Nuclear Physics B271 (1986) 540–560
North-Holland, Amsterdam

STRING FIELD THEORY*

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Received 13 May 1985

A gauge invariant field theory of the bosonic string is formulated following Siegel, Banks and Peskin, and Feigin and Fuks. The free field action is constructed from the Virasoro operators assuming only locality. The propagator is very regular at short distance. The natural jointing-splitting interaction is described abstractly and a method of construction is sketched. A way is conjectured of reformulating the theory to be independent of the vacuum geometry.

1. Introduction

String theory provides self-consistent quantizations of gravity, but there is no field theory of strings which is independent of the vacuum geometry [1]. This paper is an attempt in that direction. First a free field theory is constructed from the Virasoro operators and the assumptions of locality and gauge invariance. The joining-splitting interaction is described and a method of construction is sketched. The elements of the construction which depend on the vacuum geometry are isolated and there is some speculation on how to eliminate this dependence.

The basic steps were taken by Siegel [2], who wrote the gauge-fixed Lorentz covariant field theory of the bosonic string in the BRST formalism. The string field $\phi(x)$ is a scalar function of parametrized strings $x^{\mu}(s)$. It is represented as a state ϕ in the Hilbert space of a single parametrized string. The free field action, omitting the ghost fields, is

$$S(\phi) = \phi^{\dagger}(H-1)\phi. \tag{1}$$

$H = L_0$ is the middle Virasoro operator, the hamiltonian of the first-quantized string. The gauge transformations are generated by the raising operators of the Virasoro algebra:

$$\delta\phi = \sum_{n=1}^{\infty} L_{-n}\varepsilon_n. \tag{2}$$

* This work was supported in part by US Department of Energy grant DE-FG02-84ER-45144 and the Alfred P. Sloan Foundation.

0550-3213/86/$03.50

Banks and Peskin [3] explained how to construct the gauge invariant free field theory using the algebra of Virasoro operators. The natural transverse gauge slice is the subspace orthogonal to the gauge variations. Since $L_n = L^\dagger_{-n}$, the transverse gauge slice consists of the subspace of fields ϕ which are annihilated by the lowering operators:

$$L_{+n}\phi = 0. \tag{3}$$

The Banks-Peskin free field action is

$$S(\phi) = \phi^\dagger P_{\mathrm{tr}}(H-1)P_{\mathrm{tr}}\phi, \tag{4}$$

where P_{tr} is the projection onto the transverse gauge slice.

Banks and Peskin carried out the construction of P_{tr} level-by-level for a number of levels in the representations of the Virasoro algebra which occur in the space of string fields*. They found that P_{tr}, constructed from the Virasoro operators, was a singular function of H. Singularity in H is equivalent to nonlocality, since H depends on the total spacetime momentum p^μ of the string in the form $H = \frac{1}{2}p_\mu p^\mu + H_{\mathrm{int}}$, where H_{int} is the energy of the string vibrations. The nonlocality in $S(\phi)$ was removable, at least on the lower levels, by adding unphysical Stückelberg fields to the theory, which disappear from the action in a unitary gauge.

Following Banks and Peskin, our first step is to construct the free string field using the Virasoro operators and the principle of locality. This is done in the simplest setting, the open bosonic string in its critical dimension $D = 26$. For the open string the gauge algebra acts as a single Virasoro algebra. For the closed string the gauge algebra acts as two commuting Virasoro algebras. For the supersymmetric string the critical dimension is 10 and the gauge algebra is the Ramond-Neveu-Schwarz algebra. The propagator is given for each of these cases. It would of course be interesting to construct the propagators below the critical dimension.

2. The free string field

The space of *parametrized open strings* is the space of functions $x^\mu(s)$ from the interval $[0, \pi]$ into spacetime. The space of *physical strings* is the quotient of the space of parametrized strings by the group $\mathrm{Diff}[0, \pi]$ of reparametrizations of the interval $[0, \pi]$.

The field ϕ is a state in the Hilbert space of the conformally invariant, 1 + 1 dimensional, D component free scalar quantum field $x^\mu(s, t)$:

$$\left(\partial_t^2 - \partial_s^2\right)x^\mu = 0,$$

$$\partial_s x^\mu = 0 \qquad \text{at} \qquad s = 0, \pi. \tag{5}$$

* After this paper was written it was learned from M. Peskin that their construction of P_{tr} was completed.

The total spacetime momentum is

$$p^{\mu} = \int_0^{\pi} \frac{\mathrm{d}s}{2\pi} \dot{x}^{\mu} . \tag{6}$$

The Virasoro operators are the Fourier coefficients of the traceless stress-energy tensor

$$\begin{aligned} T_{++}(t,s) &= \tfrac{1}{2} : \partial_+ x^{\mu} \partial_+ x_{\mu} : \qquad \left(\partial_+ = \tfrac{1}{2}(\partial_t + \partial_s) \right) \\ &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{in(t+s)} L_n . \end{aligned} \tag{7}$$

The Virasoro operators satisfy

$$\begin{aligned} & L_{-n} = L_n^{\dagger} , \\ & [L_m, L_n] = (m-n) L_{m+n} + \tfrac{1}{12} c (m^3 - m) \delta_{m+n,0} , \\ & c = D = 26 . \end{aligned} \tag{8}$$

The hamiltonian is

$$H = L_0 = \tfrac{1}{2} p_{\mu} p^{\mu} + H_{\mathrm{int}} . \tag{9}$$

The energy is lowered by the L_{+n} and raised by the L_{-n}:

$$[H, L_n] = -n L_n . \tag{10}$$

The hamiltonian of the internal excitations, H_{int}, has only integer eigenvalues.

We look for a gauge invariant free field action

$$S(\phi) = \phi^{\dagger} K_{\mathrm{tr}} \phi . \tag{11}$$

The inverse propagator K_{tr} is assumed to be generated by the Virasoro operators. This represents the least possible dependence on the vacuum geometry. Unitarity implies K_{tr} should be selfadjoint. K_{tr} commutes with H, or can be made to do so by averaging:

$$K_{\mathrm{tr}} \to \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{i\theta H} K_{\mathrm{tr}} \mathrm{e}^{-i\theta H} . \tag{12}$$

The gauge invariance conditions are

$$K_{\mathrm{tr}} L_{-n} \varepsilon_n = 0 . \tag{13}$$

Feigin and Fuks [4] have given a technique for studying the locality of operators like K_{tr}. Write K_{tr} in the normal ordered form

$$K_{\mathrm{tr}} = K(H) + \sum_{n=1} \sum_{|I|,|J|=n} L_{-I} K_n^{I,J}(H) L_J . \tag{14}$$

In the natural transverse gauge the propagator is $K(H)^{-1}$. The sum is over multi-indices I,

$$\begin{aligned} I &= I_1, I_2, \dots, I_k , \\ 1 \leqslant I_1 &\leqslant I_2 \leqslant \cdots \leqslant I_k , \\ |I| &= I_1 + I_2 + \cdots + I_k , \\ L_I &= L_{I_1} L_{I_2} \dots L_{I_k} , \\ L_{-I} &= L_I^{\dagger} . \end{aligned} \tag{15}$$

Locality requires that the functions $K(H)$ and $K_n^{I,J}(H)$ be nonsingular in H. We will see in the next section that the $K_n^{I,J}(H)$ are completely determined by $K(H)$ and the commutation relations of the Virasoro algebra. Then we will find that locality is precisely equivalent to the condition

$$K(h_n) = 0, \qquad h_n = \tfrac{1}{24}(25 - n^2), \qquad n = 0,1,2,\dots . \tag{16}$$

The minimal inverse propagator has only the zeros required by locality:

$$K_{\min}(H) = \sqrt{24H - 25}\, \sinh\left(\pi\sqrt{24H - 25}\right). \tag{17}$$

The pole in the propagator at $H = 1$ is physical; the remaining poles presumably correspond to the Stückelberg fields needed in the treatment of Banks and Peskin and can be gauged away. The propagator of the string ought to be the minimal propagator, since the excitations associated with unnecessary additional poles would probably not decouple.

Siegel [2] has remarked that the ultraviolet properties of the string should be evident in a gauge invariant theory. The minimal propagator is in fact quite regular at large momentum:

$$K_{\min}(H)^{-1} \underset{p^2 \to \infty}{\sim} p^{-1} e^{-\pi\sqrt{24}\, p} . \tag{18}$$

Both the projection P_{tr} and the nonlocal inverse propagator $P_{tr}(H-1)P_{tr}$ can be derived from K_{tr}. For the projection, change variables to $\phi' = K(H)^{-1/2}\phi$:

$$P_{tr} = 1 + \sum_{n=1} \sum_{|I|,|J|=n} L_{-I} \frac{K_n^{I,J}(H)}{K(n+H)} L_J . \tag{19}$$

For the nonlocal inverse propagator change variables to $\phi'' = \sqrt{(H-1)/K(H)}\,\phi$:

$$P_{tr}(H-1)P_{tr} = (H-1) + \sum_{n=1} \sum_{|I|,|J|=n} L_{-I} \frac{(n+H-1)K_n^{I,J}(H)}{K(n+H)} L_J . \tag{20}$$

3. Construction of K_{tr}

The argument follows Feigin and Fuks with two modifications. They construct operators analytic in both H and c, while we work at a fixed value of c. They construct operators which commute with the Virasoro algebra, while we construct gauge invariant, transverse operators.

First we show by induction in n that the Virasoro algebra determines the functions $K_n^{I,J}(H)$ as linear functionals of $K(H)$. From the Kac [5] determinant formula for the Virasoro algebra it follows that the nonsingularity of all $K_n^{I,J}(H)$ is equivalent to the linear conditions $K(h_n) = 0$.

The construction of K_{tr} depends only on some general properties of the space of fields as a representation of the Virasoro algebra. First, every string field is annihilated by all L_I for $|I|$ large enough, because H is bounded below. Second, for any nonzero field ϕ the fields $L_{-I}\phi$ are all linearly independent. Third, all eigenvalues $H = h$ occur, since p^2 can take all values, and the corresponding eigenspaces vary smoothly with h.

The strategy is to approximate the space of fields by the *Verma modules*, which consist of the *transverse fields* and their *descendent fields*. The transverse fields are the fields $\hat{\phi}$ in the transverse gauge slice $L_{+n}\hat{\phi} = 0$. The descendent fields of $\hat{\phi}$ are all the linear combinations of the $L_{-I}\hat{\phi}$, $|I| > 0$. $\hat{\phi}$ and its descendents are linearly independent and span the Verma module $V(\hat{\phi})$ generated by $\hat{\phi}$. We say $\hat{\phi}$ is an *ancestor* of its descendents. The nth level of the Verma module consists of all linear combinations of the fields $L_{-I}\hat{\phi}$, $|I| = n$. In the mathematics literature the transverse fields are called *highest weight vectors*. Transverse fields occur with all eigenvalues of H, since p^2 can take all values.

Verma modules which have transverse descendents are called *degenerate*, because a transverse descendent and all its descendents are orthogonal to the entire Verma module. A Verma module which contains a transverse field $\hat{\phi}_n$ on level $n > 0$ is said to be degenerate on level n. We will see later that the Verma modules are dense in the space of fields.

We calculate $K_n^{I,J}$ by induction in n. Suppose that for all $m<n$ the matrix of functions $K_m^{I,J}(H)$ has been given explicitly as a linear functional, meromorphic in H, of the single function $K(H)$. This is true for $n=1$.

Now let K_{tr} act on a field on the nth level of a Verma module $V(\hat{\phi})$. Since $\hat{\phi}$ is transverse, $K_{\mathrm{tr}}\hat{\phi}=K(H)\hat{\phi}$, and

$$L_I L_{-J}\hat{\phi}=\begin{cases}0, & |I|>|J|\\ M_n^{I,J}(H)\hat{\phi}, & |I|=|J|\\ \sum\limits_{|K|=|J|-|I|} A_{IJ}^{K}(H)L_{-K}\hat{\phi}, & |I|<|J|.\end{cases} \tag{21}$$

The matrix $M_n^{I,J}(H)$ is polynomial in H (and c). As long as M_n is invertible, the Verma module is nondegenerate on the nth level. Write

$$K_{\mathrm{tr}}L_{-I}\hat{\phi}=(K_{n,-}+K_n)L_{-I}\hat{\phi},$$

$$K_{n,-}=K(H)+\sum_{m=1}^{n-1}\sum_{|J|,|K|=m}L_{-J}K_m^{J,K}(H)L_K,$$

$$K_{n,-}L_{-I}\hat{\phi}=\sum_{|J|=n}L_{-J}K_{n,-}^{J,I}(H)\hat{\phi},$$

$$K_n=\sum_{|J|,|K|=n}L_{-J}K_n^{J,K}(H)L_K,$$

$$K_nL_{-I}\hat{\phi}=\sum_{|J|,|K|=n}L_{-J}K_n^{J,K}(H)M_n^{K,I}(H)\hat{\phi}. \tag{22}$$

By gauge invariance $K_{\mathrm{tr}}L_{-I}\hat{\phi}=0$. Therefore

$$K_{n,-}^{J,I}(H)+\sum_{|K|=n}K_n^{J,K}(H)M_n^{K,I}(H)=0, \tag{23}$$

which determines $K_n^{I,J}$:

$$K_n^{I,J}(H)=-\sum_{|K|=n}K_{n,-}^{I,K}(H)\left(M_n^{-1}\right)^{K,J}(H). \tag{24}$$

For example, the first use of this argument gives

$$K_1^{1,1}(H)=-\frac{K(H+1)}{2H}. \tag{25}$$

By hypothesis the matrix $K_{n-}^{I,J}(H)$ is given as a linear functional of $K(H)$, so now the matrix $K_n^{I,J}(H)$ is also. Since $M_n^{I,J}(H)$ is polynomial in H, $K_n^{I,J}(H)$ is meromorphic in H. This completes the induction argument.

If $K_m^{I,J}$ is nonsingular for $m<n$ then the possibility of a singularity in $K_n^{I,J}(H)$ arises only when the matrix $M_n^{I,J}(H)$ becomes noninvertible. Kac [5] has given a formula for the determinant of $M_n^{I,J}(H)$. Up to a positive multiplicative constant,

$$\det(M_n(H)) = \prod_{p,q} (H - h_{p,q})^{P(n-pq)},$$

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m-1)},$$

$$c = 1 - \frac{6}{m(m+1)},$$

$$P(n) = \sum_{|I|=n} 1, \tag{26}$$

where the product is over positive integers p, q with $pq \leqslant n$. For $c=26$,

$$m = -\tfrac{2}{5}, \qquad h_{p,q} = \tfrac{1}{24}\left(25 - (3p+2q)^2\right). \tag{27}$$

All of the properties of Verma modules can be deduced from the determinant formula.

Suppose $\hat{\phi}$ is an eigenvector of H. If $H\hat{\phi} \neq h_{p,q}\hat{\phi}$ for all p, q then the Verma module generated by $\hat{\phi}$ is nondegenerate. On the other hand, if $H\hat{\phi} = h_{p,q}\hat{\phi}$ then on level pq of the Verma module there is a transverse field $\hat{\phi}_{p,q}$. This will be demonstrated in the next section. Since $\hat{\phi}_{p,q}$ is both a transverse and a descendent field, $K_{\mathrm{tr}}\hat{\phi}_{p,q}$ is equal to both $K(H)\hat{\phi}_{p,q}$ and 0. Therefore it is a necessary condition for locality that $K(h_{p,q} + pq) = 0$, where

$$h_{p,q} + pq = \tfrac{1}{24}\left(25 - (3p-2q)^2\right). \tag{28}$$

This collection of numbers is exactly the set $\{h_n\}$ listed in eq. (16). In the next section we will see that $K(h_n) = 0$, $n \geqslant 0$ is also a sufficient condition for locality.

It must be checked that K_{tr}, constructed to act transversely on Verma modules, acts transversely on the whole space of fields. By construction K_{tr} satisfies the transversality condition $K_{\mathrm{tr}} L_{-n} \varepsilon = 0$ for any field ε which belongs to a Verma module. But the Verma modules are dense in the space of fields, because fields can occur outside Verma modules only when there is a degenerate Verma module. For almost all eigenvalues of H there are no degenerate Verma modules. Since any gauge transformation ε can be approximated by ε' in a Verma module, and if K_{tr} is local, then $K_{\mathrm{tr}}L_{-n}\varepsilon$ can be approximated by $K_{\mathrm{tr}}L_{-n}\varepsilon'=0$. Therefore $K_{\mathrm{tr}}L_{-n}\varepsilon=0$ for all ε, and the gauge invariance of K_{tr} is proved.

In summary, there is a one-to-one correspondence between gauge invariant quadratic forms K_{tr} and operators $K(H)$ on transverse fields. K_{tr} is local if and only if $K(H)$ is a nonsingular function of H and $K(h_n)=0$ for all h_n.

4. Technical arguments

We first need to show that if $H\hat{\phi}=h_{p,q}\hat{\phi}$, then there exists a nonzero transverse vector $\hat{\phi}_{p,q}$ on level pq of the Verma module generated by $\hat{\phi}$. If $pq=n$ there is a simple zero of $\det(M_n)(H,c)$ at $H=h_{p,q}(c)$. Let $v^I(c)$ be the vector annihilated by $M_n(h_{p,q})$:

$$\sum_{|J|=n} M_n^{I,J}(h_{p,q})v^J(c)=0\,. \tag{29}$$

The corresponding field on level n is

$$\hat{\phi}_{p,q}=\sum_{|J|=n} v^J L_{-J}\hat{\phi}\,. \tag{30}$$

By the transversality of $\hat{\phi}$, $L_I\hat{\phi}_{p,q}=0$, for all I, $|I|>n$. By the definition of $M_n^{I,J}$, $L_I\hat{\phi}_{p,q}=0$, for all I, $|I|=n$. Now allow c to vary slightly from 26. Then $M_m^{I,J}(h_{p,q})$ is nonsingular for all $m<n$. It follows that $L_I\hat{\phi}_{p,q}=0$ for all I, $|I|<n$, i.e. $\hat{\phi}_{p,q}$ is a transverse descendent of $\hat{\phi}$ on level pq.

We second need to show that if $K(h_n)=0$ for all h_n then K_{tr} is nonsingular. By eq. (24), $K_n^{I,J}(H)$ is possibly singular in H only at $H=h_{p,q}$ where the matrix $M_n^{I,J}(H)$ becomes noninvertible. For $K_n^{I,J}(H)$ to be regular at $H=h_{p,q}$ the matrix $K_{n,-}^{I,J}(H)$ must annihilate the same vectors that $M_n^{I,J}(H)$ annihilates at $H=h_{p,q}$ to at least the same order in $H-h_{p,q}$.

Assume that $K(H)$ has at least a simple zero at each h_n. Suppose that $\hat{\phi}$ is a transverse field with $H\hat{\phi}=h\hat{\phi}$. Suppose that $\phi_n=\Sigma\phi_n^I L_{-I}\phi$ is a field on the nth level of the Verma module generated by $\hat{\phi}$. We will show that K_{tr} is regular on ϕ_n, by induction in n.

There are two cases to consider. First suppose that $\hat{\phi}$ is the only transverse field in the Verma module which is an ancestor of ϕ_n. There are two subcases. If ϕ_n is not itself transverse then $\Sigma(M_n^{-1})^{I,J}\phi_n^J$ is nonsingular and $K_{\mathrm{tr}}\phi_n$ is regular. If ϕ_n is transverse then $n=pq$ with $h=h_{p,q}$. $M_n^{I,J}(H)\phi_n$ has a simple zero at $H=h_{p,q}$. But the transversality of ϕ_n implies that $K_{-,n}\phi_n=K(H+n)\phi_n=K(h_{p,q}+pq)\phi_n=0$. So the zero of $K(H)$ at $H=h_{p,q}+pq$ is precisely what is needed to make the solution of eq. (22) nonsingular.

In the second case ϕ_n has transverse ancestors besides $\hat{\phi}$ in $V(\hat{\phi})$. Let $\hat{\phi}_m$ be such a transverse ancestor of ϕ_n, on level $m > 0$, chosen so that $\hat{\phi}_m$ has no transverse descendent which is also an ancestor of ϕ_n. Regard ϕ_n as a descendent of $\hat{\phi}_m$ on level $n-m$. If ϕ_n is not transverse then M_n vanishes on ϕ_n to the same order that M_m vanishes on $\hat{\phi}_m$. If ϕ_n is transverse then M_n vanishes on ϕ_n to one order more. Exactly the same holds for $K_{n,-}$ and $K_{m,-}$, by the previous paragraph. Since $m < n$ we can argue by induction that K_{n-} vanishes at least as strongly as M_n on ϕ_n. This completes the argument that $K(h_n) = 0$ for all h_n is necessary and sufficient for $K_{\rm tr}$ to be local.

5. The closed string propagator

The reparametrization group of the closed string is the group Diff(S^1) of diffeomorphisms of the circle. Its generators are the $R_n = L_n - \bar{L}_{-n}$, $R_n^\dagger = R_{-n}$. The rotation generator is $R = R_0$. The hamiltonian is $H = L_0 + \bar{L}_0 = p^2 + L_0^{\rm int} + \bar{L}_0^{\rm int}$. The transversality conditions are

$$L_{+n}\phi = \bar{L}_{+n}\phi = 0\,,$$
$$R\phi = 0\,. \tag{31}$$

The gauge invariant inverse propagator is

$$K_{\rm tr} = K(H)\delta_R + \sum_{I,\bar{I},J,\bar{J}} L_{-I}\bar{L}_{-\bar{I}}K^{I\bar{I},J\bar{J}}(H)\delta_R L_J \bar{L}_{\bar{J}}\,. \tag{32}$$

The locality condition is $K(2h_n) = 0$, $n \geqslant 0$. The minimal inverse propagator is

$$K_{\rm min}(H) = \sqrt{12H - 25}\,\sinh(\pi\sqrt{12H-25})\,. \tag{33}$$

6. The fermionic string propagator

The fields form the even G-parity sector of the Ramond-Neveu-Schwarz model. The gauge operators are $L_n^\dagger = L_{-n}$, $G_n^\dagger = G_{-n}$. The algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{8}\hat{c}(m^3 - m)\delta_{m+n}\,,$$
$$[G_m, G_n]_+ = 2L_{m+n} + \tfrac{1}{2}\hat{c}\left(m^2 - \tfrac{1}{4}\right)\delta_{m+n}\,,$$
$$[L_m, G_n] = \left(\tfrac{1}{2}m - n\right)G_{m+n}\,. \tag{34}$$

The critical dimension is $D = \hat{c} = 10$. In the bosonic (Neveu-Schwarz) sector the G_n are indexed by half-integers, in the fermionic (Ramond) sector by integers. The hamiltonian is $H = L_0$. In the fermionic sector the two-dimensional supersymmetry generator is $Q_F = G_0$, $H = Q_F^2 + \frac{1}{16}\hat{c}$.

The string field has two components ϕ_B and ϕ_F. The free field action for transverse fields is

$$S(\phi_B, \phi_F) = \phi_B^\dagger K_B(H)\phi_B + \bar{\phi}_F K_F(Q_F)\phi_F . \tag{35}$$

From the determinant formula for the Ramond-Neveu-Schwarz algebra [5, 6, 7] $K_B(H)$ must have zeros at $H = \hat{h}_n$

$$\hat{h}_n = \tfrac{1}{16}(9 - n^2), \qquad n \geqslant 0 \tag{36}$$

and $K_F(Q_F)$ must have zeros at $Q_F = q_n$,

$$q_n = i\tfrac{1}{4}n . \tag{37}$$

The minimal inverse propagator is

$$K_{B,\min} = \sqrt{H - \tfrac{9}{16}}\,\sinh\left(4\pi\sqrt{H - \tfrac{9}{16}}\right),$$

$$K_{F,\min} = \sinh(4\pi Q_F) . \tag{38}$$

The spacetime supersymmetry is not obvious.

7. The structure of the representations of the $c = 26$ Virasoro algebra

It is useful to know in exactly what patterns Verma modules are included in other Verma modules. This information was obtained by Feigin and Fuks [8], Rocha, Caridi and Wallach [9] for $c = 26$ and by Feigin and Fuks [8] for general c. For $c = 26$ the possible patterns of inclusion are the towers diagrammed in fig. 1a–c. V_n is the Verma module generated by a transverse field $\hat{\phi}_n$ with $H\hat{\phi}_n = h_n\hat{\phi}_n$. An arrow $V_m \to V_n$ means that V_m is always included in V_n. Inclusion, of course, is transitive. If V_n occurs among the string fields then so do all the V_m included in V_n.

The diagrams are constructed by the following rules. Draw an arrow $V_m \to V_n$ if $h_n = h_{p,q}$ and $h_m = h_{p,q} + pq$. Then erase the arrows which are redundant by the transitivity of inclusion. The basis for the rules is that $\det(M_{pq}(H))$ vanishes to first order at $H = h_{p,q}$, and, for $n > pq$, $\det(M_n(H))$ vanishes to the lowest possible order, $P(n - pq)$, given that $\det(M_{pq}(H))$ vanishes to first order.

At the tops of the towers are the Verma modules V_n for $n = 0, 1, 2, 3, 4, 6$. These are the Verma modules which are included within other Verma modules but do not themselves include any other Verma modules. They correspond to the numbers h_n (eq. (16)) which are not of the form $h_{p,q}$ (eq. (27)). Every Verma module which contains another Verma module contains also exactly one of these six.

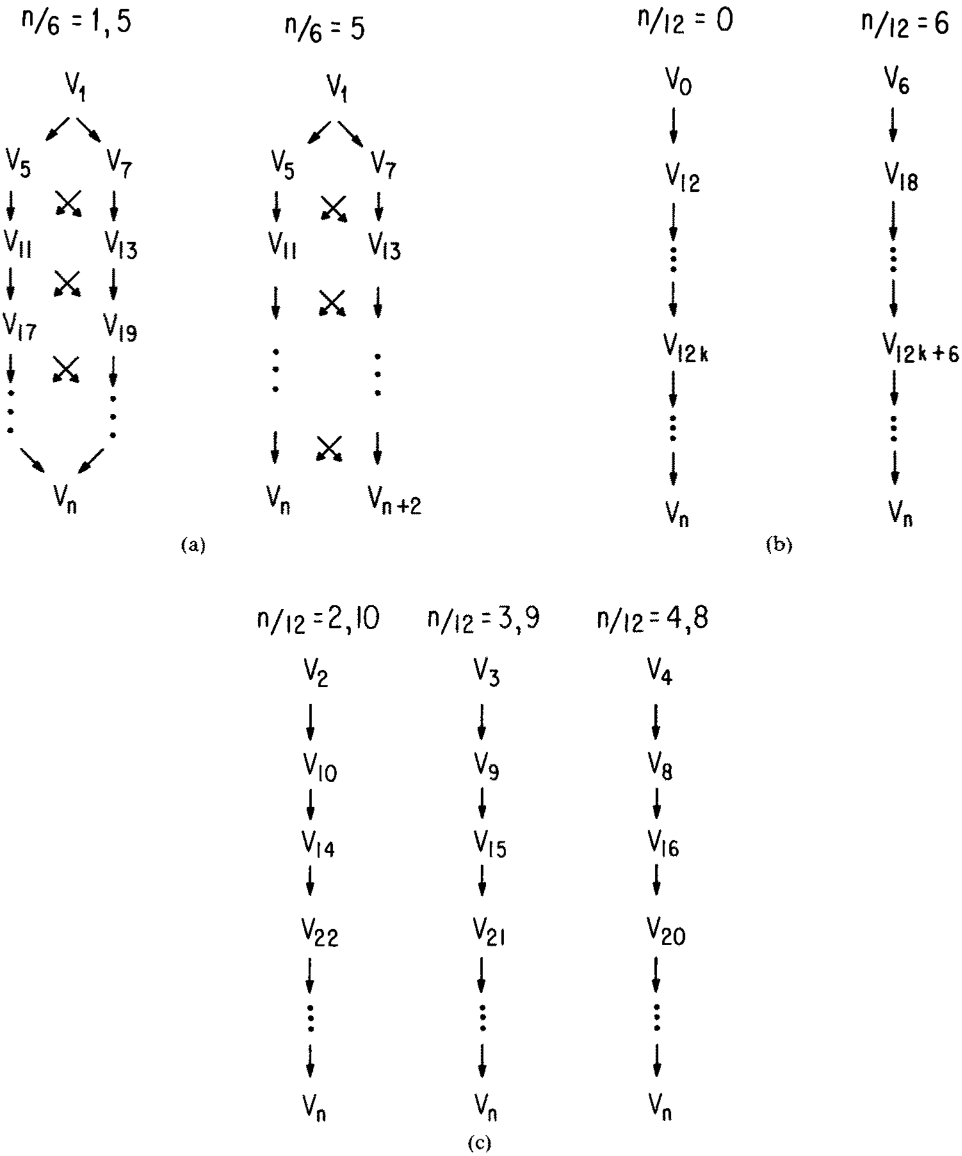

Fig. 1(a)–(c). Verma module nesting diagrams for the $\hat{c} = 26$ Virasoro algebra, $h_n = \frac{1}{24}(25 - n^2)$.

The patterns of inclusion for the Neveu-Schwarz algebra are in fig. 2a, b and for the Ramond algebra in fig. 3a, b. They are again derived from the determinant formula. For the Neveu-Schwarz Verma modules the G-parity of each tower is uniform except the tower headed by V_1 where the pattern of alternating G-parity is shown explicitly. Each Ramond Verma module splits evenly into even and odd G-parity subspaces except V_0 which has definite G-parity. The number of Verma

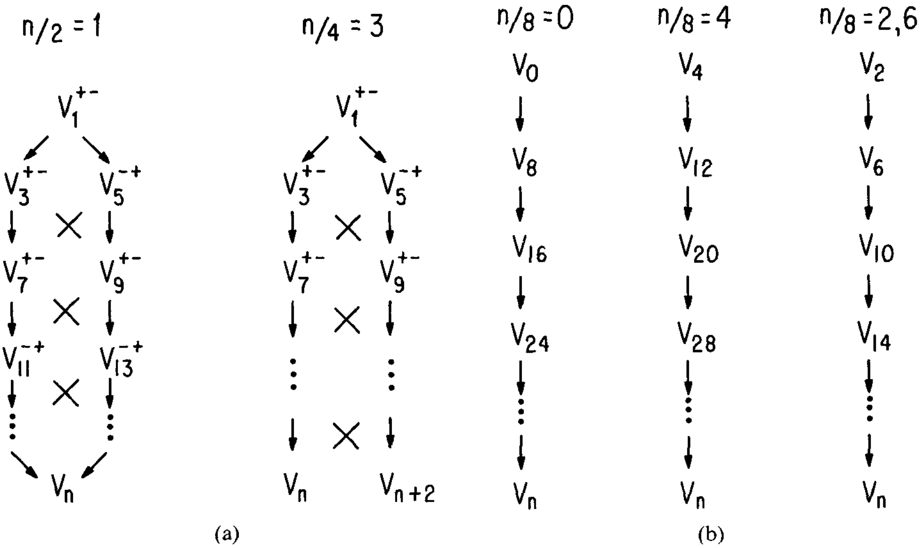

Fig. 2(a),(b). Verma module nesting diagrams for the $\hat{c} = 10$ Neveu-Schwarz algebra, $\hat{h}_n = \frac{1}{16}(9 - n^2)$.

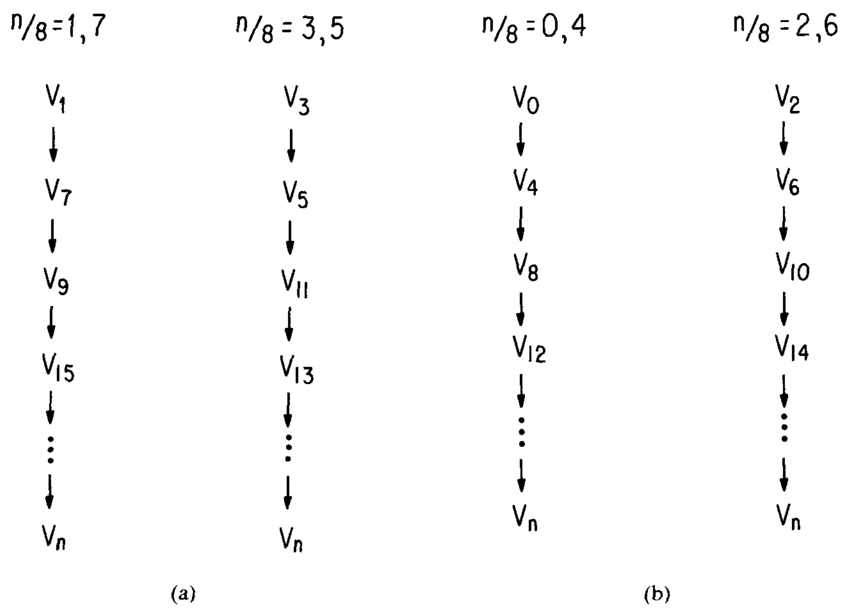

Fig. 3(a),(b). Verma module nesting diagrams for the $\hat{c} = 10$ Ramond algebra, $q_n = i\frac{1}{4}n$, $\hat{h}_n = \frac{1}{16}(10 - n^2)$.

modules V_0, counted by G-parity, is the index of Q_F as a supersymmetry generator. For even G-parity there are evident similarities between the inclusion diagrams of the Ramond and Neveu-Schwarz algebras, although the spacetime supersymmetry remains obscure.

8. Invariant operators and transverse quadratic forms

The construction of K_{tr} is patterned after the argument that Feigin and Fuks [4] used to find local operators F_{inv} which commute with the Virasoro algebra, except that Feigin and Fuks construct operators analytic in both H and c, while we work at a fixed value of c.

The Feigin-Fuks operators are written

$$F_{inv} = F(H) + \sum_{I,J} L_{-I} F^{I,J}(H) L_J,$$

$$[L_n, F_{inv}] = 0. \tag{39}$$

For fixed c, if $V_m \to V_n$ then the invariance of F_{inv} implies $F(h_m) = F(h_n)$. By the argument used in the construction of K_{tr} this is the necessary and sufficient condition that all the $F^{I,J}(H)$ be nonsingular. That is, for each of the towers in fig. 1a–c there is a constant which is the value of $F(H)$ on all of the Verma modules in the tower. These constants determine $F(H)$ up to functions which vanish on all the h_n.

The invariant operators of the physical string are generalizations of the Feigin-Fuks operators:

$$F_{inv} = F_{tr} + \sum_{I,J} L_{-I} F_{tr} \left(M_n^{-1}\right)^{I,J}(H) L_J,$$

$$[L_n, F_{inv}] = 0. \tag{40}$$

F_{tr} transverse, $[H, F_{tr}] = 0$. F_{tr} is determined on descendent fields by its action on the transverse fields. On transverse fields $F_{inv} = F_{tr} = F$. If $V_m \to V_n$ then the value of F on V_m completely determines the value on V_n. The values of F on all degenerate Verma modules are determined by its values on the six nondegenerate Verma modules at the tops of the towers: $h_0 = \frac{25}{24}$, $h_1 = 1$, $h_2 = \frac{7}{8}$, $h_3 = \frac{2}{3}$, $h_4 = \frac{3}{8}$, $h_6 = -\frac{11}{24}$. The degenerate modules are thus not observable, but there remain five unphysical poles associated with the V_n at the tops of the single towers in fig. 1b, c.

9. The string field

In the Schrödinger picture the string field ϕ is a wave function on the space of parametrized strings.

A concrete picture of the wave function is given by the elementary quantum mechanics of the harmonic oscillator. Let the canonical position be Q and the canonical momentum P. Define the destruction operator $A=\sqrt{\frac{1}{2}}\,(P+iEQ)$, satisfying $[A, A^{\dagger}]=E$. Write the hamiltonian $H=A^{\dagger}A$. In the Schrödinger picture the ground state wave function $\psi_0(Q)$ can be calculated by writing $A\psi_0=0$ and using $P=i\partial/\partial Q$:

$$\left(\frac{\partial}{\partial Q}+EQ\right)\psi_0(Q)=0\,,$$

$$\psi_0(Q)=(E/\pi)^{1/4}\mathrm{e}^{EQ^2/2}\,. \tag{41}$$

An orthogonal basis of wave functions is obtained by applying powers of the creation operator $A^{\dagger}$ to the ground state wave function.

The degrees of freedom of the open string consist of the average position q^{μ} and the normal modes of vibration $x_k^{\mu}(s)=q_k^{\mu}\sin(ks)$, $k\geqslant 1$. The q_k^{μ} are coordinates of a collection of independent harmonic oscillators of energy $E_k=k$. The string field $\phi(x)$ is a wave function of q^{μ} and of the internal coordinates q_k^{μ}.

In the internal coordinates ϕ looks like a gaussian function multiplying a sum of products of Hermite polynomials. But ϕ cannot actually be a function, because the infinite number of normal modes would cause the product of normalizing factors $(E/\pi)^{1/4}$ to diverge. What does make sense is the infinite product of factors of the form $\psi(Q)(\mathrm{d}Q)^{1/2}$, which defines a half-density on configuration space. The product of two square-integrable half-densities is an integrable density. The natural inner product on half-densities is

$$\psi_1^{\dagger}\psi_2=\int(\mathrm{d}Q)^{1/2}\psi_1(Q)^{\dagger}(\mathrm{d}Q)^{1/2}\psi_2(Q)\,. \tag{42}$$

The distinction between function and half-density is uninteresting when there are only a finite number of degrees of freedom, but when the number of degrees of freedom is infinite it no longer makes sense to regard the wave function as an ordinary function, since on infinite-dimensional configuration space there is no analog of Lebesgue measure. Without an analog of Lebesgue measure there is no way to square integrate an ordinary function. Half-densities are by definition square integrable.

The Hilbert space of the $1+1$ dimensional conformal field theory is a space of half-densities on parametrized strings. The hamiltonian H is a self-adjoint operator on half-densities.

The physical string configurations are the equivalence classes of parametrized strings under reparametrization. A string field in the transverse gauge is precisely a quantum wave function on physical string space.

The generators of reparametrization of the interval $[0, \pi]$ are the operators

$$R_k = L_k - L_{-k},$$
$$R_{-k} = -R_k = R_k^\dagger,$$
$$[R_j, R_k] = (j-k)R_{j+k} - (j+k)R_{j-k}. \tag{43}$$

R_k generates the reparametrization $s \to s + \sin(ks)$. For the closed string the infinitesimal reparametrizations form the algebra of vector fields on the circle. The generators then are the $R_n = L_n - \bar{L}_{-n}$, $-\infty < n < \infty$.

The generators R_k do not commute with the hamiltonian or even span a subspace closed under time translation. The closure is the algebra of Virasoro operators $H, R_k, [H, R_k]$. The linear combinations $L_{\pm n} = ([H, R_n] \pm nR_n)/2n$ are analogous to $P \pm iEQ$ for the harmonic oscillator. The transversality conditions $L_{+n}\hat{\phi} = 0$ are analogous to the condition $A\psi_0$ which forces $\psi_0(Q)$ to be a multiple of the ground state wave function. $L_{+n}\hat{\phi} = 0$ forces $\hat{\phi}(x)$ to behave as a modified gaussian in the directions of reparametrization. The degrees of freedom which remain form a half-density $\hat{\phi}(\hat{x})$ on the space of physical strings. The constraints are in one-to-one correspondence with the generators of reparametrization, so transversality precisely specifies the variation of the wave function in the directions of reparametrization.

The linear gauge invariance of the free field theory ensures that it describes the same field theory on physical strings whatever the choice of conformal coordinate on the world surface of the string. The choice of conformal coordinate is equivalent to choice of hamiltonian among all operators conjugate to L_0 in the Virasoro algebra. A coordinate must be chosen in order to write the theory in terms of $\phi(x)$, while in terms of the $\hat{\phi}(\hat{x})$ there are no arbitrary choices. The gauge invariance plays a different structural role in the string field theory than in ordinary gauge field theory. It is natural to write the gauge invariant gauge field theory, but there is no natural choice of gauge until a representative gauge field is chosen in the gauge equivalence class of the vacuum. In the string theory the gauge invariant formulation requires a choice among all possible equivalent hamiltonians H, but once that choice is made and the gauge invariant theory written, there is a natural choice of gauge, and the resulting field theory on physical string space is independent of the manner of writing the gauge invariant theory, independent of the choice of H, because of the gauge invariance [3]. Moreover, the space of fields on physical string space is a linear space. These considerations suggest that the string field theory should be formulated in terms of the gauge invariant variables, half-density fields on physical string space.

The correspondence between half-densities on physical strings and transverse half-densities on parametrized strings can also be seen by using the expectation values associated with a wave function to characterize it, up to a phase. The operators F of the parametrized string are the functions of $x^\mu(s)$ and $\dot{x}^\mu(s)$. The operators $\hat{F}$ of the physical string are generated by the hamiltonian and by the operators of the parametrized string which commute with the Virasoro operators. A field ϕ on the parametrized strings is determined up to a phase by its expectation values $F \to \phi^{\dagger} F \phi$. Define the wave function $\hat{\phi}$ on physical strings by the expectation values

$$\hat{\phi}^{\dagger} f(H) \hat{F} \hat{\phi} = \phi^{\dagger} f(H) \hat{F} \phi, \tag{44}$$

for $\hat{F}$ which commutes with the Virasoro operators. This gives a map from wave functions on parametrized strings to density matrices on unparametrized strings. The transverse gauge slice is a subspace of wave functions on parametrized strings which is mapped to the pure states of the unparametrized string.

Write the projection from wave functions on parametrized strings to wave functions on physical strings abstractly as

$$\hat{\phi}(\hat{x}) = \int_{\mathrm{Diff}[0,\pi]} \mathrm{d}g \mu(g) \phi(x^{g}), \tag{45}$$

where x^{g} is the string x reparametrized by g and $(\mathrm{d}g)^{1/2} \mu(g)$ is a half-density on the reparametrization group $\mathrm{Diff}[0, \pi]$. As a function of g, $\phi(x^{g})$ is actually a twisted half-density. The twist appears as the central charge $c = D$ of the Virasoro algebra. $\mu(g)$ is also a twisted half-density, with $c = -26$. This is a re-interpretation of the trace anomaly of the Faddeev-Popov ghosts of the string [10]. In the critical dimension $D = 26$ the twists cancel, the integrand of eq. (45) becomes a density in g and the integral defining $\hat{\phi}$ becomes meaningful. Presumably there is an equivalent difficulty fixing the phase of the wave function on physical strings knowing its expectation values.

The natural variables of the string field theory are the fields on physical strings. The inner product of two wave functions on physical strings is natural, so the vacuum geometry appears in the free field action only through H. The gauge invariance seems to serve only to give meaning to locality in string space. The inverse propagator $K(H)$ picks out the smooth wave functions on string space, just as on ordinary space the laplacian specifies what is a differentiable function. Through $K(H)$ the vacuum geometry gives a differentiable structure to physical string space.

10. The interaction

We will attempt to describe the interacting string field as a wave function $\hat{\phi}(\hat{x})$ on physical strings $\hat{x}$. There is always a natural inner product on half-density wave functions:

$$\hat{\phi}^{\dagger} \hat{\phi} = \int \mathrm{d}\hat{x}\, \mathrm{d}\hat{y}\, \delta(\hat{x}, \hat{y}) \hat{\phi}(\hat{x})^{\dagger} \hat{\phi}(\hat{y}). \tag{46}$$

Wave functions on string space also possess a natural trilinear form associated with the contact delta-function, the overlap integral [1,2]

$$\hat{\phi}^{\dagger}(\hat{\phi}\cdot\hat{\phi}) = \int \mathrm{d}\hat{x}\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}\,\hat{\phi}(\hat{x})^{\dagger}\delta(\hat{x},\hat{y}\cdot\hat{z})\hat{\phi}(\hat{x})\hat{\phi}(\hat{y}), \tag{47}$$

$$y\cdot z(s) = \begin{cases} y(\pi s/s_I), & 0\leqslant s\leqslant s_I \\ z(\pi(s-s_I)/(\pi-s_I)), & s_I\leqslant s\leqslant \pi. \end{cases} \tag{48}$$

The free field action is

$$\hat{\phi}^{\dagger}K(\hat{H})\hat{\phi} = \int \mathrm{d}\hat{x}\,\mathrm{d}\hat{y}\,\hat{\phi}(\hat{x})^{\dagger}\delta(\hat{x},\hat{y})K(\hat{H}_y)\hat{\phi}(\hat{y}), \tag{49}$$

where $\hat{H} = H$ is the hamiltonian on the physical fields. The interaction is the contact delta-function tempered by a form factor:

$$\hat{\phi}\hat{I}(\hat{\phi}\cdot\hat{\phi}) = \int \mathrm{d}\hat{x}\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}\,\hat{\phi}(\hat{x})^{\dagger}\delta(\hat{x},\hat{y}\cdot\hat{z})I(s_I,\hat{H}_x,\hat{H}_y,\hat{H}_z)\hat{\phi}(\hat{y})\hat{\phi}(\hat{z}). \tag{50}$$

The interaction does not depend on how the join is parametrized.

To see that the contact delta-function is independent of the vacuum geometry, rewrite it as

$$\hat{\phi}^{\dagger}(\hat{\phi}\cdot\hat{\phi}) = \int \mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{z}\,\hat{\phi}(\hat{x}_1,\hat{x}_2)^{\dagger}\delta(\hat{x}_1,\hat{y})\delta(\hat{x}_2,\hat{z})\hat{\phi}(\hat{y})\hat{\phi}(\hat{z}), \tag{51}$$

where

$$\begin{aligned} \hat{x}_1(s) &= \hat{x}(s s_I/\pi), \\ \hat{x}_2(s) &= \hat{x}(s_I + s(\pi - s_I)/\pi), \\ \hat{\phi}(\hat{x}) &= \hat{\phi}(\hat{x}_1,\hat{x}_2). \end{aligned} \tag{52}$$

For each of the four integrals in eq. (51) the integrand is a quadratic expression in half-densities and therefore is an integrable density. Thus the contact delta-function is independent of the geometry, except possibly at the interaction point s_I. In any sensible string theory the interaction point should not introduce any difficulties, since it represents only one of the infinite number of degrees of freedom of the string. Still, a careful treatment of the interaction point is needed.

In the interacting field theory, as in the free theory, the only dependence on the vacuum geometry is through the physical hamiltonian $\hat{H}$. The interaction form factor $I(s_I, \hat{H}_x, \hat{H}_y, \hat{H}_z)$ should be determined by attempting to extend the interaction to be local and gauge invariant on the space of parametrized strings. Write the interaction on parametrized string space in general normal ordered form

$$\phi^{\dagger} I_{\mathrm{tr}}(\phi \cdot \phi) = \int \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}y \, \mathrm{d}z \, \delta(x_1, y) \delta(x_2, z)$$

$$\times \phi(x)^{\dagger} \sum_{I, J, K} L_{-I}^{x} I^{I,J,K}(s_I, H_x, H_y, H_z) L_J^y L_K^z \phi(y) \phi(z) . \quad (53)$$

The L_k^x are given in terms of the L_m^y and L_n^z by the joining relation (48). The form factors $I^{I,J,K}$ are then determined from the physical form factor $I = I^{0,0,0}$ by evaluating the interaction (53) level by level in an arbitrary Verma module, and requiring it to vanish on the descendent fields. We are again interested in finding the conditions that the physical form factor must satisfy in order that I_{tr} be local.

11. On general covariance

Dependence on the vacuum geometry is now presumably isolated in the physical hamiltonian of the $1 + 1$ dimensional conformal field theory*. But the vacuum should be some particular value of the field. Therefore it is necessary to represent the vacuum as both a hamiltonian and a wave function on physical strings. The obvious wave function to associate with the hamiltonian is the ground state $\psi_0(\hat{x})$ of the conformal field theory. It satisfies the transversality conditions, so it is a wave function on physical strings. The character of the correspondence between solutions of the string field equations and hamiltonians on physical strings is not obvious, but it seems reasonable to try to extend the relationship to the general field on physical strings. The equations of motion would be the conditions for representing the hamiltonian as the middle operator in the Virasoro algebra. The simplest relationship between wave function and operator is linear. A linear map from wave functions to operators on wave functions is a trilinear form on wave functions. The natural choice is the contact delta-function.

The string field $\hat{\phi}(\hat{x})$ should now be interpreted as the fluctuation of the true string field ψ from its vacuum value ψ_0:

$$\psi = \psi_0 + \hat{\psi} = \psi_0 + Z(\hat{H}) \hat{\phi} . \quad (54)$$

The action, if it is to be completely independent of the vacuum geometry, must be

$$S = \psi^{\dagger} \psi + \psi^{\dagger}(\psi \cdot \psi) . \quad (55)$$

* The idea of representing the classical string state as a two-dimensional conformal field theory, that is, as the general nonlinear sigma model describing propagation of the string in a background geometry, has occurred to a number of people. The idea is implicit in ref. [10], especially in conjunction with those in ref. [11].

The quadratic term in the action is the natural inner product on densities. The trilinear term is the natural contact delta-function.

The equation of motion for the string field is

$$0 = \hat{\psi}^{\dagger}\psi_0 + \psi_0^{\dagger}\hat{\psi} + \hat{\psi}^{\dagger}(\psi_0 \cdot \psi_0) + \psi_0^{\dagger}(\psi_0 \cdot \hat{\psi} + \hat{\psi} \cdot \psi_0). \tag{56}$$

The hamiltonian $\hat{H}$ is determined by

$$\begin{aligned}\hat{\psi}^{\dagger}Z(H)^{-1}K(H)\hat{\psi} &= \hat{\psi}^{\dagger}\left(\frac{\partial^2 S}{\partial \psi^2}\right)\hat{\psi} \\ &= \hat{\psi}^{\dagger}\hat{\psi} + \psi_0^{\dagger}(\hat{\psi} \cdot \hat{\psi}) + \hat{\psi}^{\dagger}(\hat{\psi} \cdot \psi_0 + \psi_0 \cdot \hat{\psi}).\end{aligned} \tag{57}$$

The interaction is simply $\hat{\psi}^{\dagger}(\hat{\psi} \cdot \hat{\psi})$. Once the interaction form factor is constructed explicitly it can be checked whether this description of the vacuum is consistent with the actual interaction.

The contact relationship provides a map between string fields and some class of quantum hamiltonians on physical string space. A solution of the field theory equations of motion should correspond to a hamiltonian which can be lifted to the space of parametrized strings and there generate the Virasoro algebra from the reparametrization algebra. Then solutions of the field theory equations of motion would correspond to $1+1$ dimensional conformal field theories describing the parametrized world surface of the string. This would explain the equivalence between the classical equations of motion of the string and the equation for zero β-function in the nonlinear sigma models which describe self-consistent interactions of string and background geometry [11]. It might even be possible to interpret the equation of motion (56) for the string field ψ as the fixed point equation of a renormalization group transformation on $1+1$ dimensional reparametrization invariant quantum field theories.

If the string theory can be written in the form of eq. (55) then the theory has a very large invariance group, the group of maps of string space to itself which preserve the contact relationship. Green [12] has suggested this group as the underlying invariance group of string theory.

Even if a string field theory can be written which is free of dependence on the vacuum geometry, there remains dependence on the vacuum topology of string space. Some abstraction of the string field should exist in terms of which the field equations are purely algebraic. Such an abstraction might be found by treating the space of wave functions as an abstract separable Hilbert space. The contact interaction of a string space is some trilinear functional on Hilbert space which gives a multiplication rule turning Hilbert space into an algebra. This algebra is not necessarily associative, but duality requires that the interaction satisfy some quadratic identity. This is the same quadratic identity satisfied by the operator product coefficients of conformal field theories [13]. If this identity can be written in a form independent of the vacuum geometry, and if enough physical conditions can be put on the interaction it might be possible to show that all interactions are unitarily equivalent. Then the ground state of the string field would determine both the geometry and the topology of spacetime.

12. Conclusions

There are more problems than conclusions. The most urgent is checking whether the gauge invariance can be linear by constructing the interaction assuming linear gauge invariance and locality. It must be checked that these conditions can be satisfied, that the unphysical poles decouple to leave the Veneziano model, and that the vacuum geometry appears in the action only through the hamiltonian of the reparametrization invariant $1+1$ dimensional conformal field theory. If the construction of the interaction succeeds then the conjectures of the previous section can be tested. Another obvious problem is to find a manifestly supersymmetric field theory of the supersymmetric string. So far, the covariant theory of the supersymmetric string has been expressed in the first-quantized BRST formalism [14]. It should be possible to find more applications for the structure diagrams of the Verma modules. In particular there is a fascinating duality between the representations of the Virasoro algebra at c, h and the representations at $26-c, 1-h$ [8,9]. For the superconformal algebras the analogous duality takes $\hat{c}$ to $10-\hat{c}$, and h to $\frac{1}{2}-h$ (Neveu-Schwarz) or $\frac{10}{16}-h$ (Ramond). Finally, it is intriguing to think of this string field theory, if it makes sense, as a kind of nonlinear Schrödinger model and to wonder if the rich structure, in particular the correspondence of field and hamiltonian, might provide means to solve it exactly.

It seems remarkable that gauge invariance and locality should have so much power in string field theory. Perhaps the power of the bootstrap method in this formulation of string theory is compensation for the lack of a striking geometric interpretation.

This work was inspired by T. Banks' talk at the Argonne–Chicago Conference on Anomalies and Strings, March 28–30, 1985. I thank him for interesting conversations. I am grateful to A.B. Zamolodchikov for introducing me to the degenerate representations of the Virasoro algebra, and to him and S.H. Shenker for many discussions of the Virasoro algebra, conformal field theory and string theory. I thank J. Cohn, M. Green, C. Hull, V. Kac, A.M. Polyakov, Z. Qiu, A. Rocha-Caridi, W. Siegel, I. Singer, P. Windey and E. Witten for helpful and enlightening conversations.

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