SEMINAR 6

INTRODUCTION TO POLYAKOV'S STRING THEORY

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1. Polyakov's description of the string

A one-dimensional string in its classical motion sweeps out a surface in space–time. This world-surface, in $\mathscr{D}$ dimensions, is described by a function $x^{\mu}(\xi)$, $\mu = 1 \ldots \mathscr{D}$ depending on two real parameters $\xi = (\xi^1, \xi^2)$. We build a relativistic quantum theory of the string by making an integral

$$\int_{\text{surfaces}} \mathscr{D}x\, \mathrm{e}^{-A(x)}, \tag{1.1}$$

over all space–time trajectories. The main requirement is that the integral be invariant under reparametrization $\xi \to \eta(\xi)$; the physical properties of the string should not depend on the parameter labels we assign to the space–time events in its history.

Polyakov's proposal is to write the integral over surfaces using a Riemannian metric

$$\mathrm{d}s^2 = g_{ab}(\xi)\mathrm{d}\xi^a \mathrm{d}\xi^b$$

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J.-B. Zuber and R. Stora, eds.
Les Houches, Session XXXIX, 1982—Développements Récents en Théorie des Champs et Mécanique Statistique/Recent Advances in Field Theory and Statistical Mechanics

on the parameter space as an auxiliary variable:

$$\int_{\text{metrics}} \mathscr{D}g\, e^{-A(g)} \int_{\text{surfaces}} \mathscr{D}_g x\, e^{-A(g,x)}, \tag{1.2}$$

$$A(g,x) = \frac{1}{2}\int \frac{d^2\xi}{2\pi} \sqrt{g(\xi)}\, g^{ab}(\xi)\partial_a x^\mu \partial_b x_\mu, \tag{1.3}$$

$$A(g) = \mu_0^2 \int d^2\xi \sqrt{g(\xi)}. \tag{1.4}$$

Here

$$d^2\xi \sqrt{g(\xi)} = d^2\xi \sqrt{\det(g_{ab}(\xi))}$$

is the covariant area element, $g^{ab}(\xi)g_{bc}(\xi) = \delta^a_c$ and $\partial_a = \partial/\partial\xi^a$.

The metric $g_{ab}(\xi)$ defines the formal volume element $\mathscr{D}_g x$ through the inner product

$$(\delta x, \delta x)_g = \int d^2\xi \sqrt{g}\, \delta x^\mu(\xi)\, \delta x_\mu(\xi) \tag{1.5}$$

on infinitesimal variations of $x^\mu(\xi)$, in the same way that the finite dimensional volume element $d^2\xi\sqrt{g}$ is determined by the inner product $g_{ab}(\xi)\delta\xi^a\delta\xi^b$ on variations of ξ. The volume element $\mathscr{D}g$ on the space of metrics is similarly determined by the inner product

$$(\delta g, \delta g)_g = \int d^2\xi \sqrt{g}\, g^{ab}(\xi) g^{cd}(\xi)\, \delta g_{ac}(\xi)\, \delta g_{bd}(\xi). \tag{1.6}$$

The action and volume elements are covariantly defined, so the integral (1.2) over surfaces is at least formally invariant under reparametrization.

The particular choices (1.3–6) for the volume elements and the action can be singled out by their symmetries and by their scaling properties *in parameter space*. Everything is included which is: (i) relativistically invariant, (ii) covariant under reparametrization, (iii) polynomial in the parameter derivatives and (iv) of naive scaling dimension $\leqq 0$. Nothing else would be relevant to the continuum limit *in parameter space*.

The functional integral (1.2) characterizes the string's propagation locally in the parameters. To describe a particular kind of string (open, say, or closed) undergoing a particular process (propagation, interaction) it is necessary to specify the boundaries and other global topological properties of the surface and then to include boundary terms in the functional integral. We will not do that here, limiting attention to the local structure of the string. The technical apparatus we use can be adapted to account for the boundary and other topological effects.

It is helpful, on the other hand, to have in mind an example. This will be the complex upper half plane

$$H = \{e^{\tau + i\sigma}: \quad -\infty < \tau < \infty, \quad 0 < \sigma < \pi\}$$

which is the surface used to describe the open string. All other surfaces of interest are obtained from the upper half plane by identifying points.

We want to use the integral (1.2) over surfaces to calculate expectation values for the string to occupy an arbitrary set of points in space–time:

$$\tilde{G}(x_1, \ldots, x_n) = \left\langle \prod_{k=1}^{n} \int d^2\xi_k \sqrt{g(\xi_k)}\, \delta(x(\xi_k) - x_k) \right\rangle, \tag{1.7}$$

whose Fourier transform is

$$G(p_1, \ldots, p_n) = \left\langle \prod_{k=1}^{n} \int d^2\xi_k \sqrt{g(\xi_k)}\, e^{ip_k \cdot x(\xi_k)} \right\rangle. \tag{1.8}$$

The covariant integration over $\xi_1, \ldots, \xi_k$ ensures that the expectation values are of reparametrization invariant quantities. We learn spectral information from the space–time asymptotics of $\tilde{G}(x_1 \ldots x_n)$. Its Fourier transform has poles in the squared momenta $(p_{k_i} + \ldots + p_{k_m})^2$ at the physical masses of the particle states of the string; the residues at the poles are the scattering amplitudes. Again, we will be interested here only in local properties of the operators $\sqrt{g(\xi)}\, e^{ip \cdot x(\xi)}$.

2. Gauge fixing

The functional integral (1.2) contains an overall infinite factor due to invariance under the local gauge group of reparametrizations. This factor, which drops out of expectation values, is effectively removed by restricting the integral to a gauge slice: a subspace of metrics which meets each orbit of the local gauge group exactly once.

We will use without proof a basic fact of two-dimensional geometry: given any metric $g_{ab}(\xi)$ there is always a reparametrization $\xi \to \eta(\xi)$ which at least locally makes the metric conformally Euclidean:

$$g_{ab}(\xi) d\xi^a d\xi^b = \rho(\eta) \delta_{ab} d\eta^a d\eta^b.$$

Moreover, any further reparametrization $\eta \to \eta'(\eta)$ which preserves the conformally Euclidean form of the metric cannot be local, where local means equaling the identity $\eta' = \eta$ outside some arbitrarily small neighborhood.

Thus a good gauge slice consists of all the metrics which are conformal to a given Euclidean metric $\delta_{ab} d\eta^a d\eta^b$. A more general gauge slice is the conformal class $[\hat{g}]$ of metrics conformal to some not necessarily Euclidean metric $\hat{g}_{ab}$:

$$[\hat{g}] = \{g_{ab}(\xi) = e^{\phi(\xi)}\hat{g}_{ab}(\xi)\}. \tag{2.1}$$

All of these conformal classes are locally equivalent under reparametrization, but global topology usually prevents the existence of an everywhere Euclidean metric; then the gauge slice must be a non-Euclidean conformal class. Moreover, one conformal class is usually not enough to make a global gauge slice: also to be integrated over are a finite number of variables $m_1 \ldots m_k$, called the moduli, which parametrize the inequivalent conformal classes $[\hat{g}(m_1 \ldots m_k)]$. The upper half plane has no moduli. We leave aside the global problem of integrating over the moduli.

The integration over all metrics is now replaced with an integral over some conformal class:

$$\int_{\text{all metrics}} \mathscr{D}g = \int_{[\hat{g}]} \mathscr{D}\phi\, J(g), \tag{2.2}$$

where $J(g)$ is a Faddeev–Popov determinant needed to take account of the variable volume of the orbits of the reparametrization group.

3. Complex tensor calculus

It will be convenient to continue the investigation of the integral over surfaces using mathematical language based on a single complex parameter in place of two real ones. Once we have singled out a conformal class $[\hat{g}]$ of metrics on parameter space, we can limit ourselves to a special class of local parametrizations, the conformal parametrizations: those in which the metric is conformally Euclidean. Then we replace the two real parameters (ξ^1, ξ^2) with $z = \xi^1 + i\xi^2$. A metric in $[\hat{g}]$ now takes the form

$$g_{ab}(\xi)\, d\xi^a\, d\xi^b = \rho(z, \bar{z})|dz|^2. \tag{3.1}$$

If $\xi \to \eta(\xi)$ is any conformal reparametrization then $w = \eta^1 + i\eta^2$ is either an analytic function of z, $\partial_{\bar{z}} w = 0$, or an anti-analytic function, $\partial_z w = 0$. The complex derivatives are

$$\partial_z = \tfrac{1}{2}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} = \tfrac{1}{2}(\partial_1 + i\partial_2). \tag{3.2}$$

Conversely, if $w = \eta^1 + i\eta^2$ is (anti-) analytic in z then $\xi \to \eta(\xi)$ is a conformal reparametrization. Thus choosing a conformal class $[\hat{g}]$ of metrics is exactly equivalent to choosing a collection of local complex parametrizations, all (anti-) analytically related to each other.

Given such an analytic structure we can define classes of one component tensor fields, of the form

$$t_{z..\bar{z}..}(\mathrm{d}z)^{n+m}(\mathrm{d}\bar{z})^{m} \tag{3.3}$$

for arbitrary integers n and m. Clearly this form is not changed under conformal reparametrization, if anti-analytic reparametrization is accompanied by complex conjugation. The indices z and $\bar{z}$ in eq. (3.3) each range over only one value, since there is only one complex parameter, but they are useful to write in order to keep tract of transformation properties. A tensor with $-n$ subscripts is written with $+n$ supercirpts: e.g., $t^{zz\bar{z}}(\mathrm{d}z)^{-2}(\mathrm{d}\bar{z})^{-1}$.

It is not hard to verify that any real tensor field on the parameter surface can be written as a linear combination of complex tensors of the form (3.3). In particular, a metric $g_{ab} = \rho\delta_{ab}$ in $[\hat{g}]$ is written

$$\mathrm{d}z^2 = g_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} + g_{\bar{z}z}\,\mathrm{d}\bar{z}\,\mathrm{d}z, \tag{3.4}$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \tfrac{1}{2}\rho. \tag{3.5}$$

The other possible complex components of a real symmetric tensor vanish:

$$g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{3.6}$$

The components of the inverse metric are

$$g^{z\bar{z}} = g^{\bar{z}z} = 2\rho^{-1}, \tag{3.7}$$

$$g^{zz} = g^{\bar{z}\bar{z}} = 0. \tag{3.8}$$

A metric is useful for contracting z and $\bar{z}$ indices in pairs, so that, given a particular metric, any complex tensor of the form (3.3) can be rewritten in the more special form

$$t_{z..}(\mathrm{d}z)^{n} \tag{3.9}$$

with

$$t_{z..} = (g^{z\bar{z}})^{m}\,t_{z..\bar{z}..}. \tag{3.10}$$

We will say that tensors of the form (3.9) have rank n.

This complex tensor language makes it very easy to describe the covariant derivatives for the metric $g_{z\bar{z}}$. In the $\bar{z}$ direction the covariant derivative is just the partial derivative. Clearly

$$\nabla^{z}t_{z..} = g^{z\bar{z}}\partial_{\bar{z}}t_{z..} = g^{z\bar{z}}\nabla_{\bar{z}}t_{z..} \tag{3.11}$$

transforms as it stands as a tensor of rank $n-1$ under conformal reparametrization. The covariant derivative in the z direction is derived from that in the $\bar{z}$ direction by a series of complex conjugations, raisings and lowerings:

$$\begin{aligned}\nabla_z t_{z..} &= (g_{z\bar{z}})^n \partial_z[(g^{z\bar{z}})^n t_{z..}] \\ &= (\partial_z - n\partial_z \log\rho) t_{z..}. \end{aligned} \tag{3.12}$$

The covariant definition guarantees that $\nabla_z t_{z..}$ transforms as a tensor of rank $n+1$.

The curvature tensor is the commutator of covariant derivatives:

$$[\nabla^z, \nabla_z] t_{z..} = \frac{n}{2} R t_{z..}. \tag{3.13}$$

All tensors are one component objects, so all curvature is scalar. From eqs. (3.11) and (3.12) we calculate

$$R = \rho^{-1}(-4\partial_z\partial_{\bar{z}} \log\rho). \tag{3.14}$$

If we write the metric g_{ab} in terms of a background metric $\hat{g}_{ab}$, $g_{ab} = \mathrm{e}^{\phi}\hat{g}_{ab}$, then we can relate the covariant derivatives and curvatures of the two metrics:

$$\nabla^z = \mathrm{e}^{-\phi}\hat{\nabla}^z, \tag{3.15}$$

$$\nabla_z = \hat{\nabla}_z - n\partial_z\phi, \tag{3.16}$$

$$R = \mathrm{e}^{-\phi}(-2\hat{\nabla}^z\partial_z\phi + \hat{R}). \tag{3.17}$$

4. Faddeev–Popov fields

The determinant arising from the gauge fixing in eq. (2.2) is to be represented as a Grassmannian integral over anti-commuting Faddeev–Popov fields. We need to find which operator to take the determinant of.

An infinitesimal reparametrization is a vector field, or rank -1 tensor, $\delta z = v^z(z, \bar{z})$. It induces a variation of the metric by

$$(g_{ab} + \delta g_{ab})\mathrm{d}\xi^a \mathrm{d}\xi^b = g_{ab}(\xi + \delta\xi)\mathrm{d}(\xi^a + \delta\xi^a)\mathrm{d}(\xi^b + \delta\xi^b). \tag{4.1}$$

Expanding to first order in the variations, we find

$$\delta g_{z\bar{z}} = (\nabla_z v^z + \nabla_{\bar{z}} v^{\bar{z}}) g_{z\bar{z}}, \tag{4.2}$$

$$\delta g_{zz} = 2\nabla_z v_z. \tag{4.3}$$

The variations $\delta g_{z\bar{z}}$ and δg_{zz} are clearly orthogonal in the inner product (1.6).

Now we have two sets of variables to describe the metrics infinitesimally close to $g_{ab} = e^{\phi}\hat{g}_{ab}$: (1) arbitrary orthogonal variations $\delta g_{z\bar{z}}$, δg_{zz}; and (2) variations $\delta\phi$ along the gauge slice and infinitesimal reparametrizations v^z. In the first set of variables the volume element $\mathscr{D}g$ is

$$\mathscr{D}g = \mathscr{D}g_{z\bar{z}}\mathscr{D}g_{zz}\mathscr{D}g_{\bar{z}\bar{z}}. \tag{4.4}$$

In the second variables this becomes

$$\mathscr{D}g = \mathscr{D}_g v^z \mathscr{D}_g v^{\bar{z}} \mathscr{D}_g \phi \det \frac{\partial(g_{z\bar{z}}, g_{zz}, g_{\bar{z}\bar{z}})}{\partial(\phi, v^z, v^{\bar{z}})}, \tag{4.5}$$

where $\mathscr{D}_g\phi$ is based on the inner product

$$(\delta\phi, \delta\phi)_g = \int d^2\xi \sqrt{g}\, \delta\phi(\xi)\delta\phi(\xi) \tag{4.6}$$

and $\mathscr{D}_g v^z \mathscr{D}_g v^{\bar{z}}$ is based on

$$(v, v)_g = \int d^2\xi \sqrt{g}\, g_{ab} v^a v^b. \tag{4.7}$$

We can simplify (4.5) by noting that

$$\frac{\partial}{\partial\phi} g_{zz} = 0, \quad \frac{\partial}{\partial v^z} g_{zz} = 0$$

and that $(\partial/\partial\phi)g_{z\bar{z}}$ is essentially the identity operator. We get

$$\mathscr{D}g = \mathscr{D}_g v^z \mathscr{D}_g v^{\bar{z}} \mathscr{D}_g \phi |\det|^2 \left(\frac{\partial g_{\bar{z}\bar{z}}}{\partial v^z}\right). \tag{4.8}$$

Now we would like to drop the factor $\mathscr{D}_g v^z \mathscr{D}_g v^{\bar{z}}$ as being the volume element on the reparametrization group. But this is slightly problematic, since the inner product (4.7) which defines this volume element retains some dependence on ϕ. This implies that the volume of the gauge group is not a constant along the gauge slice, in which case it does not simply factor out of the functional integral. Assuming that the ϕ dependence of $\mathscr{D}_g v^z \mathscr{D}_g v^{\bar{z}}$ can be absorbed into $\mathscr{D}_g\phi$ we can go ahead to write the Faddeev–Popov determinant:

$$J(g) = \det(\nabla^z)\det(\nabla^{\bar{z}}), \tag{4.9}$$

where ∇^z acts from rank -1 tensors to rank -2 tensors and $\nabla^{\bar{z}}$ is its complex conjugate.

We introduce Faddeev–Popov ghost fields: $c^z(z, \bar{z})$ and its complex conjugate, to represent the infinitesimal reparametrizations; and $b_{zz}(z, \bar{z})$ and its complex conjugate, to represent infinitesimal variations perpendicular to the gauge slice. These fields anti-commute. The Faddeev–Popov determinant is

$$J(g) = \int \mathscr{D}_g c \mathscr{D}_g b \, \mathrm{e}^{-A(g,b,c)} \tag{4.10}$$

$$A(g, b, c) = \int \frac{\mathrm{d}^2\xi}{2\pi} \sqrt{g} (b_{zz} \nabla^z c^z + \text{c.c.}), \tag{4.11}$$

with Grassmannian volume elements $\mathscr{D}_g c = \mathscr{D}_g c^z \mathscr{D}_g c^{\bar{z}}$ and $\mathscr{D}_g b = \mathscr{D}_g b_{zz} \mathscr{D}_g b_{\bar{z}\bar{z}}$ derived from the inner products

$$(c, c)_g = \int \mathrm{d}^2\xi \sqrt{g} g_{z\bar{z}} c^z c^{\bar{z}}, \tag{4.12}$$

$$(b, b)_g = \int \mathrm{d}^2\xi \sqrt{g} (g^{z\bar{z}})^2 b_{zz} b_{\bar{z}\bar{z}}. \tag{4.13}$$

5. The free field integrals

The integral over surfaces now takes the form

$$\int_{\text{conformal factors}} \mathscr{D}_g \phi \, \mathrm{e}^{-A(g)} \int_{\text{F.P. ghosts}} \mathscr{D}_g b \mathscr{D}_g c \, \mathrm{e}^{-A(g,b,c)} \cdot \int_{\text{surfaces}} \mathscr{D}_g x \, \mathrm{e}^{-A(g,x)}. \tag{5.1}$$

The actions ((1.3) and (4.11)) for x^μ, b_{zz} and c^z are quadratic in the fields, so these are free fields on the parameter space. Moreover, their actions are conformally invariant:

$$A(\mathrm{e}^\phi \hat{g}, x) = A(\hat{g}, x),$$
$$A(\mathrm{e}^\phi \hat{g}, b, c) = A(\hat{g}, b, c). \tag{5.2}$$

The corresponding field equations

$$\partial_{\bar{z}} \partial_z x^\mu = 0, \tag{5.3}$$

$$\partial_{\bar{z}} c^z = 0, \quad \partial_{\bar{z}} b_{zz} = 0, \tag{5.4}$$

are consequently independent of ϕ. Thus the conformal field $\phi(\xi)$ decouples from the excitations of the free fields. But it does not necessarily decouple from the quantum fluctuations in the ground state of the free fields, because the volume elements $\mathscr{D}_g x \mathscr{D}_g b \mathscr{D}_g c$ are not independent of ϕ.

Define $S(g)$ to be the ground state action of the free fields in the metric g_{ab}:

$$e^{-S(g)} = e^{-A(g)} \int \mathscr{D}_g b \mathscr{D}_g c \int \mathscr{D}_g x \, e^{-A(g,b,c)-A(g,x)}. \tag{5.5}$$

Then eq. (5.1) can be rewritten in the form

$$\left[e^{-S(\hat{g})} \int \mathscr{D}_g \phi \, e^{-S_{\text{eff}}(\hat{g},\phi)} \right] \left[e^{S(\hat{g})} \int \mathscr{D}_{\hat{g}} b \mathscr{D}_{\hat{g}} c \mathscr{D}_{\hat{g}} x \, e^{-A(\hat{g},b,c)-A(\hat{g},x)} \right], \tag{5.6}$$

$$S_{\text{eff}}(\hat{g},\phi) = S(e^{\phi}\hat{g}) - S(\hat{g}), \tag{5.7}$$

representing the integral over surfaces as three completely decoupled field theories on the parameter space, in a common background metric $\hat{g}_{ab}(\xi)$.

The strategy now is to find an explicit formula for $S_{\text{eff}}(\hat{g},\phi)$ by making a systematic study of how the free field integrals over x^{μ}, b_{zz} and c^z depend on the classical metric g_{ab}.

We use the standard machinery of functional integration. Introduce a source $\chi_{\mu}(\xi)$ for x^{μ} and anti-commuting sources γ_z for c^z and β^{zz} for b_{zz}:

$$(\chi, x) = \int d^2 z \sqrt{g} \chi_{\mu} x^{\mu}, \tag{5.9}$$

$$(\beta, b) = \int d^2 z \sqrt{g} (\beta^{zz} b_{zz} + \text{c.c.}), \tag{5.9}$$

$$(\gamma, c) = \int d^2 z \sqrt{g} (\gamma_z c^z + \text{c.c.}). \tag{5.10}$$

The generating functional for connected diagrams is

$$e^{W} = \int \mathscr{D}_g b \mathscr{D}_g c \mathscr{D}_g x \, e^{-A(g,b,c)-A(g,x)+(\chi,x)+(\beta,b)+(\gamma,c)}. \tag{5.11}$$

Explicitly,

$$W(g,\chi,\beta,\gamma) = -S(g) + \iint \frac{d^2 z}{2\pi} \sqrt{g(z)} \frac{d^2 w}{2\pi} \sqrt{g(w)} \times \{\tfrac{1}{2}\chi_{\mu}(z)\chi^{\nu}(w)\langle x^{\mu}(z) x_{\nu}(w)\rangle - \gamma_z \beta^{ww} \langle c^z b_{ww}\rangle\}, \tag{5.12}$$

where the Green's functions satisfy

$$\frac{1}{2\pi}(-4\partial_z\partial_{\bar{z}})\langle x^\mu(z)x_\nu(w)\rangle = \delta^\mu_\nu\delta^2(z-w), \tag{5.13}$$

$$\frac{1}{2\pi}(2\partial_{\bar{z}})\langle c^z b_{ww}\rangle = \delta^2(z-w). \tag{5.14}$$

We will only need to know the singular parts at short distance:

$$\langle x^\mu(z)x_\nu(w)\rangle \sim -\delta^\mu_\nu \log|z-w|, \tag{5.15}$$

$$\langle c^z b_{ww}\rangle \sim \frac{1}{z-w}. \tag{5.16}$$

The effective action $\Gamma(g, x, b, c)$ is obtained by the Legendre transform

$$\Gamma + W = (\chi, x) + (\beta, b) + (\gamma, c), \tag{5.17}$$

$$x^\mu = \frac{2\pi}{\sqrt{g}}\frac{\delta W}{\delta\chi_\mu}, \quad b_{zz} = \frac{2\pi}{\sqrt{g}}\frac{\delta W}{\delta\beta^{zz}}, \quad c^z = \frac{2\pi}{\sqrt{g}}\frac{\delta W}{\delta\gamma_z}. \tag{5.18}$$

Note that we are relying on context to distinguish the effective fields x^μ, b_{zz} and c^z of eq. (5.18) from the integration variables x^μ, b_{zz} and c^z of eq. (5.11).

Since the fields are free, we just get back as effective action the classical action plus the ground state contribution:

$$\Gamma(g, x, b, c) = S(g) + A(g, b, c) + A(g, x). \tag{5.19}$$

The inverse Legendre transform is

$$\chi_\mu = \frac{2\pi}{\sqrt{g}}\frac{\delta\Gamma}{\delta x^\mu} = -2\nabla^z\nabla_z x^\mu, \tag{5.20}$$

$$\beta^{zz} = \frac{-2\pi}{\sqrt{g}}\frac{\delta\Gamma}{\delta b_{zz}} = -\nabla^z c^z, \tag{5.21}$$

$$\gamma_z = \frac{-2\pi}{\sqrt{g}}\frac{\delta\Gamma}{\delta c^z} = -\nabla^z b_{zz}. \tag{5.22}$$

6. Renormalization of the free field integrals

The only undefined step in the integration over x^μ, b_{zz} and c^z is the renormalization of $S(g)$. We will indicate here that this can be done in a reparametrization invariant way, but then we will actually calculate the renormalized $S(g)$ directly, using only scaling arguments and reparametrization invariance, independent of any particular renormalization scheme.

One way to renormalize $S(g)$ while maintaining reparametrization invariance is to introduce a small distance cutoff ε in parameter space, using the metric $g_{ab}(\xi)$ to define what is meant by the distance ε. The standard power counting argument for two-dimensional field theory gives that the divergent part of $S(g)$ takes the form

$$S(g)_{\text{div}} = \int \mathrm{d}^2\xi \sqrt{g}(\varepsilon^{-2}A_{-2}+\varepsilon^{-1}A_{-1}+\log\varepsilon A_0), \tag{6.1}$$

where $\sqrt{g}A_k$ is a local expression in the metric, of dimension k. The dimension of an expression A is d if $A \to \alpha^d A$ when the metric is scaled by $g_{ab} \to \alpha^{-2}g_{ab}$. By the reparametrization invariance of the cutoff, the A_k must be covariant scalars. The only such expressions are $A_{-2} = c_1$, $A_{-1} = 0$, $A_0 = c_2 R$ with $c_{1,2}$ constants. Thus $S(g)$ is renormalized by including in the bare action $A(g)$ eq. (1.4) for the metric a local cutoff dependent counter-term

$$-\int \mathrm{d}^2\xi \sqrt{g}(\varepsilon^{-2}c_1+c_2\log\varepsilon R) \tag{6.2}$$

and taking the limit $\varepsilon \to 0$. Note that the term $\sqrt{g}R$ in $A(g)$ is a total divergence (see eq. (3.14)), so plays no role in the local structure.

7. Variational formulas

In order to write Ward identities expressing the reparametrization invariance of the effective action (5.19) we will need some variational formulas. First we list them, then give derivations.

Suppose the metric $g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}+\text{c.c.}$ is varied by

$$\delta g = (\delta\phi g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}+\delta g_{zz}\mathrm{d}z\mathrm{d}z)+\text{c.c.}. \tag{7.1}$$

The corresponding variation of covariant derivatives on rank n tensors and of scalar curvature will be

$$\delta\nabla^z = -\delta\phi\nabla^z+\tfrac{1}{2}\delta g^{zz}\nabla_z+\frac{n}{2}\nabla_z(\delta g^{zz}), \tag{7.2}$$

$$\delta\nabla_z = -n\nabla_z(\delta\phi)-\tfrac{1}{2}\delta g_{zz}\nabla^z+\frac{n}{2}\nabla^z(\delta g_{zz}), \tag{7.3}$$

$$\delta R = (-2\nabla^z\nabla_z - R)\delta\phi + \nabla^z\nabla^z\delta g_{zz} - \nabla_z\nabla_z\delta g^{zz}, \tag{7.4}$$

where

$$\delta g^{zz} = -(g^{z\bar{z}})^2\delta g_{\bar{z}\bar{z}} \tag{7.5}$$

varies the inverse metric. These formulas apply to an arbitrary variation of the metric.

Next consider variations induced by an infinitesimal reparametrization

$$\delta z = v^z(z, \bar{z}). \tag{7.6}$$

The variation of the metric will be eq. (7.1) with

$$\delta\phi = \nabla_z v^z + \nabla^z v_z, \tag{7.7}$$

$$\delta g_{zz} = 2\nabla_z v_z, \quad \delta g^{zz} = -2\nabla^z v^z. \tag{7.8}$$

A rank n tensor changes by

$$\delta t_{z..} = v^z\nabla_z t_{z..} + v^{\bar{z}}\nabla_{\bar{z}} t_{z..} + n\nabla_z v^z t_{z..}. \tag{7.9}$$

In particular,

$$\delta x^\mu = v^z\nabla_z x^\mu + v^{\bar{z}}\nabla_{\bar{z}} x^\mu, \tag{7.10}$$

$$\delta c^z = v^z\nabla_z c^z + v^{\bar{z}}\nabla_{\bar{z}} c^z - \nabla_z v^z c^z, \tag{7.11}$$

$$\delta b_{zz} = v^z\nabla_z b_{zz} + v^{\bar{z}}\nabla_{\bar{z}} b_{zz} + 2\nabla_z v^z b_{zz}. \tag{7.12}$$

The remainder of this section contains the derivations.

We face an obstacle to studying a general variation (7.1) of the metric using the language of complex tensors because the change $\delta g_{zz} \neq 0$ of the conformal class of the metric changes what it means to be a complex tensor field of rank n. We need, therefore, a linear transformation we can use to convert rank n tensor fields for the conformal class of the new metric back into rank n tensor fields for the conformal class of the original metric.

First we look for a complex conformal parameter $z + \delta z(z, \bar{z})$ for the new metric (7.1). It must satisfy, for some ρ,

$$\rho(z, \bar{z})|\mathrm{d}(z + \delta z)|^2 = ((1 + \delta\phi)g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} + \delta g_{zz}\mathrm{d}z\mathrm{d}z) + \text{c.c.}, \tag{7.13}$$

which is equivalent to the condition

$$\partial_{\bar{z}}(\delta z) = -\tfrac{1}{2}g_{z\bar{z}}\delta g^{zz}. \tag{7.14}$$

Now we transform a tensor $t_{z..}(\mathrm{d}z)^n$, of rank n for the original conformal class, to

$$(1 - n\partial_z\delta z)t_{z..}[\mathrm{d}(z + \delta z)]^n. \tag{7.15}$$

Using eq. (7.14) we can write this as the infinitesimal transformation

$$\delta t_{z..}(\mathrm{d}z)^n = -\frac{n}{2}(g_{z\bar{z}}\delta g^{zz})t_{z..}(\mathrm{d}z)^{n-1}\mathrm{d}\bar{z}. \tag{7.16}$$

Note that the transformation is covariantly defined, by eq. (7.16), and gives a rank n tensor for the conformal class of the new metric, by eq. (7.15). Also note that the scalars, the rank 0 tensors, are unaffected.

Now we are in a position to describe how tensor fields vary under a reparametrization (7.6). A rank n tensor field $t_{z..}(\mathrm{d}z)^n$ goes to

$$t_{z..}(z+\delta z, \bar{z}+\delta\bar{z})[\mathrm{d}(z+\delta z)]^n, \tag{7.17}$$

i.e.

$$\begin{aligned}\delta t_{z..}(\mathrm{d}z)^n = {} & (v^z\nabla_z + v^{\bar{z}}\nabla_{\bar{z}} + n\nabla_z v^z)t_{z..}(\mathrm{d}z)^n \\ & + n(\nabla_{\bar{z}}v^z)t_{z..}(\mathrm{d}z)^{n-1}\mathrm{d}\bar{z}.\end{aligned} \tag{7.18}$$

The result is a rank n tensor for the new conformal class $[g+\delta g]$ but not for $[g]$. The formula (7.18) applied to the metric $g_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$ gives eqs. (7.1), (7.7) and (7.8) as the new metric $g+\delta g$. To see the variation of $t_{z..}$ as itself a rank n tensor for $[g]$ we use the inverse of eq. (7.16) on eq. (7.18). The result is eq. (7.9).

Next we find the corresponding variations of the covariant derivatives, using eq. (7.9) in

$$\begin{aligned}(\nabla^z + \delta\nabla^z)(t_{z..} + \delta t_{z..}) &= \nabla^z t_{z..} + \delta(\nabla^z t_{z..}) \\ (\nabla_z + \delta\nabla_z)(t_{z..} + \delta t_{z..}) &= \nabla_z t_{z..} + \delta(\nabla_z t_{z..}).\end{aligned} \tag{7.11}$$

This gives eqs. (7.2) and (7.3), at least when the variations of the metric are of the form (7.7) and (7.8). But directly from eqs. (3.15) and (3.16) we know that the contribution due to $\delta\phi$ in eqs. (7.2) and (7.3) is valid for any $\delta\phi$. The validity of the terms involving δg_{zz} and δg^{zz}, for general δg_{zz} and δg^{zz}, then follows, because any tensor δg_{zz} can be represented locally in the form (7.8), and the covariant derivates are locally defined objects. Finally, the variation (7.4) of the scalar curvature is calculated by substituting eqs. (7.2) and (7.3) in eq. (3.13).

8. The free field stress–energy tensor

We make the stress–energy tensor by varying the effective action with respect to the metric:

$$\Theta_{zz} = \frac{-4\pi}{\sqrt{g}}\frac{\delta\Gamma}{\delta g^{zz}} = \frac{4\pi}{\sqrt{g}}\frac{\delta W}{\delta g^{zz}}, \tag{8.1}$$

$$\Theta_{z\bar{z}} = \frac{-4\pi}{\sqrt{g}} \frac{\delta\Gamma}{\delta g^{z\bar{z}}} = \frac{4\pi}{\sqrt{g}} \frac{\delta W}{\delta g^{z\bar{z}}} + \text{source terms}. \tag{8.2}$$

The classical part $A(g, x) + A(g, b, c)$ in Γ is independent of ϕ, so

$$\Theta_{z\bar{z}} = \frac{2\pi}{\sqrt{g}} \frac{\delta S}{\delta\phi} g_{z\bar{z}}. \tag{8.3}$$

That is, the trace of the stress–energy tensor is due entirely to quantum effects in the ground state.

The traceless part, Θ_{zz}, of the stress–energy tensor has contributions from both the ground state and the excitations. Using the variational formulas (7.2) and (7.3) in eq. (8.1) we find that

$$\Theta_{zz} = \Theta^{g}_{zz} + \Theta^{xbc}_{zz}, \tag{8.4}$$

where the ground state contributes

$$\Theta^{g}_{zz} = \frac{-4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{zz}} \tag{8.5}$$

and the excitations

$$\Theta^{xbc}_{zz} = -\nabla_z x^\mu \nabla_z x_\mu - 2b_{zz}\nabla_z c^z - \nabla_z b_{zz} c^z. \tag{8.6}$$

Note that Θ^{xbc}_{zz} does not depend on ϕ. Using equations (8.2) for Θ_{zz} as a variation of W we can write

$$\begin{aligned}\Theta_{zz} &= \langle \Theta^{xbc}_{zz} \rangle \\ &= \langle \Theta^{xbc}_{zz} \rangle_0 + \Theta^{xbc}_{zz}.\end{aligned} \tag{8.7}$$

The bracket $\langle \cdot \rangle$ indicates expectation value in the presence of sources, using the integral (5.11) over x^μ, b and c; while $\langle \cdot \rangle_0$ means expectation value with no sources present, i.e., ground state expectation value. The Θ^{xbc}_{zz} within brackets in eq. (8.7) is the expression (8.6) in the integration variables; the Θ^{xbc}_{zz} outsides brackets is the same expression in the effective fields. To derive eq. (8.7) we use the fact that the volume elements $\mathscr{D}_g b \mathscr{D}_g c \mathscr{D}_g x$ are unchanged under a traceless variation δg_{zz} of the metric.

Using eq. (8.7), we can calculate the traceless part of the ground state stress–energy tensor from the singular behavior of the Green's functions:

$$\Theta^{g}_{zz} = -\nabla_z \nabla_w \langle x^\mu(z) x_\mu(w) \rangle|_{z=w} - (2\nabla_z + \nabla_w)\langle b_{zz} c^w \rangle|_{z=w}. \tag{8.8}$$

To make sense of this we need a covariant regularization of the singularities at $z = w$, but the result will be finite. To see this, consider a change of scale $g_{ab} \to \alpha^{-2} g_{ab}$. This has no effect on the covariant derivatives and Green's functions in eq. (8.8), so it is equivalent to scaling the short distance cutoff by $\varepsilon \to \alpha\varepsilon$. Thus a divergence in eq. (8.8) would have to be a local covariant expression in the metric, a rank 2 tensor, of dimension ≤ 0. The lowest dimension candidate, $\nabla_z \nabla_z R$, has dimension $+2$. We conclude that Θ_{zz} is finite and dimensionless, i.e., scale invariant.

We might remark that eq. (8.8) for Θ^g_{zz} offers a route towards calculating the background action $S(\hat{g}(m_1 \dots m_k))$ in the full integral over surfaces (5.6) when the surface has nontrivial topology and there are inequivalent conformal classes $[\hat{g}(m_1 \dots m_k)]$ parametrized by moduli $m_1 \dots m_k$. A variation δm_j of the moduli can be represented as a variation $\delta_j g^{zz}$ of the background metric. Then, by eq. (8.5),

$$\frac{\delta S}{\delta m_j} = -\frac{1}{2} \int \frac{\mathrm{d}^2 z}{2\pi} \sqrt{\hat{g}}\, \hat{g}^{z\bar{z}} \left(\hat{g}_{\bar{z}z} \frac{\delta \hat{g}^{zz}}{\delta m_j} \Theta^{\hat{g}}_{zz} + \text{c.c.} \right), \tag{8.9}$$

with Θ^g_{zz} calculated via eq. (8.8) in terms of the free field Green's functions on the surface. Integrating eq. (8.9) gives $S(\hat{g}(m_1 \dots m_k)$, up to a constant.

9. Conservation of stress–energy: the Liouville action

A covariant functional $\mathscr{F}$ of the fields and the metric is invariant under variations (7.7), (7.8), (7.10)–(7.12) arising from an arbitrary reparametrization $\delta z = v^z(z, \bar{z})$:

$$\left[-\nabla^z \frac{4\pi}{\sqrt{g}} \frac{\delta}{\delta g^{zz}} + \nabla_z \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \phi} \right] \Bigg|_{x,b,c} \mathscr{F} = \mathscr{D}_{\text{fields}} \mathscr{F}, \tag{9.1}$$

where

$$\begin{aligned} \mathscr{D}_{\text{fields}} = {} & \nabla_z x^\mu \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta x^\mu} + (2\nabla_z c^z + c^z \nabla_z) \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta c^z} + \nabla_z c^{\bar{z}} \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta c^{\bar{z}}} \\ & - (\nabla_z b_{zz} + 2 b_{zz} \nabla_z) \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta b_{zz}} + \nabla_z b_{\bar{z}\bar{z}} \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta b_{\bar{z}\bar{z}}} . \end{aligned} \tag{9.2}$$

Similarly any covariant functional $\mathscr{G}$ of the sources and the metric satisfies

$$\left[-\nabla^z \frac{4\pi}{\sqrt{g}} \frac{\delta}{\delta g^{zz}} + \nabla_z \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \phi} \right] \Bigg|_{\chi,\beta,\gamma} \mathscr{G} = \mathscr{D}_{\text{sources}} \mathscr{G}, \tag{9.3}$$

$$\mathscr{D}_{\text{sources}} = \nabla_z \chi^\mu \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \chi^\mu} - \gamma_z \nabla_z \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \gamma_z} + \nabla_z \gamma_{\bar{z}} \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \gamma_{\bar{z}}}$$
$$+ (3\nabla_z \beta^{zz} - 2\beta^{zz}\nabla_z) \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \beta^{zz}} + \nabla_z \beta^{\bar{z}\bar{z}} \frac{2\pi}{\sqrt{g}} \frac{\delta}{\delta \beta^{\bar{z}\bar{z}}}. \tag{9.4}$$

By applying eq. (9.1) to Γ we get the conservation law for stress–energy:

$$\nabla^z \Theta_{zz} + \nabla^{\bar{z}} \Theta_{\bar{z}z} = \chi_\mu \nabla_z \chi^\mu + \gamma_z \nabla_z c^z + \nabla_z(\gamma_z c^z)$$
$$+ \beta^{zz} \nabla_z b_{zz} - 2\nabla_z(\beta^{zz} b_{zz}). \tag{9.5}$$

We can easily verify eq. (9.5) for the excitation stress–energy, using eqs. (8.4), (5.20)–(5.22). The new information is the conservation of stress–energy in the ground state:

$$\nabla^z \Theta^g_{zz} + \nabla_z \left(\frac{2\pi}{\sqrt{g}} \frac{\delta S}{\delta \phi} \right) = 0. \tag{9.6}$$

Examining the expression (8.8) for Θ^g_{zz} we see that $\nabla^z \Theta^g_{zz}$ must be a local expression in the metric. Since Θ^g_{zz} has dimension 0, $\nabla^z \Theta^g_{zz}$ has dimension $+1$. The only such rank 1 tensor is $\nabla_z R$. Thus

$$\nabla^z \Theta^g_{zz} = \frac{-\lambda}{24} \nabla_z R, \tag{9.7}$$

where λ is a number we have yet to determine.

By eq. (9.6),

$$\frac{\delta S}{\delta \phi} = \frac{\lambda}{48\pi} \sqrt{g}\,(R + \mu^2), \tag{9.8}$$

where μ^2 is an arbitrary integration constant. We can integrate eq. (9.8) using eq. (3.17) for R to find

$$S_{\text{eff}}(\hat{g}, \phi) = \frac{\lambda}{24} \int \frac{d^2\xi}{2\pi} \sqrt{g}\,(\tfrac{1}{2}\hat{g}^{ab} \partial_a \phi \partial_b \phi + \hat{R}\phi + \mu^2 e^\phi - \mu^2). \tag{9.9}$$

This is the Liouville action, providing (in eqs. (5.6) and (5.7)) the effective classical dynamics for the ϕ field.

The conservation of stress–energy now reads

$$\nabla^z \Theta_{zz} = \frac{-\lambda}{24} \nabla_z R + (\text{source terms}), \tag{9.10}$$

$$(\text{source terms}) = \chi_\mu \nabla_z x^\mu + \gamma_z \nabla_z c^z + \nabla_z(\gamma_z c^z) + \beta^{zz}(\nabla_z b_{zz}) - 2\nabla_z(\beta^{zz} b_{zz}). \tag{9.11}$$

10. Ward identities and operator product expansions: $\lambda = 26 - \mathscr{D}$

The conservation law, eqs. (9.10) and (9.11), as an expression in the metric and the sources, generates Ward identities for products of Θ_{zz}^{xbc} and the fields. For example, differentiating once with respect to the source gives

$$\frac{1}{2\pi} \nabla^z \langle \Theta_{zz}^{abc} x^\mu(w) \rangle_c = \mathbf{1}(z, w) \nabla_w x^\mu, \tag{10.1}$$

$$\frac{1}{2\pi} \nabla^z \langle \Theta_{zz}^{xbc} c^w \rangle_c = \nabla_z \mathbf{1}(z, w) c^w + \mathbf{1}(z, w) \nabla_w c^w, \tag{10.2}$$

$$\frac{1}{2\pi} \nabla^z \langle \Theta_{zz}^{xbc} b_{ww} \rangle_c = -2\nabla_z \mathbf{1}(z, w) b_{ww} + \mathbf{1}(z, w) \nabla_w b_{ww}, \tag{10.3}$$

where

$$\mathbf{1}(z, w) = g^{-1/2} \delta^2(z - w), \tag{10.4}$$

is the covariant delta function. The expressions $\langle \Theta_{zz}^{xbc} \cdot \rangle_c$ denote connected expectation values in the presence of sources, arising when eq. (8.2) is differentiated with respect to the sources. In writing eqs. (10.1)–(10.3) the sources are in the end set to zero in neighborhoods of z and w, but are left arbitrary elsewhere. This means that eqs. (10.1)–(10.3) should be interpreted as a set of operator identities. The distant sources make the arbitrary state in which the identities hold.

Each Ward identity is equivalent to an operator product expansion (OPE). The basic fact we use is that

$$\frac{1}{2\pi} (2\partial_{\bar{z}}) \frac{1}{z - w} = \delta^2(z - w), \tag{10.5}$$

which is proved by an integration by parts. To put eq. (10.5) into a covariant form, define

$$K_{zz}^w = (z - w)^{-1} + f(z, w), \tag{10.6}$$

where K_{zz}^w is a rank 2 tensor in z and a rank -1 tensor in w and $f(z, w)$ is regular and analytic in both variables for z near w. It is easily verified that the singular part of K_{zz} keeps its form under conformal reparametrization. The non-singular part will not matter in OPEs. Now eq. (10.5) becomes

$$\frac{1}{2\pi} \nabla^z K_{zz}^w = \mathbf{1}(z, w). \tag{10.7}$$

We will also need the distributions

$$(\nabla_w)^n K^w_{zz} \sim \frac{n!}{(z-w)^{n+1}} + \mathrm{O}((z-w)^{-n}) \tag{10.8}$$

satisfying

$$\nabla^z(\nabla^n_w K^w_{zz}) = (-1)^n \nabla^n_z \mathbf{1}(z,w). \tag{10.9}$$

For example, if $g_{z\bar z} = \frac{1}{2}\rho$, then, in coordinate form,

$$\nabla_w K^w_{zz} \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}\partial_w \log\rho. \tag{10.10}$$

Now we can rewrite the Ward identities (10.1)–(10.3) as the OPEs:

$$\langle \Theta^{xbc}_{zz} x^\mu(w)\rangle_c \sim \frac{1}{z-w}\partial_w x^\mu, \tag{10.11}$$

$$\langle \Theta^{xbc}_{zz} c^w\rangle_c \sim \left[\frac{-1}{(z-w)^2} + \frac{1}{z-w}\partial_w\right] c^w, \tag{10.12}$$

$$\langle \Theta^{xbc}_{zz} b_{ww}\rangle_c \sim \left[\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right] b_{ww}. \tag{10.13}$$

We can easily confirm (10.11)–(10.13) using the explicit form (8.6) for Θ^{xbc}_{zz} and knowing the singular parts of the Green's functions (5.15) and (5.16). For example,

$$\langle -2b_{zz}\nabla_z c^z - \nabla_z b_{zz} c^z, c^w\rangle_c = 2\langle b_{zz}c^w\rangle \partial_z c^z + \partial_z\langle b_{zz}c^w\rangle c^z, \tag{10.14}$$

gives eq. (10.12).

Next we look at the Ward identity for $\langle \Theta^{xbc}_{zz}\Theta^{xbc}_{ww}\rangle_c$. We vary the conservation law, eqs. (9.10) and (9.11) with a variation δg^{zz} of the metric, keeping the sources fixed, using the variational formulas (7.2)–(7.4). This yields

$$\frac{1}{2\pi}\nabla^z\langle \Theta^{xbc}_{zz}\Theta^{xbc}_{ww}\rangle_c = \frac{\lambda}{12}\nabla^3_z \mathbf{1}(z,w) + [-2\nabla_z \mathbf{1}(z,w) + \mathbf{1}(z,w)\nabla_w]\Theta_{ww}. \tag{10.15}$$

The equivalent OPE is

$$\langle \Theta^{xbc}_{zz}\Theta^{xbc}_{ww}\rangle_c \sim \frac{-\lambda}{12}\nabla^3_w K^w_{zz} + (2\nabla_w K^w_{zz} + K^w_{zz}\nabla_w)\Theta_{ww}. \tag{10.16}$$

For the moment look only at the leading singularity

$$\langle \Theta_{zz}^{xbc} \Theta_{ww}^{xbc} \rangle_c \sim \frac{-\frac{1}{2}\lambda}{(z-w)^4}. \tag{10.17}$$

We can find the same singularity by a direct Feynman diagram calculation of the OPE:

$$\begin{aligned} \langle \Theta_{zz}^{xbc} \Theta_{ww}^{xbc} \rangle_c \sim\ & 2[\partial_z \partial_w \langle x^\mu(z) x_\nu(w) \rangle]^2 \\ & + 4\partial_w \langle b_{zz} c^w \rangle \partial_z \langle c^z b_{ww} \rangle \\ & + [2\langle b_{zz} c^w \rangle \partial_z \partial_w \langle c^z b_{ww} \rangle + z \leftrightarrow w] \\ & + \partial_z \langle b_{zz} c^w \rangle \partial_w \langle c^z b_{ww} \rangle \\ \sim\ & \frac{-\frac{1}{2}(26-\mathscr{D})}{(z-w)^4}. \end{aligned} \tag{10.18}$$

Thus $\lambda = 26 - \mathscr{D}$, which finishes the calculation of the effective Liouville action (9.9) for $\phi(\xi)$.

The full OPE (10.16) can be rewritten in the simple form

$$\langle \Theta_{zz}^0 \Theta_{ww}^0 \rangle_c = \frac{-\frac{1}{2}\lambda}{(z-w)^4} + \left[\frac{2}{(z-w)^2} + \frac{1}{(z-w)} \partial_w \right] \langle \Theta_{ww}^0 \rangle \tag{10.19}$$

if we define

$$\Theta_{zz}^0 = \Theta_{zz}^{xbc} - \frac{\lambda}{24} (-\partial_z \log \rho \, \partial_z \log \rho + 2\partial_z \partial_z \log \rho). \tag{10.20}$$

From eq. (10.16) we know that the rhs of eq. (10.19) is analytic in z for $z \neq w$, so in both variables by symmetry, therefore

$$\partial_{\bar{z}} \langle \Theta_{zz}^0 \rangle = 0, \tag{10.21}$$

which can be confirmed by direct calculation using eqs. (9.7) and (3.14).

11. Hilbert space interpretation: the Virasoro algebra

When the parameter surface is the upper half plane

$$H = \{ z = \mathrm{e}^{\tau + \mathrm{i}\sigma} : -\infty < \tau < \infty, 0 \leqslant \sigma \leqslant \pi \}, \tag{11.1}$$

the functional integral (5.1) can be interpreted as a theory of field operators in Hilbert space, expectation values becoming the matrix elements of τ-ordered products. To do this carefully requires taking account of boundary conditions and zero modes. Here we will only sketch the basic structure and examine the commutation relations of the free fields and the stress–energy tensor.

Start by defining

$$a_z^\mu = \partial_z x^\mu, \quad \partial_{\bar z} a_z^\mu = 0. \tag{11.2}$$

The equation of motion holds if there are no sources near z, which is to say it holds for all matrix elements of a_z^μ.

Next, double the parameter surface to include the lower half plane $\bar H$. The doubled surface is the punctured plane $\mathbb{C}-\{0\}$. Define a_z^μ in $\bar H$ by

$$a_z^\mu = \overline{a_{\bar z}^\mu}. \tag{11.3}$$

Generate operators by

$$a^\mu[f] = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} f(z) a_z^\mu \quad \text{for } \partial_{\bar z} f = 0, \tag{11.4}$$

$$a_z^\mu = \sum_{n=-\infty}^{\infty} z^{-n} a_n^\mu, \quad a_n^\mu = a^\mu[z^n]. \tag{11.5}$$

The contour C_0 in eq. (11.4) is any simple contour circling counterclockwise once around the origin. We use the OPE (5.15)

$$\langle a_z^\mu a_w^\nu \rangle \sim -\tfrac{1}{2}\delta^{\mu\nu}/(z-w)^2 \tag{11.6}$$

to calculate the commutation relations:

$$\langle [a^\mu[f], a_w^\nu] \rangle = \left(\oint_{C_{0w}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} - \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \right) f(z) \langle a_z^\mu a_w^\nu \rangle, \tag{11.7}$$

$$[a^\mu[f], a_{\nu w}] = \oint_{C_w} \frac{\mathrm{d}z}{2\pi \mathrm{i}} f(z) \frac{-\tfrac{1}{2}\delta_\nu^\mu}{(z-w)^2} \tag{11.8}$$

$$= -\tfrac{1}{2}\delta_\nu^\mu \partial_w f, \tag{11.9}$$

$$[a_m^\mu, a_{\nu n}] = -\tfrac{1}{2} m \delta_\nu^\mu \delta_{m+n,0}. \tag{11.10}$$

In eq. (11.7) the contour C_{0w} contains 0 and w while C_0 contains 0 but not w; τ-ordering is used to produce the commutator. To get eq. (11.8), the contour $C_{0w} - C_0$ is deformed to a curve C_w around w, asymptotically close.

The same arguments work for the Faddeev–Popov ghost fields. They are extended to $\bar{H}$ by

$$c^z = \overline{c^{\bar{z}}}, \quad b_{zz} = \overline{b_{\bar{z}\bar{z}}}. \tag{11.11}$$

They satisfy

$$\partial_{\bar{z}} c^z = 0, \quad \partial_{\bar{z}} b_{zz} = 0 \tag{11.12}$$

and give operators according to

$$c[S] = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} S_{zz} c^z \quad \text{for } \partial_{\bar{z}} S_{zz} = 0, \tag{11.13}$$

$$c^z = \sum_{n=-\infty}^{\infty} z^{n+1} c_n, \quad c_n = c[z^{-n-2}], \tag{11.14}$$

$$b[t] = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} t^z b_{zz} \quad \text{for } \partial_{\bar{z}} t^z = 0, \tag{11.15}$$

$$b_{zz} = \sum_{n=-\infty}^{\infty} z^{-n-2} b_n, \quad b_n = b[z^{n+1}]. \tag{11.16}$$

The OPE (5.16)

$$\langle c^z b_{ww} \rangle \sim \frac{1}{z-w}, \tag{11.17}$$

leads, by deforming contours, to the anti-commutation relations

$$[c[S], b_{ww}]_+ = S_{ww}, \tag{11.18}$$

$$[c_m, b_n]_+ = \delta_{m,n}. \tag{11.19}$$

The same procedure applies to $\langle \Theta^0_{zz} \rangle$ (10.20):

$$\langle \Theta^0_{zz} \rangle = \overline{\langle \Theta^0_{\bar{z}\bar{z}} \rangle} \quad \text{in } \bar{H}, \tag{11.20}$$

$$\partial_{\bar{z}} \langle \Theta^0_{zz} \rangle = 0, \tag{11.21}$$

$$\Theta[v] = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} v^z \langle \Theta^0_{zz} \rangle \quad \text{for } \partial_{\bar{z}} v^z = 0, \tag{11.22}$$

$$\Theta^0_{zz} = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n \quad \text{for } L_n = \Theta[z^{n+1}]. \tag{11.23}$$

The OPEs (10.11)–(10.13) and the contour deformation argument give the commutation relations

$$[\Theta[v], a^\mu_w] = (v^w \partial_w + \partial_w v^w) a^\mu_w, \tag{11.24}$$

$$[\Theta[v], c^w] = (v^w \partial_w - \partial_w v^w) c^w, \tag{11.25}$$

$$[\Theta[v], b_{ww}] = (v^w \partial_w + 2\partial_w v^w) b_{ww}. \tag{11.26}$$

Thus the operators $\Theta[v]$ represent the infinitesimal conformal reparametrizations. In particular, L_0 generates translation in the τ-direction.

From the OPE (10.19) for $\langle \Theta^0_{zz} \Theta^0_{ww} \rangle_c$ come the commutation relations

$$[\Theta[v], \Theta^0_{ww}] = (v^w \partial_w + 2\partial_w v^w)\Theta^0_{ww} - \frac{\lambda}{12} \partial^3_w v^w \tag{11.27}$$

which is equivalent to

$$[L_m, L_n] = (m-n) L_{m+n} + \left(\frac{-\lambda}{12}\right) m(m^2-1) \delta_{m+n,0}. \tag{11.28}$$

The latter are the commutation relations of the Virasoro algebra.

We see that the coefficients $-\lambda$ of the anomalous central term in the Virasoro commutation relations (11.28) is exactly the same thing as the coefficient of the conformal anomaly (9.8). They arise from the same short distance effect, since both are calculated from the same leading singularity in the OPE (10.17).

12. Transformation properties of $\sqrt{g}\, e^{ip\cdot x}$

We will look at the properties of $e^{ip\cdot x}$ for a fixed metric g_{ab}. The effective operator is

$$\langle e^{ip\cdot x}(\xi) \rangle = \langle e^{ip\cdot x}(\xi) \rangle_0 e^{ip\cdot x(\xi)}. \tag{12.1}$$

To renormalize $e^{ip\cdot x}$ we need to remove the divergence in

$$\langle e^{ip\cdot x}(\xi) \rangle_0 = \exp(-\tfrac{1}{2} p_\mu p_\nu \langle x^\mu(\xi) x^\nu(\xi') \rangle|_{\xi=\xi'}). \tag{12.2}$$

From eq. (5.15) we know that, with cutoff ε,

$$\langle x^\mu(\xi) x^\nu(\xi') \rangle|_{\xi=\xi'} \sim -\tfrac{1}{2} \delta^\mu_\nu \log \varepsilon. \tag{12.3}$$

Therefore a renormalized operator is defined by

$$\langle e^{ip\cdot x}\rangle = \lim_{\varepsilon\to 0}\langle e^{-\frac{1}{4}p^2\log\varepsilon}e^{ip\cdot x}\rangle_{\text{cutoff}}. \tag{12.4}$$

Because the renormalization is covariant, the operator satisfies

$$\left[2\nabla^z\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{zz}} - \nabla_z\frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi} - \mathbf{1}(z,w)\nabla_w\right]\langle e^{ip\cdot x}(w)\rangle_0 = 0, \tag{12.5}$$

or

$$\frac{\nabla^z}{2\pi}\langle \Theta^{xbc}_{zz}e^{ip\cdot x}(w)\rangle_c = \left[\nabla_z\frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi(z)} + \mathbf{1}(z,w)\nabla_w\right]\langle e^{ip\cdot x}(w)\rangle_0. \tag{12.6}$$

But a direct Feynman diagram calculation gives

$$\frac{\nabla_z}{2\pi}\langle -\partial_z x^\mu\partial_z x_\mu, e^{ip\cdot x}(w)\rangle_c = \left[\frac{-p^2}{4}\nabla_z\mathbf{1}(z,w) + \mathbf{1}(z,w)\nabla_w\right]\langle e^{ip\cdot x}(w)\rangle_0. \tag{12.7}$$

Thus

$$0 = \nabla_z\left[\frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi(z)} + \frac{p^2}{4}\mathbf{1}(z,w)\right]\langle e^{ip\cdot x}(w)\rangle. \tag{12.8}$$

We know from eq. (12.2) that $\langle e^{ip\cdot x}(w)\rangle$ cannot depend on $\phi(z)$ far from w. Therefore, writing $g_{ab} = e^{\phi}\hat{g}_{ab}$, we can make the ϕ dependence explicit:

$$\langle e^{ip\cdot x}(w)\rangle_g = e^{-\frac{1}{4}p^2\phi(w)}\langle e^{ip\cdot x}(w)\rangle_{\hat{g}}. \tag{12.9}$$

This means that to calculate expectation values of $\sqrt{g(\xi)}\,e^{ip\cdot x(\xi)}$ in the original integral over x^μ and g_{ab}, we should use the operator

$$\sqrt{\hat{g}(\xi)}\,e^{(1-p^2/4)\phi(\xi)}\,e^{ip\cdot x(\xi)}$$

in the integral (5.6) over x^μ, b_{zz}, c^z and ϕ with background metric $\hat{g}_{ab}$.

13. The stress–energy tensor of the classical Liouville model

The Liouville model is given by the functional integral over metrics $g_{ab} = e^{\phi}\hat{g}_{ab}$ in the background metric $\hat{g}_{ab}$:

$$e^{-S(\hat{g})}\int \mathscr{D}_g\phi\, e^{-S_{\text{eff}}(\hat{g},\phi)}, \tag{13.1}$$

$$S_{\rm eff}(\hat{g},\phi) = S(g)-S(\hat{g})$$
$$= \frac{\lambda}{24}\int\frac{d^2\xi}{2\pi}\sqrt{\hat{g}}\,[\tfrac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi+\hat{R}\phi+\mu^2 e^{\phi}-\mu^2]. \qquad (13.2)$$

This interacting quantum field theory must be solved to complete the integral (5.6) over surfaces. Here we will discuss properties of the classical Liouville model, leaving out the corrections due to quantum fluctuations.

The Euler–Lagrange equation $\delta S/\delta\phi = 0$, i.e., $R = -\mu^2$ (see eq. (9.8)), is satisfied in the absence of sources for ϕ, therefore the trace of the full stress–energy tensor (8.3) vanishes. The full traceless stress–energy tensor (8.1) can be split up

$$\Theta_{zz} = \Theta^{xbc}_{zz}+\Theta^{\hat{g}}_{zz}+\Theta^{\phi,\rm cl}_{zz} \qquad (13.3)$$

into the excitation and ground state stress–energies of the free fields in the background metric $\hat{g}_{ab}$, and the classical stress–energy of the ϕ field. The full stress–energy is conserved, because ((9.6), (9.8))

$$\nabla^z\Theta_{zz} = \frac{-\lambda}{24}\nabla_z R = 0 \qquad (13.4)$$

by the equation of motion $R = -\mu^2$ for ϕ.

The vanishing of the trace $\Theta_{z\bar{z}}$ means that the full integral (5.6) over surfaces depends only on the conformal class of $\hat{g}_{ab}$. The conservation (13.4) of the full stress–energy tensor means that (5.6) is invariant under changes of $\hat{g}_{ab}$ by reparametrization.

Let us now rewrite the conservation law (9.6) in terms of $\hat{g}_{ab}$ and ϕ without assuming the equation of motion for d:

$$\hat{\nabla}^z\Theta_{zz}+(\hat{\nabla}_z-\hat{\nabla}_z\phi)\frac{2\pi}{\sqrt{g}}\frac{\delta S}{\delta\phi} = 0. \qquad (13.5)$$

The conservation law for stress–energy [eqs. (9.6), (9.8)] in the background metric $\hat{g}_{ab}$ is

$$\hat{\nabla}^z(\Theta^{xbc}_{zz}+\Theta^{\hat{g}}_{zz}) = -\frac{\lambda}{24}\hat{\nabla}_z\hat{R}+(\text{source terms}). \qquad (13.6)$$

Putting together (13.3), (5.6) we get

$$\hat{\nabla}^z\Theta^{\phi,\rm cl}_{zz} = \frac{\lambda}{24}\hat{\nabla}_z\hat{R}+(\hat{\nabla}_z\phi-\hat{\nabla}_z)\frac{2\pi}{\sqrt{g}}\frac{\delta S}{\delta\phi} \qquad (13.7)$$

which is the conservation law for the Liouville model in the presence of the source $(2\pi/\sqrt{g})(\delta S/\delta\phi)$ for ϕ. The form of the source term comes from the variational formula for ϕ under reparametrization:

$$\delta\phi = (v^z\partial_z\phi + \hat{\nabla}_z v^z) + \text{c.c.} \tag{13.8}$$

which follows from $\sqrt{g} = e^{\phi}\sqrt{\hat{g}}$.

The anomalous conservation law (13.7) for $\Theta_{zz}^{\phi,\,\mathrm{cl}}$ implies that the Virasoro operators in the classical Liouville model satisfy anomalous commutation relations, as in eq. (11.31), with coefficient $+\lambda$. Note that the reparametrization invariance of the full functional integral can be seen as a cancellation of anomalies between the free fields x^μ, b_{zz}, c^z and the classical Liouville field.

We can also use the decomposition (13.3) in eq. (11.8) to get the Ward identities

$$\frac{\hat{\nabla}^z}{2\pi}\langle\Theta_{zz}^{xbc}, e^{ip\cdot x}(w)\rangle_c = \left[\frac{-p^2}{4}\hat{\nabla}_z\hat{\mathbf{1}}(z,w) + \hat{\mathbf{1}}(z,w)\hat{\nabla}_w\right] \times \langle e^{ip\cdot x}(w)\rangle, \tag{13.9}$$

$$\frac{\hat{\nabla}^z}{2\pi}\langle\Theta_{zz}^{\phi,\,\mathrm{cl}}, e^{(1-p^2/4)\phi}(w)\rangle_c = \left[\left(\frac{p^2}{4} - 1\right)\hat{\nabla}_z\hat{\mathbf{1}}(z,w) + \hat{\mathbf{1}}(z,w)\hat{\nabla}_w\right] \times \langle e^{(1-p^2/4)\phi}(w)\rangle, \tag{13.10}$$

which state precisely the cancellation of the anomalous weight of $e^{ip\cdot x}$ against $e^{-p^2/4\phi}$.

Finally, we give an explicit formula for $\Theta_{zz}^{\phi,\,\mathrm{cl}}$. This can be derived in two equivalent ways: (1) by making a variation δg^{zz} in eq. (13.2) using the variational formulas (7.2)–(7.4); and (2) by using the covariance of Θ_{zz} to calculate $\nabla_z(\delta/\delta\phi)\Theta_{zz}$. In either case the result is

$$\Theta_{zz}^{\phi,\,\mathrm{cl}} = \frac{\lambda}{24}(-\partial_z\phi\partial_z\phi + 2\hat{\nabla}_z\partial_z\phi). \tag{13.11}$$

14. Conclusions

The string theory, which began as a free scalar field $x^\mu(\xi)$ on parameter space, quantized in a fluctuating metric $g_{ab}(\xi)$, has become three decoupled fields in a common background metric $g_{ab}(\xi)$. The original x^μ field and the Faddeev–Popov ghosts b, c are free; while the field $\phi(\xi)$, which contains the conformal degrees of freedom of the metric $g_{ab} = e^{\phi}\hat{g}_{ab}$, is governed by the Liouville action $S_{\mathrm{eff}}(\hat{g}, \phi)$, eq. (9.9). We have seen that the Liouville action can be derived assuming only reparametrization invariance in the gauge fixing and in the free field quantization, along with elementary scaling behavior of the free fields.

It remains to quantize and solve the Liouville model. Here we will only remark on some conditions which will need to be satisfied in the quantization of the Liouville model. Essentially these state that the quantization should not modify any of the conformal properties of the classical Liouville model. In particular, there should be no quantum contribution to the trace of the stress–energy tensor, the Virasoro operators of the quantum Liouville model should satisfy the same commutation relations as in the classical model ((13.7), (11.31)):

$$[L_m^{\phi,\,\mathrm{qm}}, L_n^{\phi,\,\mathrm{qm}}] = (m-n)L_{m+n}^{\phi,\,\mathrm{qm}} + \frac{26-\mathscr{D}}{12}\, m(m^2-1)\delta_{m+n,\,0}, \tag{14.1}$$

and the operators $e^{(1-p^2/4)\phi}$ should satisfy the classical Ward identities (13.10).

We can derive these conditions from the requirement that the full string theory be independent of the choice of background metric $\hat{g}_{ab}$. The quantization of the Liouville model is summarized by writing the effective quantum action

$$S_{\mathrm{qm}}(\hat{g}, \phi) = S_{\mathrm{eff}}(\hat{g}, \phi) + \Delta S(\hat{g}, \phi), \tag{14.2}$$

which includes the quantum corrections $\Delta S(\hat{g}, \phi)$. The stress–energy tensor for the full string theory (with background metric $\hat{g}_{ab}$ understood) is

$$\Theta^{\mathrm{total}} = \Theta^{\mathrm{free}} + \Theta^{\phi,\,\mathrm{qm}}, \tag{14.3}$$

where the quantum Liouville stress–energy

$$\Theta^{\phi,\,\mathrm{qm}} = \Theta^{\phi,\,\mathrm{cl}} + \Delta\Theta^{\phi} \tag{14.4}$$

also includes a quantum correction $\Delta\Theta^{\phi}$.

The first thing to note is that the background metric $\hat{g}_{ab}$ is only an arbitrary choice of origin in the gauge slice $[\hat{g}] = \{e^{\phi}\hat{g}_{ab}\}$. Shifting

$$\hat{g}_{ab}(\xi) \to (1+f(\xi))\hat{g}_{ab}(\xi)$$

should make no difference to the theory. Thus we should have

$$\Theta_{z\bar{z}}^{\mathrm{total}} = 0. \tag{14.5}$$

We already have two of the contributions to $\Theta_{z\bar{z}}^{\mathrm{total}}$ (see eqs. (9.8) and (9.9)):

$$\Theta_{z\bar{z}}^{\text{free}} = \frac{\lambda}{24}(\hat{R}+\mu^2)\hat{g}_{z\bar{z}}, \tag{14.6}$$

$$\Theta_{z\bar{z}}^{\phi;\,\text{cl}} = \frac{-\lambda}{24}(\hat{R}+\mu^2)\hat{g}_{z\bar{z}}, \tag{14.7}$$

which cancel each other, essentially by construction:

$$S_{\text{eff}}(\hat{g},\phi) = S(g)-S(\hat{g}). \tag{14.8}$$

We therefore have the condition

$$\Delta\Theta_{z\bar{z}}^{\phi} = 0, \tag{14.9}$$

that there be no quantum correction to the trace of the stress–energy tensor in the Liouville model.

This condition (14.9) implies that the quantum correction to the effective action does not depend separately on $\hat{g}$ and ϕ but only on the combination $g = e^{\phi}\hat{g}$, so that

$$S_{\text{qm}}(\hat{g},\phi) = S(g)+\Delta S(g)-S(\hat{g}). \tag{14.10}$$

It is easy to work out that eq. (14.10) is the same as eq. (14.9) once we note that $\Theta_{z\bar{z}}$ should be calculated by varying

$$g_{z\bar{z}} \to g_{z\bar{z}}+\delta g_{z\bar{z}}$$

while at the same time sending

$$\phi \to \phi - g^{z\bar{z}}\delta g_{z\bar{z}}.$$

This is because the variation of $g_{z\bar{z}}$ should take place with the field held fixed. In the Liouville model the field appearing in correlation functions is actually the combination $\log\sqrt{g} = \phi+\log\sqrt{\hat{g}}$.

Next we show that the quantum correction $\Delta S(g)$ in eq. (14.10) must be a covariant functional of the metric $g_{ab}(\xi)$. For the full theory to be reparametrization invariant we need the conservation law

$$\hat{\nabla}^{z}\Theta_{zz}^{\text{total}}+\hat{\nabla}^{\bar{z}}\Theta_{z\bar{z}}^{\text{total}} = \text{source terms}. \tag{14.11}$$

But in the previous section we saw that eq. (14.11) is satisfied using only the classical Liouville stress–energy, again essentially by construction. The quantum corrections should therefore contribute equally to both sides of eq. (14.11):

$$\hat{\nabla}^{z}(\Delta\Theta_{zz}^{\phi}) = (\hat{\nabla}_z\phi-\hat{\nabla}_z)\frac{2\pi}{\sqrt{\hat{g}}}\frac{\delta\Delta S}{\delta\phi}. \tag{14.12}$$

When rewritten, eq. (14.12) becomes the statement that $\Delta S(g)$ is reparametrization invariant:

$$0 = \nabla^z \left(\frac{2}{\sqrt{g}} \frac{\delta \Delta S}{\delta g^{zz}} \right) - \nabla_z \left(\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \phi} \right). \tag{14.13}$$

From eqs. (14.12) and (13.7) we get that the anomalous conservation law for the traceless stress–energy tensor of the quantum Liouville model:

$$\hat{\nabla}^z \Theta^{\phi,\,\mathrm{qm}}_{zz} = \frac{\lambda}{24} \hat{\nabla}_z \hat{R} + \text{source terms}, \tag{14.14}$$

is identical to that for the classical model. By the discussion of sections 10 and 11, eq. (14.14) implies the Virasoro commutation relations (14.1). In exactly the same way we can argue that the operators $(\sqrt{\hat{g}}\, \mathrm{e}^{\phi})^{1-p^2/4}$ must keep their classical transformation properties (13.10) even after quantization, in order that the observables $\int \mathrm{d}^2\xi \sqrt{g}\, \mathrm{e}^{\mathrm{i}p \cdot x}$ remain reparametrization invariant.

These conditions on the quantization of the Liouville model are concerned with the short distance properties of the model; they are constraints on certain numbers appearing in operator product expansions. The Liouville model has the appearance of a superrenormalizable two-dimensional scalar field theory. But such a theory, canonically quantized, will always have quantum conformal anomalies at short distance. We expect, then, that the correct quantization will give a new kind of two-dimensional quantum field theory.

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Jean Bernard: the shepherd
The majority of students escaping on the mountains
Clever student hypnotized by The shepherd's bell
Stefano Ruffo