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COVARIANT QUANTIZATION OF SUPERSYMMETRIC STRING THEORIES:
The spinor field of the Ramond-Neveu-Schwarz model*

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A brief review of superconformal field theory and superstrings is presented. The spacetime spinor contribution to the fermion vertex operator is constructed and the four-fermion amplitude is calculated using the differential equation method in the SO(1,9) current algebra. The spinor field is also described as a vertex operator in the bosonization of the Ramond-Neveu-Schwarz fermions, and the two-cocycle for the fermion vertex is given.

The original formulation of superstring theory was the manifestly covariant Ramond-Neveu-Schwarz (RNS) model [1–3]. The RNS theory was never developed far enough to describe general fermionic amplitudes and therefore the spacetime supersymmetry of the model remained obscure. Only part of the fermion emission vertex was constructed [4], the spacetime spinor field of the RNS model, which we would now call the matter contribution to the fermion vertex. Spacetime supersymmetry was proved in light-cone gauge, where the spacetime spinor field becomes the complete fermion vertex [5].

This paper is part of a program [6, 7] to give a manifestly Lorentz covariant formulation of superstring theory by completing the RNS model. The motivations for embarking on this project were: (i) to develop effective methods for calculating string tree amplitudes and loop corrections in flat spacetime; (ii) to develop methods which can be used to describe strings in curved backgrounds; and (iii) to take a step towards understanding the general covariance of string theory.

In this paper we discuss the fermion vertex [6–8]. We use two methods. One, dealing with the matter contribution alone, is based on the SO(1,9) current algebra

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of the RNS fermions. Using the techniques of Knizhnik and Zamolodchikov [9], the current algebra gives differential equations for the correlation functions of the spinor fields. We use it to calculate the four point function of the spinor field. Combining with other fields to get the complete vertex, we also get the four-fermion scattering amplitude, obtained previously by other methods [10, 11]. This result was announced in ref. [6], and was also derived independently by Knizhnik in ref. [8]. The second method is a generalization of the Luther-Peschel construction of the order parameter in the Ising model [12]. The RNS fermions are bosonized [13], and fields are expressed as vertex operators [14, 15]. The covariant fermion vertex operator [7] is formed by combining the spacetime spinor field of the RNS model with operators from the Faddeev-Popov ghost sector in Polyakov's quantization of the string [16]. In this paper we give the cocycle for the operator representation of the fermion vertex.

The organization is as follows. In sect. 1 we briefly review conformal field theory and covariant bosonic string theory, and then discuss their supersymmetric extensions. In sect. 2 we discuss the SO(1,9) Kac-Moody algebra and use it for the four-point function of the spin fields. In sect. 3 we describe the spacetime properties and vertex operators, and calculate the four-fermion scattering amplitude. In sect. 4 we give the bosonic representations of all the operators and their two-cocycles.

1. (Super)conformal methods and strings

Conformal methods, and their supersymmetric extensions, are extremely useful in covariant string theory. We begin by summarizing the conformal case [17]. The conformal group in two dimensions has an infinite number of generators. If the two coordinates (x^0, x^1) are written in complex form, $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, any analytic function of z or antianalytic function of $\bar{z}$ is a local conformal transformation. The symmetry group splits into the direct product of two identical groups, one in z and one in $\bar{z}$. Hereafter, we will usually discuss the z sector alone. This sector corresponds to "right-moving" fields on the world sheet. Different string theories can be built by combining the z and $\bar{z}$ sectors in different ways [11, 19].

The Noether current associated with local conformal transformations is the traceless stress energy tensor $T_{\mu\nu}$. (It is traceless due to scale invariance.) In two dimensions $T_{\mu\nu}$ has two independent components. These can be chosen to be $T(z) = (T_{00} - iT_{01})$ and $\overline{T}(\bar{z}) = (T_{00} + iT_{01})$. Conservation and tracelessness for the stress tensor implies these components are analytic or antianalytic. We focus on $T(z)$, whose Laurent coefficients L_n are the generators of the transformations $z \to z + \varepsilon z^{n+1}$:

$$T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \qquad n \text{ integer}. \tag{1.1}$$

The L_n's obey the Virasoro algebra [20]

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}c(m^3 - m)\delta_{m+n}, \tag{1.2}$$

which is equivalent to the operator product

$$T(z_1)T(z_2) = \frac{\frac{1}{2}c}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}T(z_2) + \frac{1}{(z_1-z_2)}\,\partial_{z_2}T(z_2) + \text{non-singular as } z_1 \to z_2 . \tag{1.3}$$

The central charge c is the numerical coefficient in the trace anomaly [21] and characterizes the operator representation of the algebra.

Certain fields in the theory have an especially simple transformation law under a conformal transformation. These are called primary or conformal fields [18, 17]. If z transforms by $z \to z'(z)$, a primary field $\phi(z)$ transforms as

$$\phi(z')(\mathrm{d}z')^h = \phi(z)(\mathrm{d}z)^h . \tag{1.4}$$

By considering a scale transformation, $z'(z) = \lambda z$, h is identified with the z scaling dimension of the field ϕ. Eq. (1.4) and the following operator product of $\phi(z)$ with the stress-energy tensor of the theory

$$T(z_1)\phi(z_2) = \frac{h}{(z_1-z_2)^2}\phi(z_2) + \frac{1}{(z_1-z_2)}\,\partial_{z_2}\phi(z_2) + \text{non-singular as } z_1 \to z_2 \tag{1.5}$$

are equivalent. One can establish a one to one correspondence between conformal fields and certain states in a Hilbert space. Consider

$$|h\rangle = \lim_{z\to 0} \phi(z)|\mathrm{vac}\rangle , \tag{1.6}$$

where $|\mathrm{vac}\rangle$ is the vacuum. From eq. (1.4), it follows that

$$L_0|h\rangle = h|h\rangle , \tag{1.7}$$

$$L_n|h\rangle = 0, \qquad n > 0 . \tag{1.8}$$

These are called highest eight states and h is called the weight or dimension.

It is convenient to map operators to the w cylinder where

$$\tau + i\sigma = w = \log z. \tag{1.9}$$

These highest weight states are the asymptotic "in" states created by ϕ acting at $\tau = -\infty$ ("time" on the sheet). L_0 generates dilations in z and thus translations in τ. It can be viewed as the hamiltonian on the cylinder.

We now discuss a simple conformal field theory, the free scalar field, $X(z, \bar{z})$. The equations of motion allow us to split it into two chiral components, $X(z, \bar{z}) = X(z) + X(\bar{z})$. The two-point functions are

$$\begin{aligned} \langle X(z_1) X(z_2) \rangle &= -\ln(z_1 - z_2), \\ \langle X(\overline{z_1}) X(\overline{z_2}) \rangle &= -\ln(\overline{z_1} - \overline{z_2}), \\ \langle X(z_1) X(\overline{z_2}) \rangle &= 0. \end{aligned} \tag{1.10}$$

The stress-energy tensor is

$$T_{\text{scalar}}(z) = -\tfrac{1}{2} : \partial_z X \partial_z X :, \tag{1.11}$$

(: : denotes normal ordering). X is *not* a conformal field, as it does not obey eq. (5), nor is it a quantum field, since its correlations grow with separation. But primary fields can be made out of it.

An example is the "vertex operator" [14, 15]

$$V_\alpha(z) = :\mathrm{e}^{i\alpha X(z)}:, \tag{1.12}$$

where α is a real number. The operator product of V_α with T_{scalar} shows that it is primary, with dimension $\frac{1}{2}\alpha^2$. If X is regarded as an electrostatic potential, V_α introduces a source of "charge" α into the system. Nonzero correlation functions must have zero total charge (which may include operators at infinity that create "background charge") [22]. A correlation function of an arbitrary number of V_{α_i}'s is given by

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \dots V_{\alpha_n}(z_n) \rangle = \prod_{i \neq j} \frac{1}{(z_i - z_j)^{\alpha_i \alpha_j}} \delta_{0, \alpha_1 + \alpha_2 + \cdots + \alpha_n}. \tag{1.13}$$

These tools are useful for constructing many conformal field theories. We will apply them to bosonic string theory, and then generalize to the supersymmetric case.

The bosonic string [14] is defined by the sum over world sheets

$$\int_{\text{surfaces}} (\mathrm{d}X) \mathrm{e}^{-S(X)}. \tag{1.14}$$

$X^{\mu}(z, \bar{z})$ is a map into spacetime from the world sheet and μ runs from 1 to D, the dimension of spacetime. For the Nambu-Goto string [23], S is given by the area of the surface swept out by the sheet:

$$S(X) = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi} \sqrt{-g(z,\bar{z})}\, g^{ab}(z,\bar{z})\, \partial_a X^{\mu} \partial_b X_{\mu}, \tag{1.15}$$

where the metric g_{ab} is the induced metric from spacetime. Here, g is the determinant of g_{ab}, and z and $\bar{z}$ represent the two coordinates on the world sheet, $x^0 \pm ix^1$. In the Polyakov string [16], g_{ab} is an arbitrary metric, and must also be summed over in the path integral. This action is reparametrization invariant. To make the path integral well defined, gauge fixing is needed. Two common choices are the light-cone gauge [24] and the conformal gauge. Light-cone gauge leaves only the transverse, physical degrees of freedom, but is not manifestly Lorentz invariant. Conformal gauge is Lorentz invariant and allows the use of two-dimensional conformal methods, but leaves unphysical states which must be eliminated through residual gauge conditions. In the following we shall be using conformal gauge:

$$g_{ab} = \mathrm{e}^{\phi(z,\bar{z})} \delta_{ab}. \tag{1.16}$$

The choice of conformal gauge in string theory is analogous to choosing a covariant gauge to quantize electromagnetism. There one finds upon quantization negative metric states and a longitudinal mode. To eliminate them, one must impose Gauss' Law ($\partial_{\mu} A^{\mu} = 0$) on the physical states. Here, conformal gauge is being used, and the appropriate symmetry generating operator is the stress-energy tensor. Thus we require that $T = 0$ on physical states [21]:

$$\langle \text{phys} | T_{\text{total}} | \text{phys} \rangle = 0. \tag{1.17}$$

T_{total} is the complete stress-energy tensor, including the ghosts required for gauge fixing [16, 21, 25], i.e. $T_{\text{tot}} = T_{\text{matter}} + T_{\text{ghost}}$. The symbol T will continue to refer to T_{matter}. The condition (1.17) determines a positive metric Fock space [26] and is a set of "gauge conditions" for the Laurent coefficients L_n:

$$L_n | \text{phys} \rangle = 0, \qquad n > 0. \tag{1.18}$$

States satisfying (1.18) are the highest weight states in the conformal theory. For the bosonic string, the ghost sector can be chosen so that $L_0^{\text{ghost}} = -1$ [25]. So the L_0

gauge condition is:

$$L_0^{\mathrm{tot}}|\mathrm{phys}\rangle = (L_0 - 1)|\mathrm{phys}\rangle = 0. \tag{1.19}$$

We now turn to the fermionic string [1, 2]. In addition to the coordinates $X^\mu(z, \bar{z})$ on the world sheet, there are fermionic coordinates $\psi^\mu(z, \bar{z})$ related to X^μ by world-sheet supersymmetry [27]. Both of these fields have a vector index μ in spacetime. The sum over two geometries leads to a theory of two-dimensional supergravity and matter [28]. Going to superconformal gauge, the action becomes

$$S_{\text{gauge fixed}} = \int \frac{d^2 z}{2\pi}\left[\partial_z X^\mu \partial_{\bar{z}} X_\mu - \psi^\mu \partial_{\bar{z}}\psi^\mu - \bar{\psi}^\mu \partial_z \bar{\psi}^\mu\right]. \tag{1.20}$$

(The fermions ψ, $\bar{\psi} = \psi_1 \pm i\psi_2$, where the subscripts are spinor indices, are sheet spinors in a chiral basis.) The global supersymmetry can be expressed by combining the fields into a two-dimensional superfield $\boldsymbol{X}^\mu(z, \bar{z}, \theta, \bar{\theta})$ [28]:

$$\boldsymbol{X}^\mu(z, \bar{z}, \theta, \bar{\theta}) = X^\mu(z, \bar{z}) + \theta\psi^\mu(z, \bar{z}) + \bar{\theta}\bar{\psi}^\mu(z, \bar{z}) + \theta\bar{\theta}F^\mu(z, \bar{z}). \tag{1.21}$$

Here θ and $\bar{\theta}$ are anticommuting parameters with scaling dimension $\frac{1}{2}$, θ, $\bar{\theta} = \theta_1 \pm i\theta_2$, and F^μ is an auxiliary field. With these, the action can be written in a manifestly supersymmetric form:

$$S_{\text{gauge fixed}} = \int d^2 z\, d^2\theta\, \frac{1}{2\pi}\bar{D}\boldsymbol{X}^\mu \cdot D\boldsymbol{X}^\mu, \tag{1.22}$$

where $D = \partial_\theta + \theta\partial_z$, and $\bar{D}$ is the conjugate.

Conformal symmetry generalizes to superconformal symmetry [7, 29, 30]. The generators of the algebra are the Laurent coefficients for the super-stress-energy tensor,

$$T(z, \theta) = T_{z\theta}(z) + \theta T_{zz}(z). \tag{1.23}$$

The superconformal algebra comes from the operator product of the stress-energy tensor with itself

$$T(z_2, \theta_2)T(z_2, \theta_2) \sim \frac{\frac{1}{4}\hat{c}}{z_{12}^3} + \frac{3\theta_{12}}{2z_{12}^2}T(z_2, \theta_2) + \frac{\frac{1}{2}}{z_{12}}D_2 T$$

$$T(z_1, \theta_1)T(z_2, \theta_2) \sim \frac{\frac{1}{4}\hat{c}}{z_{12}^3} + \frac{3\theta_{12}}{2z_{12}^2}T(z_2, \theta_2) + \frac{\frac{1}{2}}{z_{12}}D_2 T + \frac{\theta_{12}}{z_{12}}\partial_2 T \ldots, \tag{1.24}$$

where $z_{12} = z_1 - z_2 - \theta_1\theta_2$, and $\theta_{12} = \theta_1 - \theta_2$. One expands in Laurent coefficients

$$T(z, \theta) = \sum_n z^{-n-3/2}\left[\tfrac{1}{2}G_n + \theta z^{-1/2}L_n\right], \tag{1.25}$$

and obtains the superconformal algebras:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{8}\hat{c}(m^3 - m)\delta_{m,-n},$$

$$[G_m, G_n]_+ = 2L_{m+n} + \tfrac{1}{2}\hat{c}(m^2 - \tfrac{1}{4})\delta_{m,-n},$$

$$[L_m, G_n] = (\tfrac{1}{2}m - n)G_{m+n}. \tag{1.26}$$

$\hat{c}$ is $\frac{2}{3}$ of c in the Virasoro algebra, eq. (1.2) above. In the *Neveu-Schwarz* [2] algebra n is half integral, and in the *Ramond* [1] algebra n is integral.

Again, in covariant gauge one must impose the requirement that

$$\langle \text{phys}|T_{\text{total}}|\text{phys}\rangle = 0. \tag{1.27}$$

There are now gauge conditions for both the L_n's and the G_n's:

$$L_n|\text{phys}\rangle = 0, \qquad G_n|\text{phys}\rangle = 0, \qquad n > 0, \tag{1.28}$$

and in analogy with the conformal case, the states satisfying (1.28) are the highest weight states in the superconformal theory. We will discuss the L_0 gauge condition shortly.

The Hilbert space of superconformal field theory divides into 2 sectors, depending on the boundary conditions of the fermionic fields. On the Neveu-Schwarz sector

$$T_{z\theta}(e^{2\pi i}z) = T_{z\theta}(z), \tag{1.29a}$$

and on the Ramond sector

$$T_{z\theta}(e^{2\pi i}z) = -T_{z\theta}(z). \tag{1.29b}$$

This leads to the two possible algebras for the Laurent coefficients G_n, and hence to the Ramond and Neveu-Schwarz algebras.

The sheet fermion $\psi^\mu(z)$ has a Laurent expansion

$$\psi^\mu(z) = \sum_{n=-\infty}^{\infty} \frac{\psi_n^\mu}{z^{n+1/2}}, \tag{1.30}$$

with n half integral for the Neveu-Schwarz sector, and integral for the Ramond sector. Imposing canonical commutation relations on the fields ψ^μ, one finds that

$$[\psi_m^\mu, \psi_n^\nu]_+ = -g^{\mu\nu}\delta_{m+n,0}. \tag{1.31}$$

The Hilbert space vacuum $|0\rangle$ is in the Neveu-Schwarz sector. It is invariant under the OSp(2|1) symmetry generated by L_0, $L_{\pm 1}$, and $G_{\pm 1/2}$. Physical states in the Neveu-Schwarz sector are obtained from conformal superfields acting on this vacuum. The L_0 condition includes L_0^{ghost}, which equals $-\frac{1}{2}$ on the Neveu-Schwarz ground state [6, 7]. This is a sum of terms from the ghosts of the original bosonic string ($L_0^{\text{ghost}} = -1$) and their superpartners ($L_0^{\text{ghost}} = +\frac{1}{2}$). The result is that [14]

$$(L_0 - \tfrac{1}{2})|\text{phys}\rangle_{\text{NS}} = 0 \tag{1.32}$$

for the fermionic string. States in the Neveu-Schwarz sector are spacetime bosons.

The Ramond sector has a fermionic character. The zero modes ψ_0^μ of the field ψ^μ with Ramond boundary conditions obey a Clifford algebra (eq. (1.31), above), and so are gamma matrices.

As a result, the states in this sector and the operators which create them transform as spacetime spinors, in the 16 and $\overline{16}$ of SO(10). These operators take the field ψ from one set of boundary conditions to the other, intertwining the Neveu-Schwarz and Ramond sectors. We call such operators *spin fields* [29] and denote them S_α, where α is a spacetime spinor index. Their action can be visualized as opening or closing a cut in the world sheet. One can make an analogy between the Ramond sector and the supersymmetric quantum mechanical construction of a massless Dirac spinor [28, 31].

World-sheet supersymmetry constrains the L_0 eigenvalues in the Ramond sector. The generator associated with supersymmetry on the sheet is G_0. Since this is a free field theory on the sheet, supersymmetry will be unbroken. This means the ground state must be annihilated by G_0,

$$G_0|\text{gnd}\rangle_{\text{R}} = 0\,. \tag{1.33}$$

But the commutation relations of the Ramond algebra state that

$$G_0^2 = L_0 - \tfrac{1}{16}\hat{c}\,. \tag{1.34}$$

This algebra is valid for just the matter fields alone, or for the matter plus ghost system. In the combined matter and ghost system, $\hat{c}_{\text{tot}} = 0$ for consistency (this determines $D = 10$), so $L_0^{\text{tot}} = 0$. The evaluation of L_0 must include the ghost ground state energy in the Ramond sector. The bosonic string ghosts here give -1, and their supersymmetric partners give $+\frac{3}{8}$. These produce a ghost contribution of $-\frac{5}{8}$ to the value of L_0^{tot} on the ground state. This implies that L_0^{matter} on the vacuum must be $\frac{5}{8}$. So the matter operator creating the Ramond ground state from the OSp(2|1) invariant Hilbert space vacuum must be a conformal field S_α, with dimension $\frac{5}{8}$ which transforms as a spinor under the spacetime SO(1, 9).

The spin fields S_α are unfamiliar objects to many particle physicists, but their close relatives have been long known in another context, that of the 2-dimensional Ising model of statistical mechanics. At the critical point, the 2D Ising model is equivalent to a theory of one massless free Majorana fermion ψ, with $h=\frac{1}{2}$. The order parameter field $\sigma(z,\bar{z})$ and disorder parameter [32] field $\mu(z,\bar{z})$ are spin fields in this fermionic system. The fermion ψ is double valued around these operators. In 10 dimensions we have 10 ψ's, i.e. ψ^{μ}. Roughly speaking, the S_{α} can be built from a product of ten order or disorder parameter fields. The order and disorder operators have $h=\frac{1}{16}$. So the product of such fields has dimension $10\cdot(\frac{1}{16})=\frac{5}{8}$, just as was found above using supersymmetry.

The combination of the Ramond and Neveu-Schwarz sectors is the Ramond-Neveu-Schwarz (RNS) fermionic string. This theory has the matter fields X^{μ}, ψ^{μ}, and S_{α}. The world-sheet field theory is completely described by giving all correlation functions. X^{μ} and ψ^{μ} are free, and so their Green functions can be read off. The Green function for X^{μ} is eq. (1.10), and

$$\langle\psi^{\mu}(z)\psi^{\nu}(w)\rangle=-\frac{g^{\mu\nu}}{z-w}. \tag{1.35}$$

The spin field S_{α} is more complicated. However, we know that its operator product with its conjugate includes the identity (as for any conformal field in a unitary theory), and it is a spacetime spinor. The operator product with its conjugate is:

$$S_{\alpha}(z)S^{\beta}(w)\sim\frac{-\delta_{\alpha}^{\beta}}{(z-w)^{5/4}}+\text{less singular}. \tag{1.36}$$

The tensors which raise and lower indices are antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$, the charge conjugation matrices for SO(10). The power $\frac{5}{4}$ is known since it is twice the dimension of S_{α}. The spinorial nature of S_{α} means we know how it transforms under a SO(10) current $J^{\mu\nu}$:

$$J^{\mu\nu}(z)S_{\alpha}(w)\sim\frac{(\gamma^{\mu\nu})_{\alpha}^{\beta}S_{\beta}(z)}{z-w},\qquad J^{\mu\nu}(z)=:\psi^{\mu}(z)\psi^{\nu}(z):, \tag{1.37}$$

where $\gamma^{\mu\nu}=\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}$ is the rotation generator for a ten-dimensional spinor. This means that the operator product of two S's contains J. To get the ψ, S_{α} operator product, consider the three-point function

$$\langle S_{\beta}(z_1)\psi^{\mu}(z_2)S_{\alpha}(z_3)\rangle. \tag{1.38}$$

Taking $z_1\to\infty$, $z_3\to 0$, eq. (1.38) becomes the expectation value of ψ^{μ} with Ramond boundary conditions. This picks out the zero mode of ψ^{μ}, the gamma matrices, and so leads to

$$\psi^{\mu}(z)S_{\alpha}(w) \sim \frac{(\gamma^{\mu})_{\alpha}^{\beta}S_{\beta}(z)}{(z-w)^{1/2}}. \tag{1.39}$$

This also means that ψ^{μ} appears in the operator product of two S's. Combining these results gives

$$S_{\alpha}(z)S^{\beta}(w) \sim \frac{-\delta_{\alpha}^{\beta}}{(z-w)^{5/4}} + \frac{(\gamma_{\mu})_{\alpha}^{\beta}\psi^{\mu}(z)}{(z-w)^{3/4}} + \frac{(\gamma^{\mu\nu})_{\alpha}^{\beta}\psi^{\mu}(z)\psi^{\nu}(z)}{(z-w)^{1/4}}. \tag{1.40}$$

The most general operator that can appear in the operator product expansion of two S_{α}'s is a composite of Neveu-Schwarz fermion operators, objects with integral or half integral weight.

These operator products are not enough to get the full correlation functions of an arbitrary number of S_{α}'s. There is a general method for finding such correlation functions, developed by Knizhnik and Zamolodchikov [9], which can be applied to these generalized Ising spins.

2. Kac-Moody algebra

We note that the ψ^{μ} are vectors under the SO(1,9) algebra of the Lorentz group. Using Wick rotation, this becomes an SO(10) algebra. This SO(10) is made of the zero modes of a current algebra of the conformal field theory. The presence of a Kac-Moody algebra, a chiral current algebra, in certain conformal theories provides a differential equation which the correlation functions satisfy, as Zamolodchikov and Knizhnik have shown [9]. We outline their method below and apply it to determine S_{α} correlations.

The ψ^{μ} system has a chiral SO(10) symmetry. The corresponding hermitian chiral currents are:

$$J^{\mu\nu}(z) = i:\psi^{\mu}(z)\psi^{\nu}(z): = J^{a}, \qquad a = 1,\dots,45 \text{ for SO(10)}, \tag{2.1}$$

where : : stands for normal ordered product with respect to the ψ's.

The $J^{a}(z)$'s can be expanded in a Laurent series:

$$J^{a}(z) = \sum_{-\infty}^{+\infty} \frac{J_{n}^{a}}{z^{n+1}} \tag{2.2}$$

and the components J_{n}^{a} will then obey a Kac-Moody algebra

$$[J_{m}^{a}, J_{n}^{b}] = if^{abc}J_{m+n}^{c} + \tfrac{1}{2}km\delta_{ab}\delta_{m+n,0}. \tag{2.3}$$

In eq. (2.3), k is an integer, and is related to the currents by

$$\mathrm{Tr}(\lambda^a \lambda^b) = k\delta^{ab}, \tag{2.4}$$

where the λ^a are the representation matrices of the group in the representation of the ψ's. In our case, $k=2$. The J_0^a obey the SO(10) Lie algebra.

Given these currents, the Sugawara stress-energy tensor can be constructed [33]:

$$T = \frac{1}{2\kappa} :JJ:. \tag{2.5}$$

Here normal ordering is with respect to the J's. T can also be expanded in Laurent coefficients L_n, which obey a Virasoro algebra. The Kac-Moody algebra can be extended to the semi-direct product of the Kac-Moody and this Virasoro algebra, with the additional relation,

$$[L_m, J_n^a] = -nJ_{m+n}^a. \tag{2.6}$$

Eq. (2.5) translates into an expression relating Laurent coefficients:

$$2\kappa L_n = \sum_{m=-\infty}^{+\infty} :J_m^a J_{n-m}^a:. \tag{2.7}$$

This can be imposed as an operator equation. One can define primary fields $\phi(z)$ for the combined algebra [9] satisfying:

$$L_n\phi(0)|0\rangle = 0, \qquad J_n^a\phi(0)|0\rangle = 0 \qquad \text{for } n>0, \tag{2.8}$$

$$L_0\phi(0)|0\rangle = h\phi(0)|0\rangle, \qquad J_0^a\phi(0)|0\rangle = t^a\phi(0)|0\rangle. \tag{2.9}$$

(t^a are the matrices representing the group generators in the representation of ϕ and group indices on ϕ are suppressed.) The states $\phi(0)|0\rangle$ are the highest weight states in the combined algebra.

The properties of highest weight states can determine the c of the theory constructed from the currents, and the value of proportionality between the stress-energy tensor and the currents, κ [9, 34, 35]. Start with a highest weight state and compute $[L_m, L_{-m}]$ in terms of the currents for $m=1,2$, using eqs. (1.2), (2.3) and (2.6). Then take the expectation value of L_0. This gives 3 equations for 3 unknowns: κ, c, h. One finds primary fields in a particular representation of the group have conformal dimension h given by

$$h = \frac{C_{\mathrm{rep}}}{C_{\mathrm{adj}} + k}. \tag{2.10}$$

The constant κ is determined to be

$$\kappa = \tfrac{1}{2}(C_{\mathrm{adj}} + k). \tag{2.11}$$

The value of c for a Sugawara stress-energy tensor is

$$c = \frac{k d_{\mathrm{adj}}}{C_{\mathrm{adj}} + k}, \tag{2.12}$$

where d_{adj} is the dimension of the adjoint representation, and C_{rep} is defined by $t^a t^a = C_{\mathrm{rep}} I$. Putting in the values for SO(10), we see that $c = 5$, for $k = 2$. Since the value of c for one free fermion field is $\frac{1}{2}$, the free fermion stress tensor should have $c = 5$. The Sugawara stress tensor (eq. (2.5)) need not coincide with the free fermion stress tensor [35]. The matching of the values of c indicates that it does in this case. As can be easily checked, eq. (2.10) correctly gives the conformal weight h for the S_α's.

The properties (eqs. (2.8)–(2.9)) of primary fields can also be used to reduce the infinite sum of J^a_m's in the equality between T and the currents J, eq. (2.7). For instance, consider $m = -1$. Acting on a highest weight state, the equation becomes

$$(-\kappa L_{-1} + J^a_{-1} J^a_0)\phi(0)|0\rangle = 0. \tag{2.13}$$

Using the properties

$$[L_{-1}, \phi(z)] = \frac{\partial}{\partial z}\phi(z), \tag{2.14}$$

$$[J^a_{-1}, \phi(z)] = \frac{t^a}{z}\phi(z), \tag{2.15}$$

eq. (2.13) inserted inside of a correlation function of several primary fields yields the differential equation [9]:

$$\left\{\kappa \frac{\partial}{\partial z_i} - \sum_{j \neq i} \frac{t^{a_i} t^{a_j}}{z_i - z_j}\right\}\langle \phi(z_1)\phi(z_2)\dots\phi(z_n)\rangle = 0. \tag{2.16}$$

This can be used, for instance, to calculate a four-point function of primary fields. Due to SL_2 invariance, one can consider

$$\langle \phi(\infty)\phi(1)\phi(x)\phi(0)\rangle \tag{2.17}$$

without loss of generality. (Here $\phi(\infty)$ is to be considered as $\lim_{w\to\infty} w^{2h}\phi(w)$.) A simple example demonstrating the basic ideas is the four ψ correlation function:

$$\langle \psi^{\mu_1}(\infty)\psi^{\mu_2}(1)\psi^{\mu_3}(x)\psi^{\mu_4}(0)\rangle . \tag{2.18}$$

This must be a singlet under SO(10). There are three invariants in the tensor product of four SO(10) vectors, so we can write

$$\langle \psi^{\mu_1}(\infty)\psi^{\mu_2}(1)\psi^{\mu_3}(x)\psi^{\mu_4}(0)\rangle = G_1(x)\mathrm{Inv}_1 + G_2(x)\mathrm{Inv}_2 + G_3(x)\mathrm{Inv}_3 , \tag{2.19}$$

where the G_i's are functions of x and the Inv_i's are linearly independent invariant tensors. The Inv_i's can be chosed as products of delta functions. Writing the G_i's as a vector $\boldsymbol{G} = (G_1, G_2, G_3)$, the differential equation (2.16) becomes

$$\kappa \frac{\partial}{\partial x}\boldsymbol{G} = 9\frac{\partial}{\partial x}\boldsymbol{G} = \left(\frac{1}{x}\boldsymbol{A} + \frac{1}{x-1}\boldsymbol{B}\right)\boldsymbol{G}, \tag{2.20}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are matrices that do not necessarily commute. In this case

$$\boldsymbol{A} = \begin{bmatrix} -9 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2.21}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -9 \end{bmatrix}. \tag{2.22}$$

The two-point functions give various limits as $x \to \infty$, $x \to 1$, and $x \to 0$. This information can be combined with the restriction of analyticity, asymptotic falloff at infinity, and knowledge of the operator product expansion to determine a general form of solution:

$$\begin{aligned} \boldsymbol{G} &= \frac{1}{x(x-1)}\left[\boldsymbol{a}'x + \boldsymbol{b}'(x-1) + \boldsymbol{c}'x(x-1)\right] \\ &= \frac{1}{x(x-1)}\left[\boldsymbol{a} + \boldsymbol{b}x + \boldsymbol{c}x^2\right], \end{aligned} \tag{2.23}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are constant vectors. Solving for these vectors, and normalizing with the two-point function, one finds for the four-point function

$$\langle \psi^{\mu_1}(\infty)\psi^{\mu_2}(1)\psi^{\mu_3}(x)\psi^{\mu_4}(0)\rangle = \delta^{\mu_1,\mu_2}\delta^{\mu_3,\mu_4}\frac{1}{x} - \delta^{\mu_1,\mu_3}\delta^{\mu_2,\mu_4} + \delta^{\mu_1,\mu_4}\delta^{\mu_2,\mu_3}\frac{1}{1-x}, \tag{2.24}$$

as expected. Now, consider the correlation function of four S_α's:

$$\left\langle S_{\alpha_1}(\infty) S_{\alpha_2}(1) S_{\alpha_3}(x) S_{\alpha_4}(0)\right\rangle . \tag{2.25}$$

The S_α's are primary fields of dimension $h=\frac{5}{8}$, and lie in a 16 of SO(10). There are three SO(10) singlets in $16 \otimes 16 \otimes 16 \otimes 16$. The most likely candidates for the Inv_i's, the three different pairings of gamma matrices, are not linearly independent, due to properties of gamma matrices in ten dimensions (the sum of cyclic permutations of three of the indices is zero). The solution uses only two of them. Putting two combinations in, and taking limits to get overall factors of x and $1-x$ as before, one finds:

$$\frac{\partial}{\partial x}\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \left(\frac{1}{x-1}\begin{bmatrix} -\frac{1}{12} & 0 \\ \frac{1}{3} & -\frac{3}{4} \end{bmatrix} + \frac{1}{x}\begin{bmatrix} -\frac{3}{4} & \frac{1}{3} \\ 0 & -\frac{1}{12} \end{bmatrix}\right)\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \tag{2.26}$$

where G_1, G_2 are defined as

$$\left\langle S_{\alpha_1}(\infty) S_{\alpha_2}(1) S_{\alpha_3}(x) S_{\alpha_4}(0)\right\rangle = G_1(x)\gamma^{\mu}_{\alpha_1\alpha_2}\gamma^{\mu}_{\alpha_3\alpha_4} + G_2(x)\gamma^{\mu}_{\alpha_1\alpha_4}\gamma^{\mu}_{\alpha_2\alpha_3}. \tag{2.27}$$

The solution is

$$\begin{aligned}&\left\langle S_{\alpha_1}(\infty) S_{\alpha_2}(1) S_{\alpha_3}(x) S_{\alpha_4}(0)\right\rangle \\ &\qquad = \tfrac{1}{2}x^{-3/4}(x-1)^{-3/4}\left[(1-x)\gamma^{\mu}_{\alpha_1\alpha_2}\gamma^{\mu}_{\alpha_3\alpha_4} - x\gamma^{\mu}_{\alpha_1\alpha_4}\gamma^{\mu}_{\alpha_2\alpha_3}\right].\end{aligned} \tag{2.28}$$

The normalization is not fixed by this method, but can be found using the three-point function.

In addition, one can find the correlation function for two right-handed and two left-handed spin fields:

$$\begin{aligned}&\left\langle S_{\alpha_1}(\infty) S^{\beta_1}(1) S_{\alpha_2}(x) S^{\beta_2}(0)\right\rangle \\ &\qquad = G_1(x)\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} + G_2(x)\delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2} + G_3(x)M^{\beta_1}_{\alpha_1}M^{\beta_2}_{\alpha_2}.\end{aligned} \tag{2.29}$$

The differential equation

$$\frac{\partial}{\partial x}\boldsymbol{G}(x) = \left(\frac{1}{x-1}\begin{bmatrix} 0 & 0 & -\frac{13}{48} \\ 0 & -\frac{5}{4} & -\frac{2}{3} \\ -\frac{1}{9} & 0 & -\frac{5}{18} \end{bmatrix} + \frac{1}{x}\begin{bmatrix} -\frac{5}{4} & -\frac{1}{12} & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & -\frac{1}{9} & -\frac{13}{36} \end{bmatrix}\right)\boldsymbol{G}(x) \tag{2.30}$$

yields the solution

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \text{const}\, x^{-5/4}(x-1)^{-5/4} \begin{bmatrix} 1 - 5/4x + 1/4x^2 \\ x^2 \\ x - x^2 \end{bmatrix}. \tag{2.31}$$

These results agree with previous calculations [10, 11]. This technique is rather general, and applies to many other situations.

3. Spacetime supersymmetry and vertex operators

Although the RNS model has sheet supersymmetry and spacetime fermions, its spectrum is not supersymmetric in spacetime. It is not yet the ten-dimensional superstring. To get the superstring, a projection $\Gamma = (-1)^F = 1$ is made in the Ramond and Neveu-Schwarz sectors, where F is the fermion number on the world sheet. This is the G-parity projection of Gliozzi, Olive and Scherk [4].

This projection originates from the requirement of modular [14] invariance in loop diagrams. Loop diagrams involve traces over correlation functions. Since Neveu-Schwarz and Ramond sectors have antiperiodic and periodic boundary conditions (in σ) respectively, it is necessary to sum over both σ boundary conditions in calculating a diagram. Modular invariance requires symmetry under interchange of τ and σ. Therefore one must sum over both periodic and antiperiodic boundary conditions in τ as well. Periodic boundary conditions correspond to a factor of $(-1)^F$ in the trace, so adding both together provides a factor of $(1 + (-1)^F)$. This is a projection on the $\Gamma = 1$ sector.

In the Neveu-Schwarz sector, $\Gamma = 1$ selects even world-sheet fermion number operators and eliminates the tachyon. In the Ramond sector, Γ anticommutes with all the zero modes of the fermion field ψ^μ, acting as γ_{11} on the Ramond sector ground state. The projection picks out left-handed Majorana-Weyl spinors. The resulting spectrum of the lowest-lying states is a ten-dimensional massless supermultiplet. The chirality of the theory in spacetime depends on the various ways the z and $\bar{z}$ parts of the theory are combined. The different combinations produce different types of superstring theories.

Physical processes in string theories can be interpreted as particles being emitted or absorbed by the string. Vertex operators describing these emission or absorption processes are local operators on the world sheet. Scattering amplitudes for particles are built from correlation functions of these vertex operators. A vertex operator in any kind of string theory must be independent of all coordinate transformations on the sheet. For example, a particular z or θ value has no physical meaning. Different values of all the sheet parameters must be integrated over, and the resulting integral should be independent of all transformations on the world sheet. This means that the integral of the vertex operator should have conformal weight zero ($h = 0$). So the vertex operator must be a conformal field with $h = 1$ or a superconformal field with $h = \frac{1}{2}$.

Vertex operators for the supersymmetric string can describe fermions or bosons. These are made from fields in the Neveu-Schwarz and Ramond sectors which have the correct (super)conformal properties. The fields staying within the Neveu-Schwarz or Ramond sectors must respect superconformal symmetry. The spin fields intertwining the two sectors must be conformal fields (and obey other constraints as well [7]). These will create bosons and fermions respectively.

The vertex operator respecting superconformal symmetry is the supersymmetric version of the vertex operator for the bosonic string [14]. Replacing the coordinate $X^\mu(z)$ in the bosonic string by a superfield, the lowest mass operator even under the Γ projection is

$$\int \mathrm{d}\theta\, V_{\mathrm{NS}}(p, z, \theta) = \zeta_\mu \int \mathrm{d}\theta\, DX^\mu \mathrm{e}^{-iP_\mu X^\mu(z,\theta)} = \zeta_\mu(\partial_z X^\mu + iP_\nu \psi^\mu \psi^\nu)\mathrm{e}^{-iP_\mu X^\mu(z)}. \quad (3.1)$$

Conformal weight 1 requires $P^2 = 0$. V_{NS} does not change the boundary conditions of the field ψ on the sheet, and has a vector index. It corresponds to a massless bosonic vector particle. Calculation of the correlation functions of the Neveu-Schwarz vertex is straightforward, since it involves only the fields X^μ and ψ^μ.

The fermion vertex which creates Ramond ground states is more difficult to obtain, as are its correlation functions. The fermion vertex operator must have dimension 1, since it respects conformal symmetry. So the dimension-$\frac{5}{8}$ spin field needs to be combined with some operator of dimension $\frac{3}{8}$. This operator turns out to be a ghost spin field Σ. Changing boundary conditions actually forces the inclusion of Σ in the vertex. The vertex operator not only twists the boundary conditions on ψ^μ, but also those of all the other sheet fermions. So the ghosts for super-reparameterization symmetry must also be twisted, by Σ. The resulting operator is [6,7]

$$V_{\mathrm{R}}(P, z) = \Sigma(z) S_\alpha(z) \mathrm{e}^{-iP_\mu x^\mu(z)} \cdot u^\alpha(P), \quad (3.2)$$

where u_α obeys the Dirac equation. Counting scaling dimensions, Σ has dimension $\frac{3}{8}$, S_α has dimension $\frac{5}{8}$, and $P^2 = 0$. This operator is closely connected with the supersymmetry generator in spacetime, as might be expected. This is not the whole story (for a detailed discussion see ref. [7]). There are an infinite number of operators, all equivalent under a "picture changing" operation. This additional structure is crucial for calculating the fermion scattering amplitudes, but we will not need explicit formulas here.

One way of doing calculations with the Ramond vertex is to factorize its correlation functions into separate correlation functions of the ghost fields, spin fields and momentum dependent exponentials. The spin field correlation functions can then be calculated separately and multiplied by those of the ghosts and of the momentum factor. The ghost fields can be represented as exponentials of free scalar fields with a background charge [6, 7]. As an example, we calculate the four-fermion scattering amplitude.

The form of the fermion vertex we will use is

$$V_{\alpha_1}(z_1) = e^{ik_1 \cdot X(z_1)} S_{\alpha_1}(z_1) \Sigma(z_1) . \tag{3.3}$$

Although there are many forms of the fermion vertex, the four-point function can be written in terms of V_{α_1} [7]. So it is necessary to evaluate

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle = F_4 . \tag{3.4}$$

Factorizing into the various component fields this becomes

$$\begin{aligned} &\langle e^{ik_1 \cdot X(z_1)} e^{ik_2 \cdot X(z_2)} e^{ik_3 \cdot X(z_3)} e^{ik_4 \cdot X(z_4)} \rangle \\ &\quad \times \langle S_{\alpha_1}(z_1) S_{\alpha_2}(z_2) S_{\alpha_3}(z_3) S_{\alpha_4}(z_4) \rangle \langle \Sigma(z_1) \Sigma(z_2) \Sigma(z_3) \Sigma(z_4) \rangle . \end{aligned} \tag{3.5}$$

We can use the SL_2 invariance to take three of the points to constants:

$$z_1 \to \infty , \qquad z_2 \to 1 , \qquad z_4 \to 0 \tag{3.6}$$

and choose $z_3 = x$. After substituting for the various terms (eqs. (1.13), (2.28)) we get

$$\begin{aligned} F_4 \sim\ & (1-x)^{k_2 \cdot k_3} x^{k_3 \cdot k_4} [x(x-1)]^{-3/4} [x(x-1)]^{-1/4} \\ & \times \left[(1-x) \gamma^{\mu}_{\alpha_1 \alpha_2} \gamma^{\mu}_{\alpha_3 \alpha_4} - x \gamma^{\mu}_{\alpha_1 \alpha_4} \gamma^{\mu}_{\alpha_2 \alpha_3} \right] . \end{aligned} \tag{3.7}$$

The amplitude is the integral of this over the remaining free variable x, with the appropriate measure factor (the jacobian from the transformation (3.6), and factors of $\sqrt{g}$ at each vertex):

$$A = -\tfrac{1}{2} g^2 \int_0^1 \mathrm{d}x \, x^{-1+k_2 \cdot k_3} (1-x)^{-1+k_3 \cdot k_4} \left[(1-x) \gamma^{\mu}_{\alpha_1 \alpha_2} \gamma^{\mu}_{\alpha_3 \alpha_4} - x \gamma^{\mu}_{\alpha_1 \alpha_4} \gamma^{\mu}_{\alpha_2 \alpha_3} \right] . \tag{3.8}$$

Recognizing these integrals as beta functions,

$$A = \tfrac{1}{2} g^2 \left\{ B(1 + k_2 \cdot k_3, k_3 \cdot k_4) \gamma^{\mu}_{\alpha_1 \alpha_2} \gamma^{\mu}_{\alpha_3 \alpha_4} - B(k_2 \cdot k_3, k_3 \cdot k_4 + 1) \gamma^{\mu}_{\alpha_{1\varphi}} \gamma^{\mu}_{\alpha_2 \alpha_3} \right\} . \tag{3.9}$$

This can be rewritten in terms of the Mandelstam variables, $s = -2k_1 \cdot k_2 = -2k_3 \cdot k_4$,

$t=-2k_1\cdot k_4=-2k_2\cdot k_3$:

$$A=\tfrac{1}{2}g^2\left\{B\left(1-\tfrac{1}{2}t,-\tfrac{1}{2}s\right)\gamma^{\mu}_{\alpha_1\alpha_2}\gamma^{\mu}_{\alpha_3\alpha_4}-B\left(-\tfrac{1}{2}t,1-\tfrac{1}{2}s\right)\gamma^{\mu}_{\alpha_1\alpha_4}\gamma^{\mu}_{\alpha_2\alpha_3}\right\}. \quad (3.10)$$

This has crossing symmetry, as can be verified by inspection, and is the same result as obtained previously [10, 11].

Using the loop algebra to calculate the correlation functions of the spin fields separately is one way to approach the problem. Although it keeps spacetime symmetries manifest, it becomes difficult to use for more than four spin fields. The number of unknowns in the differential equation becomes unwieldy. An alternative is to take a hint from the connections between spin fields and the Ising model and the properties of fermions and bosons in two dimensions. These suggest bosonizing all the fields in the theory. It turns out to be useful to not only bosonize the spin fields, but to include the ghosts.

4. Bosonization

It has long been known that 2 fermions can be combined to form one scalar [36]. In the present situation, there are 10 fermion fields, each with a value of c of $\frac{1}{2}$, which, roughly speaking, can be combined to form five free scalar fields, each with $c=1$. One prescription for nonabelian groups, due to Witten [37], associates currents with group elements. The commutation relations follow directly. The bosonization presented here is more closely related to the original abelian bosonization and was used for the vertex operator construction [38–41].

This approach uses exponentials of free scalar fields to represent currents. Cocycles guarantee the commutation relations and provide a local theory. Correlation functions can be evaluated using free scalar theory. Bosonization has been applied to currents and vectors in the past, and for spinors in special cases [13, 19, 41, 42]. (In the E_8 construction, for example, the currents contain the SO(16) spinor representations, and spinors in another context have been bosonized by Goddard and Olive.) Here, the fermion vertex is bosonized, which requires a combination of the above techniques.

The bosonized theory can be identified with the original theory if the new operators have the correct conformal weights and operator products, and if the c value for the system is correct. The value of c for 5 free scalar fields is 5, and the conformal weights can be read off with the knowledge of scalar field two-point functions ($\frac{1}{2}\alpha^2$ for $e^{i\alpha\phi}$). The cocycles give the correct operator product coefficients and the associativity required for a local quantum field theory.

We are looking for descriptions of three operators: J^a, ψ^{μ}, and S_{α} in terms of five scalar fields $\phi_j(z)$. The SO(10) properties of the currents J^a, vectors ψ^{μ} and spinors S_{α} are encoded in their weight vectors. The weights of SO(10) lie on a five-dimensional lattice since SO(10) is rank five. The bosonized operator with weight vector α is c_α:$e^{i\alpha\cdot\phi(z)}$:, where c_α provides proper (anti)commutation relations. The (length)^2 of a weight vector α is

$$\alpha\cdot\alpha=\sum_i(\alpha^i)^2. \tag{4.1}$$

Starting with the currents gives a natural relation between the c_α's and the structure constants of the group. The currents J^a obey, by definition,

$$\left[J_0^a, J_0^b\right]=if^{abc}J_0^c, \tag{4.2}$$

with $f^{abc}=f^{[abc]}$ the structure constants of SO(10). The currents can be put in a basis of Cartan generators, H^i, and raising the lowering operators E_a, with the following commutation relations:

$$[H^i, H^j]=0,$$

$$\left[H^i, E_a\right]=a^iE_a, \tag{4.3}$$

and

$$[E_a, E_b]=\varepsilon(a,b)E_{a+b},$$

if E_{a+b} is a raising or lowering operator, zero otherwise.

The weights of the SO(10) adjoint representation (the roots) are

$$\beta_a=(\pm1,\pm1,0,0,0) \tag{4.4}$$

and permutations. Their exponentials provide forty of the currents. The other five currents are $\partial_z\phi_j$, corresponding to the Cartan generators H^j [38–40]. The currents which are exponentials are written as

$$J^{[\mu\nu]}=J^a=\,:e^{i\beta_a^j\phi_j(z)}:c_a. \tag{4.5}$$

The zero modes of these operators must satisfy the commutation relations, eq. (4.3). So the factor c_a must obey

$$c_ac_b=\varepsilon(a,b)c_{a+b}. \tag{4.6}$$

This is one of the defining relations for a cocycle $\varepsilon(a,b)$, in the basis where the structure constants f^{abc} are all ±1. The other relations $\varepsilon(a,b)$ must obey are associativity,

$$\varepsilon(a,b)\varepsilon(a+b,c)=\varepsilon(a,b+c)\varepsilon(b,c) \tag{4.7}$$

(guaranteeing the associativity of the operator product), and a symmetry property under interchange of arguments.

An explicit representation for the cocycles has been given by Segal [40]. He defines cocycles for currents, which are on even lattices (lattices which have vectors of $(\text{length})^2 = 2$). There are two cocycles present in the theory, one which gives the Schwinger term in the Kac-Moody algebra, and one which gives the commutation relations. The Schwinger term in the algebra is taken care of by the operators in the fields $\phi_j(z)$ which produce c-numbers in the operator product. The other two-cocycle, dependent only on the group lattice, appears in the commutation relations above. It is

$$\varepsilon(\alpha, \beta) = (-1)^{\sigma(\alpha, \beta)}, \tag{4.8}$$

where

$$\sigma(\alpha, \alpha) = \tfrac{1}{2}\langle \alpha, \alpha \rangle, \tag{4.9}$$

$$\sigma(\alpha, \beta) + \sigma(\beta, \alpha) = \langle \alpha, \beta \rangle \mod 2. \tag{4.10}$$

Here, $\langle \alpha, \beta \rangle$ is a suitable scalar product on the lattice, and $\sigma(\alpha, \beta)$ need only be defined mod 2. One can easily check that this definition of the two-cocycle obeys eq. (4.6). Currents (of SO(10) in particular) are represented this way, with $\langle \alpha, \beta \rangle = \alpha \cdot \beta$. This causes $\varepsilon(\alpha, \beta)$, and thus the structure constants f^{abc} to be antisymmetric.

The next objects in the theory to look are the $\psi^i(z)$, SO(10) vectors. They have dimension $\frac{1}{2}$ and weight vectors

$$\rho_i = (\pm 1, 0, 0, 0, 0) \tag{4.11}$$

and permutations. The vertex representation [13, 39, 41] is

$$\psi^{\rho_i}(z) = :e^{i\rho_i \cdot \phi(z)}: c_{\rho_i}. \tag{4.12}$$

This is an *integer* lattice, not an even lattice.

The cocycle relations were defined to get commutation relations for the currents on the sheet. Since ψ is a sheet fermion, it must anticommute, not commute on the sheet. Taking this into account, a generalized inner product can be defined [41]:

$$\langle \alpha, \beta \rangle = \alpha \cdot \beta - |\alpha|^2 |\beta|^2 \mod 2, \tag{4.13}$$

which can be used to define the cocycle in (4.8)–(4.10). Since this is defined mod 2, the scalar product is unchanged for $(\text{length})^2 = 2$ vectors. The operators with $\text{length}^2 = 1$ weight vectors anticommute with each other but commute with everything else.

The third operator to construct is the spin field S_α. It is a dimension-$\frac{5}{8}$ operator and a SO(10) spinor. The 32 spinor weights (of the 16 and $\overline{16}$ representations) of

SO(10) are

$$\lambda_\alpha = \tfrac{1}{2}(\pm, \pm, \pm, \pm, \pm). \tag{4.14}$$

The number of minus signs in λ_α is even or odd, depending on the spinor representation. The weight λ_α has (length)^2 $\frac{5}{4}$. The natural candidate for S_α is:

$$S_\alpha = :e^{i\lambda^j_\alpha \phi_j(z)}: c_\alpha . \tag{4.15}$$

There is no obvious way to generalize the inner product for Segal's construction and keep anticommutation elations on the sheet and in spacetime. Anticommutation relations are guaranteed if this spin field can be made (length)$^2 = 1$. The answer comes from realizing that on the sheet, it isn't meaningful to consider a matter spin field without a ghost spin field. These two comprise the fermion vertex, which *does* have integer (length)^2 weight vectors associated with it, on an expanded lattice that includes a weight for the ghosts.

To construct the integer lattice, we represent the ghost spin fields Σ as exponentials of a free scalar field.

$$\Sigma_\pm(z) = :e^{\pm i\phi_{gh}(z)/2}:. \tag{4.16}$$

The field $\phi_{gh}(z)$ has an opposite sign metric

$$\langle \phi_{gh}(z)\phi_{gh}(w)\rangle = +\ln(z-w) \tag{4.17}$$

and there is a background charge present. So the conformal weight of $:e^{iq\phi_{gh}(z)}:$ is $-\frac{1}{2}q(q+2)$. Thus $\Sigma_{-1/2}$, with dimension $\frac{3}{8}$, is the ghost spin field discussed earlier. Its chiral conjugate, $\Sigma_{1/2}$, has dimension $-\frac{5}{8}$. Combining these operators with the matter spin fields, we get operators of dimensions 1 and 0, which are the spin field parts of two representatives of the fermion vertex:

$$\begin{aligned} V^{\alpha\pm} = \Sigma_\pm \cdot S^{\alpha\pm} &= :\exp\left(i\lambda^j_\alpha \phi_j(z) \pm \tfrac{1}{2} i\phi_{gh}(z)\right): c_{\alpha\pm} \\ &\equiv :e^{i\bar{\lambda}_\alpha \cdot \bar{\phi}(z)}: c_{\alpha\pm}, \end{aligned} \tag{4.18}$$

where

$$\bar{\lambda} = (\lambda, \lambda_{gh}), \qquad \lambda_{gh} = \pm\tfrac{1}{2}, \tag{4.19}$$

$$\bar{\phi} = (\phi, \phi_{gh}). \tag{4.20}$$

The six-dimension weight lattice of this system is integral and lorentzian, with inner product $(+++++-)$, i.e.

$$\bar{\lambda}\cdot\bar{\lambda} = \lambda\cdot\lambda - \lambda_{gh}\lambda_{gh}. \tag{4.21}$$

So cocycles are defined for the $V^{\alpha\pm}$'s using the cocycle for integer $(\text{length})^2$ weight vectors, eqs. (4.8)–(4.10), (4.13). The vertex operators obey anticommutation relations on the sheet.

This is a generalization of the approach used by Narain [42]. He used weight vectors lying on even euclidean or lorentzian lattices of dimension $8n+2q$, with signature $(++++++++)$ n times, and $(+-)$ q times, for operators obeying commutation relations. Here, weight vectors lie on integer lattices of dimension $4n+2q$ with metric $n(++++)+q(+-)$. It includes operators obeying commutation or anticommutation relations, depending on the $(\text{length})^2$ of the weights. Instead of using the connection between the roots of E_8 and SO(16) spinor weights, this uses the properties of SO(8) and its isomorphisms between the two-spinor and one-vector representations.

The nonlocal model containing all of the above operators is defined on an integral, but not self-dual lattice. After the Γ projection to a local theory, an integral self-dual lattice results. The Γ projection is done on the *product* of ghost and matter spin fields [7], and

$$\Gamma V^{\alpha\pm} = \prod_i \text{sign}(\lambda_i)\,. \tag{4.22}$$

The requirement $\Gamma = 1$ picks out an integral self-dual sublattice. The existence of cocycles for self-dual integral lattices has been shown by Goddard and Olive [41]. They show that an identification can be made between the prefactors c_a and gamma matrices. (It is interesting to note that there is a similarity between the requirements of the metric of the lattices, and the dimensions where Majorana-Weyl fermions can be defined.)

Using the weight lattice is not Lorentz invariant, as only SU(5) (of the five pairs of fermions) is manifest, but it is efficient.

More in line with the original Luther-Peschel doubling scheme is to add ten new fermion fields, each contributing a c of $\frac{1}{2}$. This "squares" the correlation function, giving a set of ten fields with $c = 1$ each. Calculations are easy for these ten free scalar fields. At the end, the "square root" must be taken (this is the difficult step). The advantage of this approach is that the SO(10) symmetry (Lorentz covariance) is manifest as a subgroup of SU(10) throughout. In this case, perhaps one of Narain's lattices could be used, with SO(1,9) manifest. (Obeying commutation relations rather than anticommutation relations on the sheet would be alright, since these objects would be the squares of the spin fields.)

5. Conclusion

We have identified the spin fields of Corrigan et al. [10, 11] as the SO(10) vertex operators with spinor weights. We have given two methods of calculating their correlation functions: the Lorentz invariant linear differential equation (eq. (2.16)) and bosonization. The Lorentz invariant approach, utilizing the SO(10) loop algebra, may be useful for deepening understanding, but is impractical to apply to correlation functions of more than 4 operators. (The number of invariants in the differential equation grows quickly.) The bosonization approach is not Lorentz invariant in intermediate stages, but it is practical. The role of the ghosts in simplifying this construction is noteworthy. The SO(10) loop algebra demonstrates a special connection between spacetime Lorentz symmetries in 10 dimensions and sheet conformal symmetry. Perhaps this will provide more clues for relationships between world-sheet "string-like" properties and spacetime ones.

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