a smooth Calabi-Yau manifold belong to this category.

An important consideration for fermionic strings on orbifolds is whether spacetime supersymmetry is preserved. This depends on whether the point group P for the $2n$-dimensional orbifold is a subgroup of SU(n) [3,13]. For the case of a Z_N orbifold, P is generated by the rotation θ of (3.13), which is in SU(n) if $\sum_{i=1}^{n} k_i = N$. This condition was needed in sect. 2 to construct the dimension-$\frac{1}{2}$ twist superfields $\mathcal{T}$ which create massless spacetime bosons. If spacetime supersymmetry is broken, then the lowest-lying bosonic modes of the fermionic string in the twisted sectors are typically tachyonic rather than massless [32]. The construction of vertex operators for the massless bosons will be slightly different in this case.

4. Bosonic correlation functions

4.1. Z_2 TWISTS

We are now ready to apply the technique outlined in sect. 2 to calculate the correlation function of four twists. In this section, we concentrate on the correlations of the σ's; in the next, we combine these with the rest of the twist superfields to compute fermionic string twist field correlations and operator product coefficients.

First we will calculate the four Z_2 twist correlation. To do so we take a slight detour from the main line of development. We will use the stress tensor method to calculate the amplitude, but on a covering space of the z-plane where X is single-valued. The procedure illustrates some beautiful classical mathematics which underlies the twist fields. For Z_2 twists, the coordinate field X may be taken to be real, and there are two fixed points $f_0 = 0$, $f_1 = \frac{1}{2}\cdot 2\pi R$ of the twist operation on the spacetime circle (for convenience we consider a one-dimensional spacetime). There is a selection rule on what combination of twists can appear. Each twist field corresponds to a conjugacy class of the orbifold space group; the correlation function must be neutral overall, which means the product of all the conjugacy classes of the twists must be the conjugacy class of the identity. As discussed above, Z_2 twisting has two conjugacy classes, $\{(-1,\text{even})\}$ and $\{(-1,\text{odd})\}$. Neutrality forces us to have an even number of each type of twist. The correlations thus have a symmetry under any of the transformations $\sigma_0 \to -\sigma_0$, $\sigma_1 \to -\sigma_1$, $\sigma_0 \leftrightarrow \sigma_1$, which generate the dihedral group D_4 (the symmetries of the square). This is the symmetry group of the Ashkin-Teller model [21]. By $SL_2(C)$ invariance we can fix the location of any three of the twists to 0, 1, and ∞, and by the dihedral group symmetry, the most general possible configuration of twists is

$$Z_{\varepsilon_0\varepsilon_1}(x) = \lim_{z_\infty\to\infty} |z_\infty|^{1/4}\left\langle \sigma_0(z_\infty)\sigma_{\varepsilon_1}(1)\sigma_{\varepsilon_0+\varepsilon_1}(x)\sigma_{\varepsilon_0}(0)\right\rangle .$$

Here $\varepsilon_{0,1} \in \{f_0, f_1\}$ label the fixed point. The index $\pm$ on σ is superfluous here as the twist field is self-conjugate for Z_2 twists. The prefactor cancels the scaling dimension of σ to give a sensible $z_\infty \to \infty$ limit (more properly, this factor will come from whatever multiplies Z to make a scale-invariant string scattering amplitude). The classical solution ∂X_{cl} is a function holomorphic in the z-plane except for square root branch singularities at each of the four twists $\sigma_\varepsilon(z_i)$, $i = 1,\dots,4$

$$\partial X_{\text{cl}}(z) = \frac{\text{const}}{\left[(z-z_\infty)(z-1)(z-x)(z)\right]^{1/2}} . \tag{4.1}$$

The differential $dt = dz\,\partial X_{\text{cl}}$ defines a classical elliptic function, the Weierstrass function $\wp(t)$,

$$z(t) = \frac{\wp(t)-e_1}{e_2-e_1}, \qquad x \equiv \frac{e_3-e_1}{e_2-e_1}, \qquad e_1+e_2+e_3 = 0 . \tag{4.2}$$

Note that, as a holomorphic differential form on the surface, ∂X_{cl} is regular at

$z=\infty$. This function maps the torus with coordinate t two-to-one onto the sphere with coordinate z, branched over four points. These four points in z are the images of the four half-points of the torus: $z(\frac{1}{2})=0$, $z(\frac{1}{2}\tau)=1$, $z(\frac{1}{2}(1+\tau))=x$, $z(0)=\infty$. On the torus the field X is single-valued. The modulus τ of the torus is defined implicitly in terms of the cross-ratio x of the twist locations on the sphere:

$$x=\left(\frac{\vartheta_3(\tau)}{\vartheta_4(\tau)}\right)^4=\prod_{n=1}^{\infty}\left(\frac{1+u^{n-1/2}}{1-u^{n-1/2}}\right)^8, \qquad u=e^{2\pi i\tau}, \tag{4.3}$$

where $\vartheta_i(\tau)$ are the Jacobi theta functions. One computes correlation functions in an operator formalism by summing over intermediate states propagating between interactions (operators). Crossing symmetry of correlation functions on the sphere is a consequence of the theory's indifference to whether this sum is carried out in the s channel or the t channel. On the torus, this symmetry manifests itself as modular invariance of the partition function, given by the relation between x and τ. The relation of x to classical elliptic functions implies

$$x(\tau+1)=\frac{1}{x}, \qquad x(-1/\tau)=\frac{x}{x-1}. \tag{4.4}$$

Invariance under these is crossing symmetry among the various channels s, t, and u.

The path integral on the torus is over functions satisfying $X(t)=-X(-t)$ because X is antiperiodic on the z-plane; $t\leftrightarrow -t$ is the sheet interchange. To evaluate the quantum contribution to the partition function we use the Green function $g_\tau(t,t')$ on the torus. We write

$$\begin{aligned} g_\tau(t,t')&=g(t,t')+g(t,-t'),\\ g(t,t')&=-\tfrac{1}{2}\langle\partial X_{\mathrm{qu}}(t)\,\partial X_{\mathrm{qu}}(t')\rangle. \end{aligned} \tag{4.5}$$

Although X_{qu} is parity-odd on the torus, ∂X_{qu} is parity-even, hence the relative plus sign in the first line. We will also need

$$h(\bar t,t')=-\tfrac{1}{2}\langle\bar\partial X_{\mathrm{qu}}(\bar t)\,\partial X_{\mathrm{qu}}(t')\rangle. \tag{4.6}$$

By definition, the integral of the quantum Green function must have vanishing periods about any closed loop in the cut z-plane: $\Delta_{\mathscr{C}}X_{\mathrm{qu}}=0$ (cf. eq. (3.10)). Applied to eqs. (4.5), (4.6) on the torus covering, this implies

$$\begin{aligned} 0&=\int_0^1 dt\,g(t,t')+\int_0^1 d\bar t\,h(\bar t,t'),\\ 0&=\int_0^\tau dt\,g(t,t')+\int_0^{\bar\tau} d\bar t\,h(\bar t,t'). \end{aligned} \tag{4.7}$$

We build g as follows: first, look for a quantity that contains the double pole of g and is otherwise a holomorphic one-form in each of t and t'. In fact, the Weierstrass function $\wp(t-t')$ has these properties ($\wp(t-t') \sim (t-t')^{-2}$ in order that the origin of the t-plane is mapped to a branch cut at ∞ in the z-plane). The integral of this function is not single-valued; so the second step is to add a holomorphic one-form in t times a holomorphic one-form in t', which doesn't disturb the short-distance behavior, and substitute into eq. (4.7). The holomorphic forms on the torus are just the constants; thus we have

$$g(t,t') = \wp(t-t') + a_1\,,$$

$$h(\bar{t},t') = a_2\,.$$

Substituting into eq. (4.7) produces

$$a_1 = \frac{-1}{4i\,\mathrm{Im}\,\tau}\left[\int_0^\tau \wp - \bar{\tau}\int_0^1 \wp\right]. \tag{4.8}$$

Now note that $\int_0^t \wp \equiv -\zeta(t)$ is quasiperiodic, with $\zeta(t+\omega_i) = \zeta(t) + 2\eta_i$, $i=1,2$ (here $\omega_i = (1,\tau)$ are the periods of the torus), and integrating $\zeta(t)$ around the perimeter of the torus in the t-plane shows that $\eta_1\tau - \eta_2 = i\pi$. This relates the two integrals in (4.8), so that

$$g(t,t') = \tfrac{1}{2}\wp(t-t') - \tfrac{1}{2}\int_0^1 \wp - \frac{\pi}{2\,\mathrm{Im}\,\tau}\,,$$

$$h(\bar{t},t') = \frac{\pi}{2\,\mathrm{Im}\,\tau}\,. \tag{4.9}$$

The stress tensor on the torus is then $T(t) = [g_\tau(t,t') - \frac{1}{2}(t-t')^{-2}]_{t'=t}$; however we want the stress tensor on the sphere. We obtain it through the conformal mapping (4.2), remembering to take into account the conformal anomaly [4]

$$T(t) = T(z)\left(\frac{\partial z}{\partial t}\right)^2 + \tfrac{1}{12}c\left(\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right), \tag{4.10}$$

where $c=1$. Eq. (4.1) for $z(t)$ relates the derivatives of the Weierstrass function to $\wp$ itself, so one can simplify the expression for the schwarzian derivative (the second term on the r.h.s.). Using the "duplication formula" for $\wp$, $\wp(2t) = -2\wp(t) + (\wp''/2\wp')^2$, one finds

$$\frac{1}{12}\left(\frac{\wp'''}{\wp'} - \frac{3}{2}\left(\frac{\wp''}{\wp'}\right)^2\right) = -\tfrac{1}{2}\wp(2t)\,.$$

From (4.5) and (4.9), and the fact that $\lim_{t\to 0}[\wp(t) - t^{-2}] = 0$, the r.h.s. in (4.10) is simply

$$\langle T(t)\rangle = \tfrac{1}{2}\wp(2t) - I - \pi(\operatorname{Im}\tau)^{-1},$$

where

$$I = \int_0^1 \mathrm{d}t\,\wp(t) = (2\pi i)^2\left[\tfrac{1}{12} - 2\sum_{n=1}^{\infty}\frac{nu^n}{1-u^n}\right].$$

Combining these results gives

$$\langle T(z)\rangle = \left(\frac{e_2 - e_1}{\wp'(t)}\right)^2\left[\wp(2t) - I - \pi(\operatorname{Im}\tau)^{-1}\right], \tag{4.11}$$

which, using the expression (4.2) for the Weierstrass function, reduces to

$$\langle T(z)\rangle = \frac{1}{16}\frac{1}{(z-x)^2} + \frac{1}{z-x}\left[\frac{1}{8}\left(\frac{1}{x} + \frac{1}{x-1}\right)\right.$$
$$\left. + \frac{\left[-2(e_2 - e_1)x - 2e_1 - I - \pi(\operatorname{Im}\tau)^{-1}\right]}{4(e_2 - e_1)x(x-1)}\right] + \text{finite}.$$

The coefficient of the $1/(z-x)$ term is then $\partial_x \ln Z(x)$. Unfortunately, this answer is expressed in terms of the modulus of the torus τ, which is only implicitly a function of the cross-ratio x, eq. (4.3). It is a minor miracle that (4.11) can be integrated; the crucial observations are that

$$\begin{aligned}\frac{1}{4(e_2 - e_1)x(x-1)} &= -\frac{1}{4\pi^2 x\vartheta_2^4}\\ &= \frac{1}{4\pi i}\frac{\mathrm{d}\tau}{\mathrm{d}x},\end{aligned} \tag{4.12}$$

and that

$$I = (2\pi i)\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[u^{1/12}\prod_n(1-u^n)^2\right].$$

Therefore, the residue of the subleading singularity in $\langle T\rangle$,

$$\partial_x \ln Z_{\mathrm{qu}} = -\tfrac{1}{24}\left(\frac{1}{x} + \frac{1}{x-1}\right) - \frac{1}{4\pi i}\left[I + \frac{\pi}{\operatorname{Im}\tau}\right]\frac{\mathrm{d}\tau}{\mathrm{d}x},$$

integrates to

$$Z_{\text{qu}}(x) = \text{const}\cdot(\operatorname{Im}\tau)^{-1/2}|x(1-x)u|^{-1/12}\prod_{n=1}^{\infty}|1-u^n|^{-2}. \tag{4.13}$$

This is but half of the full partition function (luckily the more difficult half to evaluate). The rest comes from the sum over classical solutions, the different possible winding sectors in intermediate states. Note that these don't contribute as R, the radius of the orbifold, goes to infinity. In that limit the above is the complete answer for $\varepsilon_1 = 0$ (modulo an overall normalization to be determined shortly). For $\varepsilon_1 = 1$ the winding number is odd and the classical action damps the amplitude to zero for large R.

The classical solutions to $\bar{\partial}\partial X = 0$ on the torus are

$$X_{\text{cl}} = at + \overline{at}. \tag{4.14}$$

These will depend on which fixed point in spacetime each of the twists corresponds to; this contrasts with the quantum fluctuations, which by definition care only about the type of twist (i.e. the element of the point group) at each vertex and not how the coordinate varies from twist to twist. The Weierstrass function maps the half-points of the torus to the branch points on the sphere where the twists are; the field X must satisfy the condition $X = -X$ at these points, so

$$\frac{1}{\pi R}X_{\text{cl}}\left(t=\tfrac{1}{2}\right) = \varepsilon_0 \pmod 2,$$

$$\frac{1}{\pi R}X_{\text{cl}}\left(t=\tfrac{1}{2}\tau\right) = \varepsilon_1 \pmod 2.$$

We have fixed $X_{\text{cl}}(0) = 0$, and the preceding equation implies $(1/\pi R)X_{\text{cl}}(\frac{1}{2}(1+\tau))$ $= \varepsilon_0 + \varepsilon_1 \pmod 2$. The requirement on the constants a is that the solutions (4.14) describe strings which shift by $m_0 = 2n_0 + \varepsilon_0$ when taken around the fixed points at $z = x$ and $z = 1$, and by $m_1 = 2n_1 + \varepsilon_1$ around the fixed points $z = 0, x$; this implies

$$a = \frac{\pi i R}{\operatorname{Im}\tau}(m_1 + m_0\bar{\tau}).$$

The classical action for these solutions is

$$S_{\text{cl}} = \frac{1}{2\pi}\int d^2z\,|a|^2$$

$$= \frac{\pi R^2}{4\operatorname{Im}\tau}|m_0\tau + m_1|^2;$$

an additional factor of $\frac{1}{2}$ compensates for the fact that the torus parameter space double covers the sphere. The classical contribution to the amplitude is therefore

$$Z_{\mathrm{cl}} = \sum_{n_0, n_1 \in Z} \exp\left[-\frac{\pi R^2}{\operatorname{Im}\tau} \left| n_1 + n_0\tau + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_0\tau) \right|^2 \right].$$

The classical action is conformally invariant, so there is no anomalous term in the transformation back to the sphere. We can turn this sum over two winding numbers into a sum over winding numbers n and momenta m for a particular channel by performing a Poisson resummation in n_1;

$$Z_{\mathrm{cl}} = \left(\frac{\operatorname{Im}\tau}{R^2} \right)^{1/2} \sum_{\substack{m \in Z \\ n \in 2Z + \varepsilon_0}} (-)^{m\varepsilon_1} u^{(m/R + nR/2)^2/4} \bar{u}^{(m/R - nR/2)^2/4}. \tag{4.15}$$

Our final result for the normalized four Z_2 correlation function is then (4.13) times (4.15),

$$Z(x) = 2^{-2/3} |x(1-x)u|^{-1/12} \prod_{k=1}^{\infty} |1 - u^k|^{-2}$$
$$\times \sum_{\substack{m \in Z \\ n \in 2Z + \varepsilon_0}} (-)^{m\varepsilon_1} u^{(m/R + nR/2)^2/4} \bar{u}^{(m/R - nR/2)^2/4}. \tag{4.16}$$

Crossing symmetry of this correlation function results from its being composed of an invariant combination of modular functions on the torus. This expression for the twist amplitude contains, as discussed in sect. 2, all the operator product coefficients for the fields onto which two twists can be factorized. For Z_2 twists, all such fields live in the untwisted sector. The coefficients are revealed when we take the limit $x \to 0$ or 1, since the amplitude factorizes on the product of three-point functions, eq. (2.13). To take the $x \to 0$ limit we must perform a modular transformation (4.4) on the partition function (4.16). We find

$$\lim_{x \to 0} Z(x) = |x|^{-1/4} \sum_{\substack{m \in Z \\ n \in 2Z + \varepsilon_1}} \frac{(-)^{m\varepsilon_0}}{16^{h + \bar{h}}} x^h \bar{x}^{\bar{h}} (1 + \cdots), \qquad h, \bar{h} = \frac{1}{2}\left(\frac{m}{R} \pm \tfrac{1}{2} nR \right)^2. \tag{4.17}$$

The $h = \bar{h} = 0$ term in (4.17) gives the correct normalization for (4.16), since $|\langle \sigma\sigma \rangle|^2 = 1$. The operator product coefficients are all powers of 16; we will find similar behavior for Z_N twists below. Note that when $\varepsilon_1 = 1$ the two twists that are being factorized sit at different fixed points in spacetime. This means that the two asymptotic states are being stretched to different locations on the orbifold; correspondingly the operator product coefficient is damped exponentially in the radius of the orbifold, reflecting the large classical action required to pass between fixed points.

4.2. Z_N TWISTS

We now present a calculation of a correlation function involving four Z_N twists, generalizing the four Z_2 twist calculation just presented. Mapping the correlation function onto a smooth Riemann covering surface of higher genus (genus $N-1$ as it will turn out) becomes quite cumbersome for all but the simplest calculations. Moreover it is actually unnecessary; we will now show that one can solve the problem directly in the cut z-plane. In the case $N=2$ we will recover precisely the answer obtained above. To simplify the calculation, let us first assume that the lattice Λ for the orbifold is the direct sum of n orthogonal two-dimensional lattices, each of which is preserved by the Z_N twist. This reduces the problem to that for a two-dimensional orbifold (as noted in the previous section), which we will solve in this section. It is then easy to generalize the solution to that for an arbitrary lattice Λ. There are actually very few two-dimensional Z_N orbifolds, because the order-N rotation used to construct the orbifold must be an automorphism of some two-dimensional lattice; hence it must have order $N=2$, 3, 4, or 6. The $N=3$ and $N=6$ cases require the hexagonal lattice; the $N=4$ case requires the square lattice.

Choose complex coordinates $X=X_1+iX_2$ and $\bar{X}=X_1-iX_2$ for the two-plane. Then take θ to rotate this two-plane by an angle $2\pi k/N$ for some $k\in 1,\dots,N-1$, so the field X obeys the local monodromy conditions (2.4) with respect to the twist operators $\sigma_\pm$ associated with θ and θ^{-1}. The correlation function to be calculated contains two twists and two antitwists:

$$Z(z_i,\bar{z}_i)\equiv\langle\sigma_-(z_1,\bar{z}_1)\sigma_+(z_2,\bar{z}_2)\sigma_-(z_3,\bar{z}_3)\sigma_+(z_4,\bar{z}_4)\rangle\,. \tag{4.18}$$

Note that Z has net twist zero since $\theta^{-1}\theta\theta^{-1}\theta=1$, so it satisfies the point group selection rules mentioned in sect. 3. Each twist operator should also be labelled by an index ε denoting the fixed point of θ at which it creates a twisted state: $\sigma_{\pm,\varepsilon_i}(z_i)$. The space group will provide restrictions on which combinations of the ε_i can yield nonvanishing correlators, as in the Z_2 case. However, we will split the calculation into a quantum piece and a classical piece, as described in sect. 2, and calculate first the quantum piece of the amplitude, which is independent of this index; so we can omit the index for now. For $N=2$ there is no distinction between σ_+ and σ_-, and $Z(z_i)$ will give back the four Z_2 correlator calculated previously. For $N>2$ we can extract two types of three-point correlation functions from the four-point function. As $z_2\to z_1$ or $z_2\to z_3$ the correlation function factorizes on the exponentials $e^{ip\cdot X}$

(with quantized momentum p^i on the spacetime torus) and on other untwisted fields. Factorization on the identity operator ($p = 0$), and the fact that

$$\langle \sigma_+ \sigma_- I \rangle = \langle \sigma_+ \sigma_- \rangle = 1$$

gives the correct normalization for the two-point function for the states $|\sigma_\pm\rangle$, will fix the overall normalization of the four-point function. Letting $z_2 \to z_4$ factorizes the four-point function on states coming from the sector twisted by θ^{-2}. In particular this limit yields the three-point function $\langle \sigma_+ \sigma_+ \sigma_{--} \rangle$, where σ_{--} creates the ground state in the sector twisted by θ^{-2}. This three-point function will provide the exact string-tree-level Yukawa couplings for chiral generations coming from the twisted sectors, when we consider the heterotic string on a Z_N orbifold in sect. 5. These couplings will include contributions which are damped exponentially in the radii of the orbifold.

As described in sect. 2, the first step in constructing the quantum piece of the correlator (4.18) is to find the Green function in the presence of the four twists,

$$g(z, w; z_i) \equiv \frac{\langle -\frac{1}{2}\partial_z X \partial_w \bar{X} \sigma_-(z_1)\sigma_+(z_2)\sigma_-(z_3)\sigma_+(z_4)\rangle}{\langle \sigma_-(z_1)\sigma_+(z_2)\sigma_-(z_3)\sigma_+(z_4)\rangle}. \tag{4.19}$$

The Green function obeys the following asymptotic conditions, as discussed in sect. 2:

$$\begin{aligned} g(z, w; z_i) &\sim \frac{1}{(z-w)^2} + \text{finite} && \text{as } z \to w \\ &\sim (z - z_{1,3})^{-k/N} && \text{as } z \to z_{1,3} \\ &\sim (z - z_{2,4})^{-(1-k/N)} && \text{as } z \to z_{2,4} \\ &\sim (w - z_{1,3})^{-(1-k/N)} && \text{as } w \to z_{1,3} \\ &\sim (w - z_{2,4})^{-k/N} && \text{as } w \to z_{2,4}\,. \end{aligned} \tag{4.20}$$

The holomorphic fields for the cut z-plane in this case are:

$$\begin{aligned} \partial X^{(1)}(z) &\equiv \omega_k(z) \equiv [(z-z_1)(z-z_3)]^{-k/N}[(z-z_2)(z-z_4)]^{-(1-k/N)} \\ \partial \bar{X}^{(1)}(z) &\equiv \omega_{N-k}(z) \equiv [(z-z_1)(z-z_3)]^{-(1-k/N)}[(z-z_2)(z-z_4)]^{-k/N} \end{aligned} \tag{4.21}$$

By inspection,

$$g(z,w)=\omega_k(z)\,\omega_{N-k}(w)\left\{\frac{k}{N}\,\frac{(z-z_1)(z-z_3)(w-z_2)(w-z_4)}{(z-w)^2}\right.$$
$$\left.+\left(1-\frac{k}{N}\right)\frac{(z-z_2)(z-z_4)(w-z_1)(w-z_3)}{(z-w)^2}+A\right\} \tag{4.22}$$

is the unique function of z and w with the desired properties (4.20). The factor $\omega_k(z)\omega_{N-k}(w)$ gives the correct behavior for g as z, $w\to z_i$. The first two terms in brackets give the required double pole without residue as $z\to w$. The remainder $A=A(z_i,\bar z_i)$ is a constant (in z and w) which is left undetermined by (4.20) because it multiplies the nonsingular product of two holomorphic differentials, one in z and one in w. Also, A contains the only dependence of the Green function on the antiholomorphic coordinates $\bar z_i$. A will be determined by the *global* monodromy condition (3.10) discussed in sect. 3: the requirement that the quantum field X be strictly periodic around all closed loops $\mathscr{C}_i$ in the cut z-plane.

Before determining A, however, we will extract the differential equation or $Z(z_i)$ from $g(z,w;z_i)$ by first taking $w\to z$, then $z\to z_2$. The operator product

$$-\tfrac{1}{2}\,\partial_z X\partial_w \bar X\sim(z-w)^{-2}+T(z)+\cdots$$

tells us that

$$\frac{\langle T(z)\sigma_-\sigma_+\sigma_-\sigma_+\rangle}{\langle\sigma_-\sigma_+\sigma_-\sigma_+\rangle}=\lim_{w\to z}\left[g(z,w)-(z-w)^{-2}\right]$$
$$=\frac{1}{2}\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{1}{z-z_1}+\frac{1}{z-z_3}-\frac{1}{z-z_2}-\frac{1}{z-z_4}\right)^2$$
$$+\frac{A}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,. \tag{4.23}$$

Then the operator product

$$T(z)\sigma_+(z_2)\sim\frac{h_\sigma\sigma_+(z_2)}{(z-z_2)^2}+\frac{\partial_{z_2}\sigma_+(z_2)}{z-z_2}+\cdots$$

applied to (4.23) confirms that the scaling dimension of σ_+ (and also of σ_-) is

$h_\sigma = \frac{1}{2}(k/N)(1-k/N)$ and gives rise to the differential equation

$$\partial_{z_2} \ln Z_{\mathrm{qu}}(z_i, \bar{z}_i) = -\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{1}{z_2-z_1}+\frac{1}{z_2-z_3}-\frac{1}{z_2-z_4}\right)$$
$$+\frac{A(z_i,\bar{z}_i)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}. \tag{4.24}$$

We can use $\mathrm{SL}_2(\mathrm{C})$ invariance to fix the locations of three of the four vertex operators: $z_1=0$, $z_2=x$, $z_3=1$, $z_4\to\infty$. Then (4.24) becomes

$$\partial_x \ln Z_{\mathrm{qu}}(x,\bar{x}) = -\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{1}{x}-\frac{1}{1-x}\right)-\frac{A(x,\bar{x})}{x(1-x)}, \tag{4.25}$$

where

$$Z_{\mathrm{qu}}(x,\bar{x}) \equiv \lim_{z_\infty\to\infty} |z_\infty|^{2(k/N)(1-k/N)}\langle\sigma_-(z_\infty)\sigma_+(1)\sigma_-(x,\bar{x})\sigma_+(0)\rangle,$$

$$A(x,\bar{x}) \equiv \lim_{z_\infty\to\infty} -z_\infty^{-1}A(0,x,1,z_\infty).$$

Now we use the global monodromy conditions (3.10) to determine A. Inserting (3.10) into the appropriate correlation function implies

$$0=\oint_{\mathscr{C}_i} \mathrm{d}z\, g(z,w)+\oint_{\mathscr{C}_i} \mathrm{d}\bar{z}\, h(\bar{z},w). \tag{4.26}$$

The auxiliary correlation function

$$h(\bar{z},w;z_i) \equiv \frac{\langle -\frac{1}{2}\partial_{\bar{z}}X\partial_w\bar{X}\sigma_-\sigma_+\sigma_-\sigma_+\rangle}{\langle\sigma_-\sigma_+\sigma_-\sigma_+\rangle}$$
$$= B(z_i,\bar{z}_i)\bar{\omega}_{N-k}(\bar{z})\omega_{N-k}(w) \tag{4.27}$$

is determined (up to the constant factor B) in the same way as g was, using local monodromy. Eq. (4.26) should be satisfied for every closed loop on the z-plane, which has two Nth root branch cuts (fig. 4a). To determine the number of independent closed loops, we could map the complex plane with Nth root branch cuts to a smooth N-fold covering surface M, and compute its genus as follows: Triangulate the cut plane such that four of the vertices of the triangulation are located at the four branch points. Let the triangulation have V vertices, E edges, and F faces, where $V-E+F=2$ is the Euler character of the genus-zero sphere.

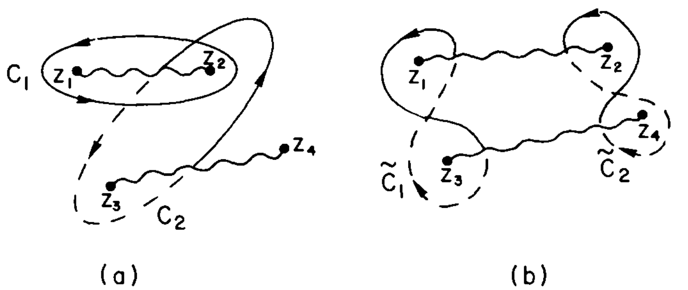

Fig. 4. Two sets of closed loops in the complex plane, for the case of four Z_N twists located at z_i. Different sheets are denoted by solid vs. dashed lines, and branch cuts by wavy lines. (a) The loops $\mathscr{C}_1$ and $\mathscr{C}_2$, which form a basis for the closed loops. (b) The loops $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$, which do not generally form a basis, but which correspond to the closed loops for the three-twist functions on which the four-point function factorizes.

Now lift the triangulation to the N-fold covering surface. The edges, faces and all vertices but the four at the branch points are replicated N times, so the Euler character of the covering surface is $4 + N((V-4) - E + F) = 2 - 2(N-1) = 2 - 2g$, and so M has genus $g = N - 1$. We might therefore expect to find $2g = 2(N-1)$ independent loops $\mathscr{C}_i$ in the z-plane, corresponding to the $2g$ generators of homology $\mathrm{H}_1(\mathrm{M};\mathrm{Z})$ for the covering surface M. We indeed find these loops, but we also find that they can all be generated from a basis consisting of only *two* loops $\mathscr{C}_1$ and $\mathscr{C}_2$, which are shown in fig. 4a, plus the loops obtained from $\mathscr{C}_1$ and $\mathscr{C}_2$ by shifting them "vertically" to different sheets of the cut plane. We will call such a loop, shifted by l sheets, $\alpha^{kl}C_{1(2)}$, where $\alpha \equiv \mathrm{e}^{2\pi i/N}$, because the functions $g(z,w)$ and $h(\bar{z},w)$ just pick up the overall phase α^{kl} when shifted by l sheets in the z-plane. This property also shows that if the condition (4.26) is satisfied for $\mathscr{C}_1$ and $\mathscr{C}_2$, then it is satisfied for all the $\alpha^{kl}C_{1(2)}$ and hence for all the closed loops. So we have two equations for the two unknowns A and B. (Note that the higher-genus covering surface has $3g-3 = 3N-6$ moduli describing it; whereas the configuration in the plane is described by one complex parameter, x. This shows that we must be dealing with a very special higher-genus surface, and also indicates why it is more economical to carry out the calculation directly in the z-plane.)

If we divide the global monodromy conditions (4.26) by $\omega_{N-k}(w)$, let $w \to \infty$ and use $\mathrm{SL}_2(\mathrm{C})$ invariance to fix the z_i as above, we find that they become

$$A\oint_{\mathscr{C}_i} \mathrm{d}z\,\omega_k + B\oint_{\mathscr{C}_i} \mathrm{d}\bar{z}\,\bar{\omega}_{N-k} = -\left(1 - \frac{k}{N}\right)\oint_{\mathscr{C}_i} \mathrm{d}z\,(z-x)\,\omega_k\,, \qquad i = 1,2\,. \quad (4.28)$$

All the contour integrals in (4.28) can be expressed in terms of the hypergeometric function

$$F(x) \equiv F\left(\frac{k}{N}, 1 - \frac{k}{N}; 1; x\right) = \frac{1}{\pi}\sin\left(\frac{\pi k}{N}\right)\int_0^1 \mathrm{d}y\, y^{-k/N}(1-y)^{-(1-k/N)}(1-xy)^{-k/N}$$

and its derivative:

$$\oint_{\mathscr{C}_1} \mathrm{d}z\,\omega_k = 2\pi i\alpha^{-k/2}F(x)\,, \qquad \oint_{\mathscr{C}_2} \mathrm{d}z\,\omega_k = 2\pi i F(1-x)\,,$$

$$\oint_{\mathscr{C}_1} \mathrm{d}\bar{z}\,\bar{\omega}_{N-k} = 2\pi i\alpha^{-k/2}\bar{F}(\bar{x})\,, \qquad \oint_{\mathscr{C}_2} \mathrm{d}z\,\bar{\omega}_{N-k} = -2\pi i\bar{F}(1-\bar{x})\,,$$

$$-\left(1-\frac{k}{N}\right)\oint_{\mathscr{C}_1} \mathrm{d}z\,(z-x)\,\omega_k = 2\pi i\alpha^{-k/2}x(1-x)\frac{\mathrm{d}F(x)}{\mathrm{d}x}\,,$$

$$-\left(1-\frac{k}{N}\right)\oint_{\mathscr{C}_2} \mathrm{d}z\,(z-x)\,\omega_k = 2\pi i x(1-x)\frac{\mathrm{d}F(1-x)}{\mathrm{d}x}\,. \tag{4.29}$$

Solving eqs. (4.28) for A, we find that

$$A(x,\bar{x}) = x(1-x)\,\partial_x \ln I(x,\bar{x})\,, \tag{4.30}$$

where the expression

$$I(x,\bar{x}) \equiv F(x)\bar{F}(1-\bar{x}) + F(1-x)\bar{F}(\bar{x}) \tag{4.31}$$

also turns out to be the action for the holomorphic fields (4.21). This form for $A(x,\bar{x})$ allows us to integrate the differential equation (4.25) for the quantum piece of the four-point function:

$$Z_{\mathrm{qu}}(x,\bar{x}) = \mathrm{const}\,|x(1-x)|^{-2(k/N)(1-k/N)}I(x,\bar{x})^{-1}\,.$$

We have used the $x \leftrightarrow \bar{x}$ symmetry of $Z(x,\bar{x})$ to fix the $\bar{x}$-dependence of the integration constant.

The next step is to construct properly normalized classical solutions $X_{\mathrm{cl}}(z,\bar{z})$ and $\bar{X}_{\mathrm{cl}}(z,\bar{z})$, compute their action, and sum over the appropriate set of these solutions. The z and $\bar{z}$ derivatives of the classical solutions are holomorphic and antiholomorphic fields for the cut plane, so we can write

$$\partial X_{\mathrm{cl}}(z) = a\omega_k(z)\,, \qquad \bar{\partial} X_{\mathrm{cl}}(\bar{z}) = b\bar{\omega}_{N-k}(\bar{z})\,;$$

$$\partial \bar{X}_{\mathrm{cl}}(z) = \bar{a}\omega_{N-k}(z)\,, \qquad \bar{\partial} \bar{X}_{\mathrm{cl}}(\bar{z}) = \bar{b}\bar{\omega}_k(\bar{z})\,; \tag{4.32}$$

where the constants a, b, $\bar{a}$ and $\bar{b}$ are determined by the global monodromy conditions (3.11) discussed in sect. 3. As in the determination of A in the quantum calculation, it suffices to satisfy (3.11) for just the two loops $\mathscr{C}_1$ and $\mathscr{C}_2$. This is because ω_k and $\bar{\omega}_{N-k}$, like g and h, pick up the phase α^{kl} when shifted by l sheets on the cut plane. So if $\Delta_{\mathscr{C}_1} X_{\mathrm{cl}} = v$, say, with v belonging to some coset of Λ, then $\Delta_{\alpha^{kl}\mathscr{C}_1} X_{\mathrm{cl}} = \alpha^{kl} v$, and one can check that $\alpha^{kl} v$ (which is the vector v rotated by the lattice automorphism θ^l) belongs to the same coset as v. (For example, the Z_3 cosets shown in fig. 3b clearly rotate into each other under rotations by $\frac{2}{3}\pi l$.) Thus satisfying the global monodromy conditions for $\mathscr{C}_1$ and $\mathscr{C}_2$ automatically satisfies them for $\alpha^{kl}\mathscr{C}_1$ and $\alpha^{kl}\mathscr{C}_2$ and hence for all the closed loops, since the rest can be generated from these loops.

We first construct two classical solutions $X_{\mathrm{cl},1}$ and $X_{\mathrm{cl},2}$ (and their tangent space conjugates $\bar{X}_{\mathrm{cl},1}$ and $\bar{X}_{\mathrm{cl},2}$) which have simple global monodromy:

$$\Delta_{\mathscr{C}_i} X_{\mathrm{cl},j} = \Delta_{\mathscr{C}_i} \bar{X}_{\mathrm{cl},j} = 2\pi\delta^{ij}, \qquad i = 1,2. \tag{4.33}$$

We will multiply these solutions by the appropriate coset vectors to get the properly normalized solution. Let a_i, b_i, $\bar{a}_i$, and $\bar{b}_i$ be the coefficients for $X_{\mathrm{cl},i}$ and $\bar{X}_{\mathrm{cl},i}$ in (4.32). They are determined using (4.33) and the integrals (4.29):

$$\begin{aligned} a_1 &= -\alpha^k \bar{a}_1 = -i\alpha^{k/2}\bar{F}(1-\bar{x})/I(x,\bar{x}), \\ a_2 &= \bar{a}_2 = -i\bar{F}(\bar{x})/I(x,\bar{x}), \\ b_1 &= -\alpha^k \bar{b}_1 = -i\alpha^{k/2}F(1-x)/I(x,\bar{x}), \\ b_2 &= \bar{b}_2 = +iF(x)/I(x,\bar{x}). \end{aligned} \tag{4.34}$$

It is important to keep track of the phases here. The global monodromy conditions (3.11) for $\mathscr{C}_1$ and $\mathscr{C}_2$, along with (3.12) to determine the cosets for v_1 and v_2, reveal that the coefficients for X_{cl} in (4.32) are

$$a = v_1 a_1 + v_2 a_2, \qquad b = v_1 b_1 + v_2 b_2,$$

$$\bar{a} = \bar{v}_1 \bar{a}_1 + \bar{v}_2 \bar{a}_2, \qquad \bar{b} = \bar{v}_1 \bar{b}_1 + \bar{v}_2 \bar{b}_2,$$

where

$$v_1 \in (1-\theta)(f_{\epsilon_2} - f_{\epsilon_1} + \Lambda), \qquad v_2 \in (1-\theta)(f_{\epsilon_2} - f_{\epsilon_3} + \Lambda). \tag{4.35}$$

Note also that the space group selection rule for this correlation function reads

$$f_{\epsilon_2} + f_{\epsilon_4} - f_{\epsilon_1} - f_{\epsilon_3} \in \Lambda. \tag{4.36}$$

We have absorbed a factor of 2π into the definition of Λ for later convenience; i.e. now X is identified with $X + 2\pi\Lambda$.

To compute the classical action

$$S_{\rm cl}=\frac{1}{4\pi}\int_{\rm C}\mathrm{d}^2z\,(\partial X_{\rm cl}\,\bar\partial\bar X_{\rm cl}+\bar\partial X_{\rm cl}\,\partial\bar X_{\rm cl})$$

requires the integral

$$\int_{\rm C}\mathrm{d}^2z\,|\omega_k|^2=\int_{\rm C}\mathrm{d}^2z\,|z|^{-2k/N}|z-x|^{-2(1-k/N)}|z-1|^{-2k/N}$$
$$=\frac{\pi^2}{\sin(\pi k/N)}I(x,\bar x)\,.$$

(The integral over the z-plane can be evaluated by splitting it up into holomorphic and antiholomorphic contour integrals using a method that Kawai, Lewellen and Tye [33] used to relate open and closed string tree amplitudes. In our case there are two terms, each the product of a holomorphic and an antiholomorphic integral of the kind evaluated in eq. (4.29).) After some algebra, the action for the classical solutions becomes

$$S_{\rm cl}(v_1,v_2)=\frac{\pi}{4\tau_2\sin(\pi k/N)}\left[v_2\bar v_2+\tau_1(v_1\bar v_2\bar\beta+\bar v_1v_2\beta)+|\tau|^2v_1\bar v_1\right],\quad(4.37)$$

where the phase $\beta=-i\alpha^{-k/2}$ and we have defined the modulus $\tau(x)$ of a "fake torus" by

$$\tau(x)\equiv\tau_1+i\tau_2\equiv\frac{iF(1-x)}{F(x)},\qquad\bar\tau(\bar x)\equiv\tau_1-i\tau_2\equiv\frac{-i\bar F(1-\bar x)}{\bar F(\bar x)}.\quad(4.38)$$

In the Z_2 case ($k/N=\frac{1}{2}$), the hypergeometric functions can be expressed in terms of elliptic theta functions:

$$F(x)=F\left(\tfrac{1}{2},\tfrac{1}{2};1;x\right)=\vartheta_3^2(\tau),$$
$$F(1-x)=F\left(\tfrac{1}{2},\tfrac{1}{2};1;1-x\right)=\vartheta_3^2(-1/\tau).$$

The modular transformation property $\vartheta_3(-1/\tau)=(-i\tau)^{1/2}\vartheta_3(\tau)$ shows that the variable τ defined in (4.38) for $N=2$ is indeed the modulus of the torus which we used previously to construct the four Z_2 twist amplitude. For $N>2$, τ just provides a compact notation.

We sum $e^{-S_{\rm cl}(v_1,v_2)}$ over the cosets for v_1 and v_2 given by eq. (4.35), and then multiply by $Z_{\rm qu}$ to give the final result:

$$Z_{\varepsilon_i}(x)=\mathrm{const}\frac{|x(1-x)|^{-2(k/N)(1-k/N)}}{\tau_2(x,\bar x)|F(x)|^2}\sum_{v_1,v_2}e^{-S_{\rm cl}(v_1,v_2)}.\quad(4.39)$$

In order to extract operator product coefficients of the form $C_{\sigma_+\sigma_-\mathcal{O}[X]}$ it is useful to Poisson resum this expression in the variable v_2, which converts a sum over a lattice into a sum over the dual lattice. We expect to end up with a sum over momenta p for untwisted states in the intermediate s channel. These momenta lie on Λ^*, the lattice dual to Λ. However, the sum in v_2 is not over Λ but over the coset $(1-\theta)(f_{\varepsilon_2}-f_{\varepsilon_3}+\Lambda)$. The peculiar phase β and the factor $\sin(\pi k/N)$ appearing in (4.37) play a key role in converting the v_2 sum into one over Λ. Note that $1-\theta=1-e^{2\pi i k/N}=-2\beta^{-1}\sin(\pi k/N)$. So if we let $v_2=-2\beta^{-1}\sin(\pi k/N)(f_{\varepsilon_2}-f_{\varepsilon_3}+q)$ in (4.37), the action $S_{\rm cl}(v_1,q)$ will be summed over $q\in\Lambda$. Poisson resummation in q then yields

$$Z_{\varepsilon_i}=\frac{\mathrm{const}|x(1-x)|^{-2(k/N)(1-k/N)}}{V_\Lambda\sin(\pi k/N)|F(x)|^2}\times\sum_{\substack{p\in\Lambda^*\\ v\in\Lambda_c}}\exp\left(-2\pi i\left(f_{\varepsilon_2}-f_{\varepsilon_3}\right)\cdot p\right)w^{(p+v/2)^2/2}\overline{w}^{(p-v/2)^2/2},\qquad(4.40)$$

where $w(x)\equiv\exp[\pi i\tau(x)/\sin(\pi k/N)]$, the coset $\Lambda_c\equiv(1-\theta)(f_{\varepsilon_2}-f_{\varepsilon_1}+\Lambda)$, and V_Λ is the volume of the unit cell for the lattice Λ. As $x\to 0$, the leading term in $Z(x)\sim|x|^{-2(k/N)(1-k/N)}$ (for $f_{\varepsilon_1}=f_{\varepsilon_2}$) factorizes the four-point function on $|\langle\sigma_+\sigma_- I\rangle|^2=1$, as explained above. This fixes the normalization: const $=V_\Lambda\sin(\pi k/N)$.

The expression (4.40) for the case $k/N=\frac{1}{2}$ can be compared with the previous Z_2 result (4.16). In this case one substitutes $F(x)=\vartheta_3^2(\tau)$, $w=e^{\pi i\tau}$; $p=(m_1/R,m_2/R)$, $m_{1,2}\in Z$; $f_{\varepsilon_2}-f_{\varepsilon_3}=\varepsilon_0\cdot\frac{1}{2}R$ and $v\in(2Z+\varepsilon_1)R$ (due to the normalization of $\Lambda=\{nR,n\in Z\}$ – the previous calculation included an extra factor of 2π). Using these facts and the expression (4.3) for the cross-ratio x in terms of theta functions, it is easy to check that (4.40) reduces to the square of (4.16). The square is simply because the twist field σ used here twists one *complex* coordinate and so it is actually the product of two uncorrelated twist fields of the type used in the previous calculation.

The results (4.39) and (4.40) easily generalize to an arbitrary Z_N twist θ (given by (3.13)) acting on an arbitrary $2n$-dimensional lattice Λ. For each complex coordinate we repeat the above analysis with k replaced by k_i, $i=1,\dots,n$. The only coupling between the n different complex dimensions is in the sum over cosets for v_1 and v_2, which are still given by the general expressions (4.35). So each term in the sum over cosets in (4.39) is simply replaced by the product of n such terms, with $k\to k_i,\tau\to\tau_{(i)}$ in each term. (Recall that $\tau(x)$ depends on k/N.) The gaussian integral needed for the Poisson resummation in q is likewise the product of n two-dimensional integrals, so the same replacements in (4.40) as were made in (4.39) also make (4.40) valid for an arbitrary lattice.

To extract the operator product coefficients $C_{\sigma_+\sigma_- e^{ip\cdot X}}$ and $C_{\sigma_+\sigma_+\sigma_{--}}$ one needs the behavior of $w(x)$ as $x \to 0$ and ∞, respectively. The requisite asymptotics for the hypergeometric function are

$$
\begin{aligned}
&F(x) \sim 1, \qquad F(1-x) \sim \frac{1}{\pi}\sin\left(\frac{\pi k}{N}\right)(-\ln x + \ln\delta), && x \to 0, \\
&F(x) \sim \alpha^{k/2}\frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}x^{-k/N} - \alpha^{-k/2}\frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}x^{-(1-k/N)}, && x \to \infty, \\
&F(1-x) \sim \frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}x^{-k/N} + \frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}x^{-(1-k/N)}, && x \to \infty.
\end{aligned}
\tag{4.41}
$$

Here $\ln\delta(k/N) \equiv 2\psi(1) - \psi(k/N) - \psi(1-k/N)$; the specific values of δ needed for the four two-dimensional orbifolds are $\delta(\frac{1}{2}) = 2^4$, $\delta(\frac{1}{3}) = 3^3$, $\delta(\frac{1}{4}) = 2^6$, $\delta(\frac{1}{6}) = 2^4 3^3$. Using the definition of τ (4.38), one finds that $w(x) \sim x/\delta$ as $x \to 0$, and $\tau \to \tau_\infty = \sin(\pi k/N) + i|\cos(\pi k/N)|$ as $x \to \infty$. So for example the operator products

$$C_{\sigma_+\sigma_-(e^{ip\cdot X}+e^{-ip\cdot X})} = \delta^{-p^2/2} \tag{4.42}$$

are obtained (modulo phases) from the limit

$$
\lim_{z\to 0} Z(x) = |x|^{-2(k/N)(1-k/N)} \sum_{\substack{p\in\Lambda^* \\ v\in\Lambda_c}} \frac{\exp\left(-2\pi i\left(f_{\epsilon_2} - f_{\epsilon_3}\right)\cdot p\right)}{\delta^{h+\bar h}} x^h \bar x^{\bar h}(1+\cdots),
$$

$$
h, \bar h = \tfrac{1}{2}\left(p \pm \tfrac{1}{2}v\right)^2,
$$

which generalizes equation (4.17) for the Z_2 case. By $e^{ip\cdot X} + e^{-ip\cdot X}$ in (4.42) we mean the operator whose two-point function has been normalized to $1\cdot|z-w|^{-2p^2}$. Again the operator product coefficients are damped exponentially in the radii of the orbifold when the two twists being factorized sit at different fixed points in spacetime ($f_{\epsilon_1} \neq f_{\epsilon_2}$). However, these coefficients do *not* give rise to Yukawa couplings for $p, v \neq 0$, because massless fermionic string states always have zero momentum and winding number in the compactified directions.

In the Z_2 case the $x \to \infty$ limit of $Z(x)$ provides no new information, due to crossing symmetry. (The factorization is on untwisted states in all three channels, s, t, and u.) For $N > 2$, however, we obtain three-twist operator product coefficients. In the limit $x \to \infty$ (for $N > 2$), $\tau_2(x)$ approaches a positive constant, $|\cos(\pi k/N)|$, so the "unresummed" expression (4.39) for $Z(x)$ can be factorized on the twisted states. The classical contributions to the three-twist functions can also be calculated directly, by the same techniques as used above for the four-point function. The disadvantage is that the overall normalization is left undetermined; this additional information *is* provided by factorization of the four-point function. We sketch very briefly the direct three-twist calculation, in order to see what to expect from the four-twist factorization. One finds that there is always exactly one holomorphic field $\partial X^{(1)}$ *or* one antiholomorphic field $\bar{\partial} X^{(1)}$ (whereas both existed for the four-point function), and correspondingly there is only one independent closed loop (say $\tilde{\mathscr{C}}$) with which to normalize the field. The classical solutions have the form $X_{\mathrm{cl}}(z)$ and $\bar{X}_{\mathrm{cl}}(\bar{z})$, or vice-versa, and may be thought of as *holomorphic instantons* [14]. Because there is only one normalization condition, the sum over classical solutions is over a single lattice, as opposed to the double lattice sum in (4.39) for the four-twist case.

We would therefore like to show that the double lattice sum for $Z(x)$ factors into two single sums as $x \to \infty$. First we make a change of basis for the lattices: The closed loops $\mathscr{C}_1$ and $\mathscr{C}_2$, used to normalize the four-twist classical solutions, differ from the loops $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$ which normalize the solutions for the two three-twist functions on which we want to factorize (see fig. 4). Of course we can generate $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$ from $\mathscr{C}_1$ and $\mathscr{C}_2$, and in fact fig. 4 shows that

$$\tilde{\mathscr{C}}_1 = \mathscr{C}_1 - \mathscr{C}_2 , \qquad \tilde{\mathscr{C}}_2 = \alpha^k \mathscr{C}_1 + \mathscr{C}_2 \tag{4.43}$$

for the two-dimensional orbifold examples. This suggests that we rewrite the sum in (4.39) in terms of the lattice vectors $\tilde{v}_1$ and $\tilde{v}_2$ which give the change in X_{cl} around the loops $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$, rather than around $\mathscr{C}_1$ and $\mathscr{C}_2$. By the linearity of the contour integrals defining the v's, they satisfy

$$\tilde{v}_1 = v_1 - v_2 , \qquad \tilde{v}_2 = \alpha^k v_1 + v_2 . \tag{4.44}$$

In terms of $\tilde{v}_1$ and $\tilde{v}_2$ the classical action for $x \to \infty$ simplifies to

$$S_{\mathrm{cl}}^{(\infty)}(\tilde{v}_1, \tilde{v}_2) = \frac{\pi}{4|\sin(2\pi k/N)|}\left(\tilde{v}_1^2 + \tilde{v}_2^2\right), \tag{4.45}$$

so the $\tilde{v}_1$ and $\tilde{v}_2$ sums appear to decouple in this limit. However, whereas the cosets for v_1 and v_2 were summed over independently in (4.39), the sums over $\tilde{v}_1$ and $\tilde{v}_2$ are constrained in general. This is because the loops $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$ do not in general form a basis from which all the closed loops can be generated (whereas $\mathscr{C}_1$ and $\mathscr{C}_2$

do). In particular, inverting (4.43) to get

$$\mathscr{C}_1 = \frac{1+\alpha^{-k}}{2(1+\cos(2\pi k/N))}\left(\tilde{\mathscr{C}}_1 + \tilde{\mathscr{C}}_2\right), \qquad \mathscr{C}_2 = \frac{1+\alpha^{-k}}{2(1+\cos(2\pi k/N))}\left(-\alpha^k\tilde{\mathscr{C}}_1 + \tilde{\mathscr{C}}_2\right) \tag{4.46}$$

shows that $\mathscr{C}_1$ and $\mathscr{C}_2$ cannot be generated from $\tilde{\mathscr{C}}_1$ and $\tilde{\mathscr{C}}_2$ (plus their sheet-shifted copies $\alpha^{kl}\tilde{\mathscr{C}}_1$ and $\alpha^{kl}\tilde{\mathscr{C}}_2$), except for the Z_3 case, because only then is $[2(1+\cos(2\pi k/N))]^{-1}$ an integer.

The physical reason why the $\tilde{v}_i$ sums are constrained is that several states in the doubly-twisted sector (states located at different fixed points of θ^2) can contribute to the factorized amplitude. In fact, the constrained double sum over $\tilde{v}_1$ and $\tilde{v}_2$ can be split into a number of double sums, one for each contributing fixed point of θ^2. Each of these double sums then factors into two single sums and allows us to extract the operator product coefficients which involve each of the intermediate states. For instance, we find in the two-dimensional examples of Z_3 (Z_4, Z_6) that 1 (2, 3) fixed points of θ^2 should contribute in the intermediate channel. Because some of the doubly-twisted physical states are linear combinations of states located at different fixed points (see sect. 3), this is equivalent to the correlation $\langle \sigma_{+,\varepsilon_1}\sigma_{+,\varepsilon_2}\sigma_{-\,-,\varepsilon_3}\rangle$ being nonzero for 1 (1 or 2, 2) value(s) of ε_3, given ε_1 and ε_2. These results are obtained using the space group selection rule described in the previous section, which here can be written as

$$f_{\varepsilon_1} + f_{\varepsilon_2} - (1+\theta)f_{\varepsilon_3} \in \Lambda\,,$$

where f_{ε_3} and θf_{ε_3} are the fixed points of θ^2 corresponding to a given doubly-twisted physical state. Correspondingly, the double lattice sum for the four-point function can be shown to split into 1 (2, 3) sums which then factorize. These sums turn out to be over cosets of Λ with respect to the rotation θ^2 rather than θ. The correctly normalized three-twist function is

$$C_{\sigma_{+,\varepsilon_1}\sigma_{+,\varepsilon_2}\sigma_{--,\varepsilon_3}} = \langle \sigma_{+,\varepsilon_1}\sigma_{+,\varepsilon_2}\sigma_{--,\varepsilon_3}\rangle$$

$$= \sqrt{V_\Lambda\left|\tan\frac{\pi k}{N}\right|}\,\frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}|1-2k/N|\right)}{\Gamma(|1-2k/N|)}\sum_{\tilde{v}}\exp\left[-\frac{\pi\tilde{v}^2}{4|\sin(2\pi k/N)|}\right],$$

$$\tilde{v} \in (1-\theta^2)\left(f_{\varepsilon_3} - f_{\varepsilon_2} + \Lambda\right), \qquad \frac{k}{N} \neq \tfrac{1}{2}. \tag{4.47}$$

(For $k/N = \frac{1}{2}$ the "twist field" σ_{--} is merely the identity operator, and (4.47) is replaced by $\langle \sigma_{\varepsilon_1}\sigma_{\varepsilon_2} I\rangle = \delta_{\varepsilon_1\varepsilon_2}$, using (4.17).) If the fixed point f_{ε_3} of θ^2 is not fixed by

θ, then (4.47) should contain a second sum with f_{ε_3} replaced by θf_{ε_3}. This operator product coefficient is one of our chief results. The classical contribution does agree with the results of the direct calculation which we sketched above.

Finally, we note that the Green functions g and h for the propagation of ∂X in the presence of twists may be used to evaluate any correlation function of non-twist vertices, in the presence of twists. To see this let us integrate g, h to obtain the Green function for X itself:

$$G(z,w) = \left\langle X_{\mathrm{qu}}(z,\bar{z})\bar{X}_{\mathrm{qu}}(w,\bar{w})\right\rangle$$

$$= \int_{z_0}^{z}\int_{w_0}^{w} g(z,w) + \int_{\bar{z}_0}^{\bar{z}}\int_{w_0}^{w} h(\bar{z},w) + \int_{z_0}^{z}\int_{\bar{w}_0}^{\bar{w}} \bar{h}(z,\bar{w}) + \int_{\bar{z}_0}^{\bar{z}}\int_{\bar{w}_0}^{\bar{w}} \bar{g}(\bar{z},\bar{w}).$$

Because the (quantum) Green functions g and h are defined to have vanishing global monodromy (cf. eq. (3.10)), G is well-defined (otherwise it would depend on a choice of integration contour around the twists). The choice of base points z_0, w_0 will drop out of vertex correlation functions. The quantum contribution to a correlation of $\mathcal{O}[X]$'s is then given by the set of free field contractions just as in the absence of twist fields, but now using G for the contractions instead of $-\ln|z-w|^2$. Note that G appears undifferentiated only in contractions with exponentials of X. Terms involving, say, z_0 will thus always appear in the form $\Sigma_i p_i G(z_i, w_j)$. Inserting the expression for G above, the base point dependence vanishes by momentum conservation. Thus the calculation of the quantum contribution to a correlation function such as

$$\left\langle \sigma_1(z_1)\dots\sigma_M(z_M) e^{-ik\cdot X_{\mathrm{qu}}(z)} e^{ik\cdot X_{\mathrm{qu}}(w)}\right\rangle = \left\langle \sigma_1(z_1)\dots\sigma_M(z_M)\right\rangle e^{k^2 G(z,w)}$$

reduces to the calculation of pure twist correlations. As discussed in sect. 2, the full correlation function is a sum of contributions for each classical solution X_{cl}. The fact that $\langle X\rangle = X_{\mathrm{cl}}$ is nonzero must then be taken into account when performing the free-field contractions. Correlation functions involving excited twist fields such as $\tau_{\pm}, \tau'_{\pm}$ can be obtained from ∂X correlations by moving the ∂X's near the twist fields σ_i and using the operator product relations (2.5). The pure twist correlation function is very much like the partition function for higher genus surfaces in that X correlations are excitations above the ground state of the twists, and the ground state correlation function appears as a factor in any other correlation function. In fact, we saw above that calculation of the four Z_2 twist correlation is quite similar to the calculation of a partition function on the torus.

5. Twists for fermionic strings

In the last section, we succeeded in calculating the four bosonic twist field correlation function, and extracted operator product coefficients for products of twist fields. The extension of this calculation to the fermionic string is straightforward because the parts of the twist that act on world-sheet spinors are trivial in the bosonized representation; their main role is to ensure that, when combined with the σ correlation (4.40), the full correlation function has no fractional powers of x out front. The result needed is

$$\left\langle e^{\alpha_4 \cdot H}(z_\infty) e^{\alpha_3 \cdot H}(1) e^{\alpha_2 \cdot H}(x) e^{\alpha_1 \cdot H}(0) \right\rangle = z_\infty^{-\alpha_4^2} x^{\alpha_2 \cdot \alpha_1} (1-x)^{\alpha_2 \cdot \alpha_3}. \tag{5.1}$$

In particular we will want the correlation of four $s_\pm$ fields

$$\left\langle s_-(z_\infty) s_+(1) s_-(x) s_+(0) \right\rangle = z_\infty^{-(k/N)^2} x^{-(k/N)^2} (1-x)^{-(k/N)^2} \times \text{cocycles}. \tag{5.2}$$

The values taken by the α's depends on both the type of string theory – type II or heterotic – and on which picture we choose. The picture used for the calculation is irrelevant in the end; the same answer for scattering amplitudes will hold for all of them. However the actual calculation may be simpler to perform in one picture than in another. This is certainly true in our case, as use of only the vertices V_0, eq. (3.6), requires evaluation of the correlation functions of excited twist fields τ. Excited twist correlation functions can be worked out from Green functions like $g(z,w)$, but there will be many terms to evaluate in the product of four twists. We can get by with the ground state twist correlations already calculated in sect. 4 if we simply change pictures.

Consider first the superstring. The superfield vertex is

$$V_{(0,0)}(z,\bar{z}) = \int d\theta\, d\bar{\theta}\, \hat{\mathscr{T}}_{ss} e^{ik_\mu \mathscr{X}^\mu}(z,\bar{z},\theta,\bar{\theta}), \tag{5.3}$$

taking $\hat{\mathscr{T}}_{ss}$ from eq. (2.18). The superstring has both a local analytic and antianalytic supersymmetry, so we can picture-change both the left- and right-movers. Each picture-change takes the highest component of a superfield to the lowest component. Whereas the theta integrals in the superfield picture pick out the higher components of the superfield (2.16), after both picture-changes the vertex becomes

$$V_{(-1,-1)} = e^{-\phi-\bar{\phi}} \tilde{s}_+ s_+ \sigma_+ e^{ik_\mu \mathscr{X}^\mu}(z,\bar{z}),$$

which is a considerable simplification. The superfield picture is more complicated because it is reached from this one by application of T_F and $\bar{T}_F$, each of which is a sum of terms for each complex dimension. We are not free to picture-change all of the vertices in a correlation function, but only two, in order to soak up the ghost background charge. Thus for the four-point function we are left with

$$\left\langle V^{-}_{(-1,-1)}(z_\infty,\bar z_\infty)V^{+}_{(0,0)}(1)V^{-}_{(-1,-1)}(x,\bar x)V^{+}_{(0,0)}(0)\right\rangle .$$

We have converted the two vertices containing $\hat{\mathscr{T}}_-$ as this is the most convenient choice in what follows. Let us write this as

$$\left\langle V^{-}_{(-1,-1)}(z_\infty,\bar z_\infty)\left(e^{\phi}T_F e^{\bar\phi}\bar T_F V^{+}_{(-1,-1)}(1)\right)V^{-}_{(-1,-1)}(x,\bar x)\left(e^{\phi}T_F e^{\bar\phi}\bar T_F V^{+}_{(-1,-1)}(0)\right)\right\rangle \tag{5.4}$$

in terms of the $(-1,-1)$ picture vertices and the picture-changing operator. The fermionic stress tensor is $T_F(z)=-\frac{1}{4}(\partial X\bar\psi+\partial\bar X\psi)-\frac{1}{2}\partial\mathscr{X}\cdot\Psi$, and similarly for $\bar T_F(\bar z)$.

Recall that when we evaluated the operator product of T_F with $\mathscr{T}^0_+$, only the $\partial X\bar\psi$ term was singular and not the $\partial\bar X\psi$ term. Since both operator products in (5.4) involve $\mathscr{T}^0_+$ and not $\mathscr{T}^0_-$, the contribution of the higher component $\mathscr{T}^1_+$ of the twist superfield vanishes by H and $\tilde H$ (ψ and $\tilde\psi$) charge conservation. This leaves only the part of T_F involving $-\frac{1}{2}\partial\mathscr{X}\cdot\Psi$, which does not affect the twist field part of the vertex. Thus the theta integral of the superfield vertices yields the $ik\cdot\Psi e^{ik\cdot\mathscr{X}}\mathscr{T}^0_+$ terms as the only nonvanishing contribution to the correlation function. It is important for what follows that this introduces explicit powers of momentum. The nonvanishing part of the vertex correlation function is thus

$$\begin{aligned}
Z^{\text{tot}}_{\varepsilon_i}(x) &\equiv \langle V_4\dots V_1\rangle\\
&=\left(\left\langle e^{-\phi}(z_\infty)e^{-\phi}(1)\right\rangle\left\langle c(z_\infty)c(1)c(0)\right\rangle\times\text{c.c.}\right)\\
&\quad\times\left\langle e^{ik_4\cdot\mathscr{X}}(z_\infty)e^{ik_3\cdot\mathscr{X}}(1)e^{ik_2\cdot\mathscr{X}}(x,\bar x)e^{ik_1\cdot\mathscr{X}}(0)\right\rangle\\
&\quad\times\left(\left\langle k_3\cdot\Psi(1)k_1\cdot\Psi(0)\right\rangle\times\text{c.c.}\right)\left\langle\mathscr{T}^0_-(z_\infty)\mathscr{T}^0_+(1)\mathscr{T}^0_-(x,\bar x)\mathscr{T}^0_+(0)\right\rangle .
\end{aligned}\tag{5.5}$$

The ghost parts of the correlation are trivial, providing a factor $|z_\infty|^2$. The $\mathscr{X}$'s give

$$\langle e^{ik_4\cdot\mathscr{X}}\dots e^{ik_1\cdot\mathscr{X}}\rangle=\left|x^{k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\right|^2$$

and the Ψ's produce a factor of $(k_1\cdot k_3)^2$. The twist correlation functions (4.40) and (5.2) cancel the factor of $|z_\infty|^2$. In (4.40) we make the replacements discussed in sect. 4 which convert the two-dimensional result to one for a six-dimensional orbifold. The full correlation function (5.5) is then integrated over the complex plane to give the scattering amplitude for four massless twisted bosons:

$$\int d^2x\, Z^{\text{tot}}_{\epsilon_j} = \tfrac{1}{4}u^2 \int d^2x\, |x|^{-2-s}|1-x|^{-2-t} P_{\epsilon_j}(x,\bar{x}),$$

$$P_{\epsilon_j}(x,\bar{x}) \equiv \sum_{\substack{p\in\Lambda^* \\ v\in\Lambda_c}} \prod_{i=1}^{3} |F_{(i)}(x)|^{-2}$$

$$\times \exp\left(-2\pi i\left(f_{\epsilon_2}-f_{\epsilon_3}\right)\cdot p_{(i)}\right) w(x)^{(p_{(i)}+v_{(i)}/2)^2/2}\,\overline{w}(\bar{x})^{(p_{(i)}-v_{(i)}/2)^2/2}. \tag{5.6}$$

Here $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$, and $u=-(k_1+k_3)^2$; i runs over the three complex dimensions of the orbifold; and $w(x)$ is defined following eq. (4.40).

Clearly one cannot evaluate the integral in (5.6) explicitly; however, one may deduce its basic structure as a function of the momenta, using asymptotics for $P_{\epsilon_j}(x,\bar{x})$. For $x\to\infty$ the behavior depends on the form of the twist θ; that is, on the fractions k_i/N. Since $\Sigma k_i/N=1$, at most one of the k_i/N can be greater than $\frac{1}{2}$, say k_1/N. So there are two possibilities, $k_1/N<\frac{1}{2}$ or $k_1/N>\frac{1}{2}$; the amplitude (5.6) behaves differently in the two cases. (If some $k_i/N=\frac{1}{2}$, then the complex field X^i is not twisted by θ^2, and (4.41) is not valid; these cases can be treated separately.) Using the asymptotics (4.41) for $F_{(i)}(x)$ and also $\Sigma k_i/N=1$ one has

$$\begin{aligned} P_{\epsilon_j}(x,\bar{x}) &\sim 1, && x\to 0,1;\\ &\sim |x|^2, && x\to\infty \quad \left(\frac{k_1}{N}<\tfrac{1}{2}\right);\\ &\sim |x|^{2-2(2k_1/N-1)}, && x\to\infty \quad \left(\frac{k_1}{N}>\tfrac{1}{2}\right). \end{aligned} \tag{5.7}$$

Note that as $x\to 0$, the amplitude factorizes on the untwisted sector. Here the integral in (5.6) behaves like

$$\int d^2x\, |x|^{-2-s} \sim \frac{1}{s}$$

at low momentum. Thus there is a u^2/s pole in the S-matrix. Similarly, in the crossed channel $x\to 1$ there will be a u^2/t pole. These poles indicate the exchange of massless particles (gravitons, dilatons, etc.) in the untwisted sector. However, in the u channel where one factorizes on the doubly-twisted sector, the $1/u$ pole (for $k_1/N<\frac{1}{2}$) in the integral is cancelled by the explicit factor of u^2. For $k_1/N>\frac{1}{2}$ this pole is shifted from $u=0$ to $u=2(2k_1/N-1)$, so the amplitude is proportional to

u^2 rather than u at low momentum. This softer behavior will be important when we study amplitudes involving $(0,2)$ modes of the heterotic string. After subtracting out the massless poles arising from lower order exchange, the superstring amplitude vanishes at zero momentum; there are therefore no four-twist contact terms in the low energy effective action.

This last result is of some importance, for it says that the effective action for superstrings compactified on orbifolds contains no potential terms for the massless twisted states. From a field theoretic viewpoint, giving an expectation value to these massless modes would correspond to smoothing out the orbifold singularities (known as "blowing up" the fixed points of the orbifold). A flat potential means that this can be done while preserving the equations of motion, in accord with previous arguments [11, 14]. Note, however, that we have calculated the exact (string tree-level) potential in the orbifold limit – no approximations have been made. In terms of a nonlinear (sigma) model description, blowing up the fixed points would correspond to the addition to the action of the twist field at zero momentum, with a small coefficient. The absence of a potential for the blowing up mode means that the twist operator is truly marginal, and the perturbed nonlinear model still would have vanishing beta function. Although we have only calculated explicitly the four scalar field contribution to the effective action, in any higher point calculation the picture-changing procedure used in (5.4) applies. If there are n twist superfields and n antitwist superfields, we leave two of the antitwist superfields in the $(-1,-1)$ picture and again find a factor of $(\text{momentum})^4$ from the Ψ correlations which multiplies the unintegrated amplitude. The integration produces $1/k_i \cdot k_j$ poles from the regions of integration where two or more vertices collide; here we may apply the operator product expansions for the twist fields $\sigma_\pm$ and $s_\pm$. Subtracting out these poles again leaves an amplitude that vanishes as $(\text{momentum})^2$. Thus the superstring effective potential for equal numbers of twist fields and antitwist fields will be flat to all orders in the number of twist fields.

We now turn to the derivation of four twisted string scattering amplitudes in the heterotic string. We will be able to relate four-point amplitudes for the various twisted heterotic states to the basic four-twist superstring amplitude (5.6) by making simple modifications. In the superstring, states must be highest weight under the fermionic generators in T_F and $\bar{T}_F$ as well as the Virasoro generators; the additional restrictions mean fewer physical vertices. The left-handed parts of the heterotic string are not restricted in this way; they need only be conformal fields, rather than superconformal fields, so there are more massless states than for the superstring. Moreover, there are several options for the expectation value of the $E_8 \otimes E_8$ gauge field. (We focus on this case rather than $\mathrm{Spin}(32)/Z_2$.) First consider the case $\langle \omega \rangle = \langle A \rangle$; the spin connection is embedded in a canonical SU(3) subgroup of one E_8. In this case, the orbifold has the $(2,2)$ supersymmetry of the superstring. Note, however, that the conditions which define physical vertices are different because this extra supersymmetry is *global* and does not contribute to the BRST charge (and therefore to the physical state conditions) as it does in the superstring. The singly-twisted sector of the Hilbert space (in the (-1) picture for the right-handed modes) contains the following massless states:

$$(3,\mathrm{grav}) \qquad \tilde{\alpha}^{i}_{-k_i/N}\bar{\lambda}^{i}_{-(1/2-k_i/N)}|\hat{\mathscr{T}}^{0}_{+}\rangle, \qquad i=1\ldots3\ (\text{no sum on } i),$$

$$(10) \qquad \lambda^{a}_{-1/2}|\hat{\mathscr{T}}^{0}_{+}\rangle, \qquad a=7\ldots16,$$

$$(1) \qquad \bar{\lambda}^{1}_{-(1/2-k_1/N)}\bar{\lambda}^{2}_{-(1/2-k_2/N)}\bar{\lambda}^{3}_{-(1/2-k_3/N)}|\hat{\mathscr{T}}^{0}_{+}\rangle,$$

$$(16) \qquad s^{\alpha}|\hat{\mathscr{T}}^{0}_{-}\rangle, \qquad \alpha=1\ldots16. \tag{5.8}$$

The first set of these is a multiplet of gravitational scalars; in fact, the trace $\sum_i\tilde{\alpha}^{i}_{-k_i/N}\bar{\lambda}^{i}_{-(1/2-k_i/N)}|\hat{\mathscr{T}}^{0}_{+}\rangle$ is the (2,2) supersymmetric "blowing-up mode" of the superstring; i.e. among the physical heterotic states are those which are allowed in the type II theory. The remaining states are a **27** of E_6, decomposed under the manifest SO(10) of the fermionic formulation of the gauge algebra. In addition each of the above states should carry an index denoting the fixed point of θ at which it resides, and each is accompanied by a four-dimensional superpartner. It is a simple exercise to write down the conformal fields that create these states.

Other possible embeddings of the spin connection depend on the details of the model. These embeddings result in (0,2) supersymmetric conformal field theories, as opposed to the (2,2) solutions considered above. In general, they will not contain matter fields which correspond to anything in a type II superstring background. For instance, in the Z orbifold described in [13] (a Z_3 twist acting on three orthogonal two-dimensional spacetime lattices), it is possible to embed the spin connection in both E_8's simultaneously: $\langle\omega\rangle=\langle A\rangle_1=\langle A\rangle_2$. In other words, we choose $\tilde{k}_{i,1}/N=\tilde{k}_{i,2}/N=k_i/N=\frac{1}{3}$, $i=1,2,3$, and 0 otherwise. This choice breaks the gauge group to $(E_6\otimes SU(3))^2$. The massless states in the twisted sector are singlets under both E_6's but transform as $\bar{\mathbf{3}}$'s under both SU(3)'s:

$$\bar{\lambda}^{i}_{-1/6,1}\bar{\lambda}^{j}_{-1/6,2}|\hat{\mathscr{T}}^{0}_{+}\rangle.$$

Correspondingly, the lowest dimension twist field must twist both sets of gauge fermions:

$$\hat{\mathscr{T}}^{0}_{+}(i,j)=e^{-\phi}(z)\Big(\tilde{t}^{(i)}_{+,1}\prod_{k\neq i}\tilde{s}^{(k)}_{+,1}(\bar{z})\Big)\Big(\tilde{t}^{(j)}_{+,2}\prod_{l\neq j}\tilde{s}^{(l)}_{+,2}(\bar{z})\Big)\prod_{m=1}^{3}s^{(m)}_{+}(z)\sigma^{(m)}_{+}(z,\bar{z}), \tag{5.9}$$

where

$$\tilde{s}^{(i)}_{+(1,2)} = e^{i\tilde{H}_{i(1,2)}/3}, \qquad \tilde{t}^{(i)}_{-(1,2)} = e^{-2i\tilde{H}_{i(1,2)}/3}.$$

These fields have dimension $h, \bar{h} = 1$, and are (-1) picture vertices. The lower dimension field where only unexcited twists appear for the gauge field fermions does not satisfy the separate projections $(-)^{\tilde{F}_1} = (-)^{\tilde{F}_2} = (-)^F = 1$ (besides, it doesn't have the right dimension).

Let us first consider the (2,2) supersymmetric backgrounds. The vertex for the blowing-up mode described above was given in eq. (3.6) and is in fact the same vertex (5.3) that appears in the type II theory, apart from the term

$$\prod_{i=1}^{3} \mathscr{T}_0^{(i)} ik_\mu \tilde{\Psi}^\mu e^{ik_\mu \mathscr{X}^\mu}(z, \bar{z}), \tag{5.10}$$

which appears in the type II vertex but not in the heterotic vertex. Superstring amplitudes, like the above four-point calculation, will therefore give us directly the corresponding scattering amplitudes for the heterotic blowing-up modes in the low-momentum limit, because (5.10) vanishes as $k_\mu \to 0$. The heterotic vertex is of course written in the (0) "picture" on the bosonic (left-moving) side because there is no local superconformal invariance and hence no picture-changing. On the other hand, for the superstring we were able to use the picture-changing trick on both the left- and right-moving sides in order to simplify the calculation and to extract four explicit powers of momentum; this led us to conclude that the superstring amplitude contains no contact terms. The term (5.10) implies that the heterotic amplitude is not exactly the same as the corresponding superstring amplitude. However, if one amplitude is subtracted from the other, then the difference contains at least one factor of $(\text{momentum})^2$ from the left-movers, since the term (5.10) must appear at least twice in the difference in order to contribute. The difference also contains another common factor of $(\text{momentum})^2$ from using the (-1) picture on the right-hand side for both calculations. These two factors are enough to ensure that the difference cannot give rise to contact terms. Therefore our arguments about the flatness of the potential to all orders for the blowing-up modes of the superstring apply to the heterotic blowing-up modes as well.

The picture-changing trick used for the superstring avoids the arduous task of evaluating excited-twist correlation functions. One can calculate amplitudes for either string in the superfield (0) picture for all of the vertices; it just requires more work to evaluate them. In so doing, *global* superconformal invariance (OSp(2, 1)) may be used to relate correlations of highest components of superfields to those with lower components. One finds that the highest components of the twist superfields contribute total derivatives to the correlation function, which can be integrated by parts onto the spacetime factor $|x|^{-s}|1-x|^{-t}$ to give the extra factors of momentum. For the heterotic amplitude for four blowing-up modes one finds explicitly

$$\int d^2x\, Z_{\varepsilon_j}^{\mathrm{tot}} = \int d^2x\, |x|^{-2-s}|1-x|^{-2-t}\left[\tfrac{1}{4}u^2 - \tfrac{1}{8}u\frac{(s(1-\bar{x})-t\bar{x})^2}{\bar{x}(1-\bar{x})}\right]P_{\varepsilon_j}(x,\bar{x}). \tag{5.11}$$

Taking the first term in brackets gives the superstring amplitude (5.6). The second term in brackets gives the difference between the two. Using (5.7) it is seen to be linear in s, t, and u at low energy, so it also does not give rise to a contact term, as we argued above.

In any case, the result is again that the (2, 2) modes which correspond to resolving the orbifold singularities have a flat potential. In the superstring calculation, this was a consequence of the fact that certain excited twist fields did not contribute to the correlation function; this forced the theta integrals of the superfield vertices to pick out the highest component $ik\cdot\Psi e^{ik\cdot\mathscr{X}}$ of the spacetime part of the vertex, bringing down explicit factors of momenta. It was these extra powers of momenta which led to the absence of contact terms in the effective action for the superstring blowing-up modes and also for the (2, 2) heterotic string blowing-up modes. For the (0, 2) modes (those which have no counterpart in the superstring), no such argument applies; the left-moving (antianalytic) contributions to the twist correlation functions are not just total derivatives in $\bar{x}$. Thus the potential for (0, 2) fields is *not* generically flat because there aren't enough powers of spacetime momentum in the amplitude to prevent the appearance of a contact term in the effective action; $\int d^2x\,\langle V_4 \ldots V_1\rangle \neq 0$ at zero momentum for the (0, 2) modes.

For example, the scattering amplitude for four massless **27** scalars is simple to work out. Take the four states to be

$$\lambda^{a_1}_{-1/2}|\hat{\mathscr{T}}^0_+\rangle, \qquad \bar{\lambda}^{a_1}_{-1/2}|\hat{\mathscr{T}}^0_+\rangle, \qquad \lambda^{a_2}_{-1/2}|\mathscr{T}^0_-\rangle, \qquad \bar{\lambda}^{a_2}_{-1/2}|\mathscr{T}^0_-\rangle, \tag{5.12}$$

with $a_1 \neq a_2$ (i.e. all in the **10** of SO(10)). Then the calculation is exactly the same as the previous superstring calculation except that the correlation

$$\langle k_3\cdot\tilde{\Psi}(1)k_1\cdot\tilde{\Psi}(0)\rangle = -k_1\cdot k_3$$

is replaced by

$$\bar{z}_\infty\langle e^{-i\tilde{H}_{a_2}}(\bar{z}_\infty)e^{-i\tilde{H}_{a_1}}(1)e^{i\tilde{H}_{a_2}}(\bar{x})e^{i\tilde{H}_{a_1}}(0)\rangle = 1,$$

so the amplitude is just the superstring amplitude (5.6) divided by a factor of $\frac{1}{2}u$. The u/s and u/t poles now result from gauge boson rather than graviton exchange (since the s and t channels are gauge non-singlet); subtracting them out leaves a four-boson contact term (for $k_1/N < \frac{1}{2}$), which can be completed into an E_6-invariant $\mathbf{27}^2\overline{\mathbf{27}}^2$ coupling. In fact this term is related by supersymmetry to Yukawa couplings for three **27**'s, which we will discuss shortly. For $k_1/N > \frac{1}{2}$ the first u-channel pole in the integral for the four-point amplitude is massive, at $u = 2(2k_1/N - 1)$ rather than at $u = 0$; this means that subtraction of the gauge boson exchanges leaves an amplitude proportional to u rather than a contact term. We will find that the Yukawa couplings correspondingly vanish in this case. The amplitudes involving the non-blowing-up modes in the multiplet of gravitational scalars in (5.8) can also be calculated but require excited bosonic twist fields and so are more complicated.

The above analysis also applies to (0,2) orbifolds (spin connection ≠ background gauge field). Here there are no left-right symmetric "blowing-up modes" at all, and in general twisted vertex scattering amplitudes will generate a non-flat effective potential for all the matter fields coming from the twisted sector. Consider for example the (0,2) orbifold obtained from the Z orbifold by embedding the spin connection once in each E_8. The only massless twist fields are the $\hat{\mathscr{T}}^0_+(i, j)$ of eq. (5.9). The four-point amplitude

$$\left\langle \hat{\mathscr{T}}^0_-(i_4, j_4)\hat{\mathscr{T}}^0_+(i_3, j_3)\hat{\mathscr{T}}^0_-(i_2, j_2)\hat{\mathscr{T}}^0_+(i_1, j_1)\right\rangle$$

depends on the combination of (i_m, j_m) chosen, since these determine which channels are gauge-singlet, etc. The choice $j_1 = i_3 = i_4 = j_4 \neq i_1 = i_2 = j_2 = j_3$ leaves all three channels gauge non-singlet, and this amplitude works out to be exactly the same as that for the four **10**'s in (5.12); namely the superstring amplitude (5.6) divided by $\frac{1}{2}u$. The contact term generated here is also the same (modulo different group theory factors). The exact calculations here substantiate the results of the instanton calculations of [14], that (0,2) models based on smooth Calabi-Yau manifolds are generically not solutions to the classical string equations of motion.

Finally, we discuss the calculation of Yukawa couplings in the low-energy effective action for strings on orbifolds. The twist field operator product coefficients give them directly, because Yukawa couplings are simply the three string scattering amplitudes at zero momentum. In the superstring and the heterotic string, Yukawa couplings are related to the four-boson contact terms we described above by four-dimensional supersymmetry; they both derive from the same Φ^3 term in the superpotential. Looking for this term in the superpotential *via* Yukawa couplings is therefore equivalent to looking for it *via* contact terms and avoids the need to make field theory subtractions in the four-point amplitude. (Similarly, Φ^n terms in the superpotential can be seen by computing amplitudes with 2 fermions and $n-2$ bosons rather than $2(n-1)$ bosons.)

Again we consider the superstring first. Here we choose the picture

$$\langle V_{(-1/2,-1)} V_{(-1,0)} V_{(-1/2,-1)} \rangle, \tag{5.13}$$

where any two of the vertices are singly-twisted (+) and the third is doubly-anti-twisted ($--$). All the vertex locations can be fixed using $SL_2(C)$ invariance (since they all have dimension $h, \bar{h}=1$ there is actually no z dependence whatsoever). The spacetime fermion vertices have the form (cf. sect. 2)

$$V^{+}_{(-1/2,-1)} = e^{-\phi/2-\bar{\phi}}\mathcal{S}\prod_{i=1}^{3}\exp\left(i\left[\left(-\tfrac{1}{2}+\frac{k_i}{N}\right)H^i+\frac{k_i}{N}\tilde{H}^i\right]\right)\sigma_{+}^{(i)}\cdot e^{ik\cdot\mathcal{X}},$$

$$V^{--}_{(-1/2,-1)} = e^{-\phi/2-\bar{\phi}}\mathcal{S}\prod_{i=1}^{3}\exp\left(i\left[\left(\tfrac{1}{2}-\frac{2k_i}{N}\right)H^i+\left(1-\frac{2k_i}{N}\right)\tilde{H}^i\right]\right)\sigma_{--}^{(i)}\cdot e^{ik\cdot\mathcal{X}};$$

$$(5.14)$$

whereas the spacetime bosons look like

$$V^{+}_{(-1,0)} = e^{\bar{\phi}}\bar{T}_F\left(e^{-\phi-\bar{\phi}}\prod_{i=1}^{3}\exp\left(i\left[\frac{k_i}{N}H^i+\frac{k_i}{N}\tilde{H}^i\right]\right)\sigma_{+}^{(i)}\cdot e^{ik\cdot\mathcal{X}}\right),$$

$$V^{--}_{(-1,0)} = e^{\bar{\phi}}\bar{T}_F\left(e^{-\phi-\bar{\phi}}\prod_{i=1}^{3}\exp\left(i\left[\left(1-\frac{2k_i}{N}\right)H^i+\left(1-\frac{2k_i}{N}\right)\tilde{H}^i\right]\right)\sigma_{--}^{(i)}\cdot e^{ik\cdot\mathcal{X}}\right).$$

$$(5.15)$$

In these expressions, double twists have charges $2k/N$ in the exponents instead of k/N; $\mathcal{S}$ is a spin field for the four uncompactified dimensions. If one of the fractions k_i/N is greater than $\frac{1}{2}$, say k_1/N, one finds that the doubly-twisted vertices in (5.14) and (5.15) have the *wrong* conformal weight; they must be modified by the replacements $(\frac{1}{2}-2k_1/N)\to(\frac{3}{2}-2k_1/N)$ and $(1-2k_{2,3}/N)\to -2k_{2,3}/N$. This modification turns out to be related to the different $x\to\infty$ asymptotics (5.7) we found for $P_{\epsilon_j}(x,\bar{x})$ when $k_1/N>\frac{1}{2}$. Those asymptotics led to a vanishing contact term for four twisted **27**'s in the heterotic string when $k_1/N>\frac{1}{2}$. The modified vertex operators in (5.14) and (5.15) will lead to a vanishing of the corresponding Yukawa couplings. It is a useful exercise to check that each of the fields in (5.14) and (5.15) does have conformal weight 1. We can now see that the three-twist superstring Yukawa couplings vanish simply by fermion (H and $\tilde{H}$) charge conservation. For $k_1/N<\frac{1}{2}$ the H charges balance in (5.13), but not the $\tilde{H}$ charges. (The picture-changing operator in (5.15) can balance the charge for one of the $\tilde{H}_i$ but not the other two.) For $k_1/N>\frac{1}{2}$ even the H charge is not conserved. Of course all these couplings were expected to vanish, because there were no contact terms in the corresponding four-boson amplitudes. The Yukawa couplings for three blowing-up modes of the heterotic string also vanish, because the two types of vertex operator are identical at zero momentum.

On the other hand, the vertex operators for (0,2) modes on (2,2) orbifolds, and for all modes on (0,2) orbifolds of the heterotic string, involve different sets of $\tilde{H}$ charges. Gauge-invariant combinations of these vertices will generically conserve $\tilde{H}$ charge, so for $k_1/N<\frac{1}{2}$ these modes will typically have Yukawa couplings. For example, the nonvanishing Yukawa couplings for three twisted **27**'s of E_6 states are essentially just the bosonic twist operator product coefficients $C_{\sigma_{+,\epsilon_1}\sigma_{+,\epsilon_2}\sigma_{-,\epsilon_3}}$ calculated in sect. 4 (eq. (4.47)); the various other free field correlators appearing in (5.13) simply make the full correlation function $SL_2(C)$-invariant, and the cocycles which we have ignored should contribute the signs needed to build the symmetric E_6-invariant tensor d_{abc}. The generalization of the operator product coefficient (4.47) to a six-dimensional Z_N orbifold is needed here:

$$C_{\sigma_{-,\epsilon_1}\sigma_{-,\epsilon_2}\sigma_{-,\epsilon_3}} = \sqrt{V_\Lambda}\sum_{\tilde{v}}\prod_{i=1}^{3}\sqrt{\left|\tan\frac{\pi k_i}{N}\right|\frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{2}|1-2k_i/N|\right)}{\Gamma(|1-2k_i/N|)}}$$

$$\times\exp\left[-\frac{\pi\tilde{v}_{(i)}^2}{4|\sin(2\pi k_i/N)|}\right],$$

$$\tilde{v}\in(1-\theta^2)(f_{\epsilon_3}-f_{\epsilon_2}+\Lambda),\qquad \frac{k_i}{N}\neq\tfrac{1}{2},\tag{5.16}$$

plus a second sum with $f_{\epsilon_3}\to\theta f_{\epsilon_3}$ if f_{ϵ_3} is not fixed by θ. For completeness we note that the Yukawa couplings between two twisted **27**'s and one untwisted **27**, and those between one twisted **27** and two untwisted **27**'s, always vanish by the point group selection rule of sect. 3: $\theta\cdot\theta\cdot 1\neq 1\neq\theta\cdot 1\cdot 1$. (This was also noted in [34].) The couplings of three untwisted **27**'s are of course simple to work out using the appropriate untwisted vertex operators which are invariant under the combined spacetime and gauge twist. In particular there are no exponentially suppressed contributions to these couplings. We also note that for $k_1/N>\frac{1}{2}$ *all* the Yukawa couplings of the type (twist, twist, double antitwist) vanish, so all the corresponding cubic terms in the superpotential vanish. However, other three-twist couplings are generally nonzero, as are higher-order terms in the superpotential.

These results can be compared with calculations of Yukawa couplings for the heterotic string compactified on smooth Calabi-Yau manifolds [35, 36], in which the treatment is perturbative in the nonlinear (sigma) model coupling. This corresponds to taking the large R limit in (5.16), where R is the characteristic length of the lattice Λ (the overall size of the orbifold). To make contact with the calculations in [35, 36], which depend on the various moduli of the Calabi-Yau manifold, we must also take some lengths r_i to zero. These lengths are the characteristic sizes of certain noncompact complex manifolds which are glued in to repair the orbifold singularities in the construction of the corresponding smooth Calabi-Yau manifold. So in the limit $r_i\to 0$ one recovers the orbifold.

Here we specialize to the case of the Z manifold [3, 36] and its limiting Z orbifold [13], with the standard embedding of the spin connection. The point group for the Z orbifold is generated by a Z_3 rotation θ which acts on each of three orthogonal two-dimensional lattices, so $k_i/N = \tilde{k}_i/N = \frac{1}{3}$, $i = 1,2,3$: This model has 36 generations of **27**'s of E_6; 9 of these come from the untwisted sector and 27 from the twisted sector. (There are 27 fixed points for the rotation θ acting on the six-torus.) The states in the antitwisted sector are the antiparticles of the states in the twisted sector. We first summarize the results of [36] for the Z manifold. The ith generation is associated with a cohomology class in H^2 for the manifold, which can be represented by a closed two-form F_i. Near the orbifold limit, the two-forms associated with the 9 untwisted **27**'s are just the two-forms on the torus which are left invariant by θ, namely $\mathrm{d}X^i \wedge \mathrm{d}\bar{X}^j$, $i, j = 1,2,3$; the two-forms for the 27 twisted **27**'s are localized near the 27 fixed points. A Yukawa coupling for three **27**'s is given by an "intersection number" – the integral over the complex manifold of the wedge product of the three closed two-forms which represent the three **27**'s: $\int F_i \wedge F_j \wedge F_k$. If the F_i are nomalized to represent integer cohomology classes, then this number is an integer, independent of the moduli of the Z manifold. Most of these integers vanish. In fact *all* the ones involving two untwisted **27**'s and one twisted **27**, or one untwisted **27** and two twisted **27**'s, vanish. The only nonvanishing integers for three twisted **27**'s occur when all three are the same **27**. Finally, if all three **27**'s are untwisted, labelled by $(i_1, j_1), (i_2, j_2), (i_3, j_3)$, the integers vanish unless $i_1 \neq i_2 \neq i_3 \neq i_1$ and $j_1 \neq j_2 \neq j_3 \neq j_1$. (This is easily seen by integrating the product of the corresponding two-forms $\mathrm{d}X^{i_m} \wedge \mathrm{d}\bar{X}^{j_m}$ [36].)

The intersection numbers for the F_i tell us which Yukawa couplings vanish (perturbatively), but not the correct normalization of the nonvanishing ones, because the above normalization of the F_i which represent the chiral generations does not give the conventional normalization of their kinetic terms. Instead the normalization matrix depends on intersection numbers of the form $\int F_i \wedge J \wedge J$ and $\int F_i \wedge F_j \wedge J$, where the Kähler form J is a linear combination of the F_i with coefficients which are the moduli of the Z manifold: $J = \Sigma_i m_i F_i$. So the nonzero Yukawa couplings pick up dependence on the moduli of the manifold through the normalization matrix. In the orbifold limit the moduli for the 27 twisted F_i vanish, and the Kähler form becomes simply the Kähler form for the six-torus, which is a linear combination of 3 of the 9 untwisted F_i: $J = R^2 \Sigma_{i=1}^3 \mathrm{d}X^i \wedge \mathrm{d}\bar{X}^i$. Using the intersection numbers listed above, the entries in the normalization matrix for the twisted **27**'s are all seen to vanish in the orbifold limit. In other words, the nonvanishing three-twist Yukawa couplings all become infinite in this limit. The nonvanishing Yukawa couplings for three untwisted **27**'s, on the other hand, remain finite.

Now let us compare these results with the orbifold calculations. We leave the reader to check that the three-untwisted Yukawa couplings do vanish if and only if the corresponding intersection numbers vanish. We have already noted why the couplings with two untwisted **27**'s and one untwisted **27** and the couplings with one untwisted **27** and two twisted **27**'s all vanish. Finally the three-twist Yukawa couplings are given by (5.16). They are nonzero when all three-twist fields create states at the same fixed point, $f_{\varepsilon_1} = f_{\varepsilon_2} = f_{\varepsilon_3}$ (corresponding to the nonzero intersection numbers for the twisted F_i), but they are also nonzero for other combinations of fixed points which satisfy the space group selection rule $f_{\varepsilon_1} + f_{\varepsilon_2} - 2f_{\varepsilon_3} \in \Lambda$. (For the Z orbifold one can check that, given any two fixed points f_{ε_1} and f_{ε_2}, the selection rule is satisfied for precisely one other fixed point f_{ε_3}.) The latter couplings are exponentially suppressed for large R, behaving like $R^3 e^{-R^2}$, because the origin is not included in the coset $(1-\theta^2)(f_{\varepsilon_3} - f_{\varepsilon_2} + \Lambda)$ for $f_{\varepsilon_2} \neq f_{\varepsilon_3}$. (The world-sheet must stretch between different fixed points.) In general, the $\tilde{v} \neq 0$ terms in (5.16) represent corrections to the Yukawa couplings of [35, 36] which are nonperturbative in the inverse string tension α'. The $\tilde{v} = 0$ terms, which occur only when all 3 fixed points are the same, correspond to the perturbative contributions found in [35, 36]. Actually the correspondence is rather loose, because even the large R limit of the orbifold cannot be reached from a large, smooth Z manifold (i.e. a weakly coupled nonlinear model) without passing through a strongly-coupled regime where the perturbative calculations break down. Indeed, we find *finite* three-twist Yukawa couplings for the orbifold at finite R, in contrast to the divergences found by taking the limit $r_i \to 0$ in [36]. (The couplings do diverge like R^3 as $R \to \infty$, though.)

6. Conclusions

We have succeeded in giving a complete specification of orbifold conformal field theory. Twist fields are dealt with via the stress-energy they induce. The classical stress-energy is computed from classical solutions in the presence of twists; the quantum piece is obtained from the connected Green functions. Integration of the stress tensor gives the twist correlation functions. Correlations of untwisted fields in the presence of twists can be obtained using the appropriate Green functions and classical solutions. These correlations determine the operator product coefficients, which are simple functions of the orbifold fixed point geometry. This data can be used to construct the low-energy effective action which generates the S-matrix. If an orbifold with otherwise acceptable properties is someday discovered, one can calculate the couplings in the effective theory to see whether they are realistic. The stress tensor method used here has also been applied successfully [24] to the calculation of loop amplitudes in flat space backgrounds; the procedure developed there would in fact seem to apply straightforwardly to any theory with a stress tensor in the Sugawara form. Therefore we foresee no difficulties in combining the methods of [24] and this paper for the purpose of computing loop corrections to processes on orbifold backgrounds. Although computations in orbifold conformal field theory are rather more complicated than in flat space, we hope that we have conveyed a sense that one really can calculate their properties in great detail.

We would also like to find a handle on all the other conformal field theories which provide solutions to the string equations of motion. A step in this direction might be to look at those which have orbifolds as one boundary of their parameter (moduli) space of shapes. For backgrounds with (2, 2) global world-sheet supersymmetry, there is no obstruction to perturbing away from the orbifold limit by resolving its curvature singularities. However, this can only be done in (2, 2) backgrounds, and only for those perturbations which preserve this symmetry. Backgrounds without this symmetry (i.e. (0, 2) orbifolds), and left-right asymmetric modes in the (2, 2) case, cannot be perturbed away from the orbifold limit while preserving the spacetime equation of motion (conformal invariance). In other words, the dynamics of the heterotic string seem to drive the nearby (0, 2) backgrounds to the orbifold limit. Study of orbifolds might provide a laboratory for dissecting the structure of the general conformal field theory [37]. It certainly provides a rich collection of solutions of the string equations.

Similar results to those presented in this paper have been described recently in [38].

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Note added in proof

After submitting this paper for publication we received a paper by Bershadsky and Radul, where the same techniques have been applied to bosonic string propagation on branched covers of the sphere.

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