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Notes on String Theory
and
Two Dimensional Conformal Field Theory

Daniel Friedan

Enrico Fermi Institute and Department of Physics
University of Chicago, Chicago, Illinois 60637

ABSTRACT

These lecture notes cover topics in the covariant first quantisation of supersymmetric string: super Riemann surfaces, superconformal quantum field theory in two dimensions, the superconformal world surface of the string, the superconformal ghosts on the world surface, the BRST invariant fermion vertex and the spacetime supersymmetry current on the world surface.

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CONTENTS

The basic problem in covariant first quantized string theory is to construct the world surface of the string as a local two dimensional conformally invariant quantum field theory. The problem divides in two parts. A conformal field theory is completely defined by the operator product expansions of its quantum fields, which can be determined at arbitrarily small distance. So the first task is to describe the local structure of the world surface. Once the conformal field theory is defined by its local properties, its global behavior can be checked to determine the consistency of the string loop expansion.

These notes are about the superconformal invariance of the world surface of supersymmetric string. The main topics are the construction of the vertex operators for emission of spacetime fermions and the demonstration of spacetime supersymmetry in the covariant first quantization. Only the local structure of the world surface is described; explicit global information is given only for the two sphere, in order to calculate tree amplitudes. The tree amplitudes illustrate how global facts such as spacetime supersymmetry and BRST invariance are obtained from local information coded in operator products of chiral fields and, in particular, conformal currents. The translation from local to global information is based on the analyticity of chiral quantum fields in two dimensions. The chiral fields on the string world surface include the super stress-energy tensor, the Fadeev-Popov ghost fields and their anomalous currents, the BRST superconformal current, and the conformal current for spacetime supersymmetry.

These notes are meant to be read in conjunction with the lectures of Stephen Shenker[1], and describe work done with him, Joanne Cohn, Emil Martinec and Zongan Qiu[2-6]. Only a few references are given, and then only to relatively recent work. The references are definitely not meant to convey the history of the subject. A more complete introduction to the literature can be found in reference 5. Some of the ideas of conformal field theory and covariant bosonic string theory are discussed in reference 7 from the point of view which is taken here.

Many of the arguments and calculations in these notes are presented rather telegraphically. The industrious reader might treat the gaps as exercises or problems.

Section 1 is a sketch of the general strategy of covariant first quantization; section 2 develops the most basic properties of super Riemann surfaces; section 3 sketches superconformal field theory; section 4 describes the superconformal world surface of fermionic string and the superconformal ghosts; section 5 is a general discussion of two dimensional free tensor quantum fields satisfying first order equations of motion; section 6 applies the general results of section 5 to the superconformal ghosts and constructs the BRST current; and section 7 constructs the fermion vertex and the spacetime supersymmetry current.

1. Introduction

A theory of gravity, such as string theory, should at least provide a manifestly Lorentz covariant scheme for calculating scattering amplitudes in flat spacetime. Covariant first quantization of strings could also be useful as a step towards understanding the underlying structure of string.

A manifestly relativistic first quantization of string can be carried out using the language of two dimensional conformal quantum field theory to describe sums over world surfaces of first quantized strings. The analog in particle theory is the relativistic calculation of scattering amplitudes in first quantization by representing Feynman diagrams as sums over particle world lines (joined at interaction vertices).

1.1 Covariant quantisation of bosonic strings

The basic ideas of covariant first quantization of strings are realized in the bosonic theory[8]. A world surface is given by its location in spacetime, $x^\mu(z,\bar z)$, and by an intrinsic metric $g_{ab}(z,\bar z)$ on the parameter space of the complex variable z, with line element $ds^2 = g_{zz}dz^2 + g_{z\bar z}dzd\bar z + g_{\bar z z}d\bar z dz + g_{\bar z\bar z}d\bar z^2$. The intrinsic metric makes it possible to write a sum over world surfaces $\int dx\, dg\, e^{-S(g,x)}$ which is both local in parameter space and invariant under reparametrizations, and whose action

$$S(g,x) = \int d^2z\sqrt{g}\,\left(\mu^2 + \lambda R^{(2)} + g^{ab}\partial_a x^\mu \partial_b x_\mu + \cdots\right) \tag{1.1.1}$$

can be expanded in powers of the two dimensional derivatives.

The reparametrizations of the world surface act as a gauge group in the functional integral over surfaces. A natural gauge fixing condition is $g_{ab} = \rho(z,\bar z)g^{(m)}_{ab}$. where $g^{(m)}_{ab}$ is some background metric. In this gauge the integral over metrics becomes an integral over the conformal factors $\rho(z,\bar z)$ and over the conformal classes of metrics, represented by a collection of background metrics $g^{(m)}_{ab}$ which are indexed by a finite number of moduli $m = (m^1, m^2, \ldots)$. The conformal classes of two dimensional surfaces are the Riemann surfaces.

A Fadeev-Popov determinant is introduced into the functional integral because of the gauge fixing. The determinant is calculated by a Grassmann integral over conjugate ghost fields $b(z), c(z)$ which are chiral fermion fields on the world surface, of spins 2 and -1 respectively, corresponding to variations of the gauge condition and to infinitesimal reparametrizations of the world surface. The gauge fixed functional integral has the form

$$\begin{gathered}\sum_{topologies} e^{-4\pi\lambda(Euler\#)} \int_{moduli} dm \int_{fields} dx\, db\, dc \\ \exp\left\{-\int d^2z\left(\partial x\bar\partial x + b\bar\partial c + \bar b\partial\bar c\right)\right\}\end{gathered} \tag{1.1.2}$$

when the action is written in conformal coordinates $(z,\bar z)$ with $g^{(m)}_{zz} = 0$, $g^{(m)}_{z\bar z} = \frac{1}{2}$, and interactions of dimension > 2 are dropped from the two dimensional action because they are irrelevant (nonrenormalizable) in the continuum limit of parameter space. The coefficient $e^{-4\pi\lambda}$ is the string coupling constant. In the sum over surfaces, the Euler number indexes the string loop expansion.

Note that the conformal factor ρ is left out of 1.1.2. The classical action in 1.1.2 is independent of ρ, but this conformal invariance does not persist in the two dimensional quantum field theory of x^μ, b, c if there is a net conformal anomaly, which always happens except in the critical dimension $d = 26$. In the critical dimension ρ drops from the surface dynamics, leaving 1.1.2. In noncritical spacetime dimensions the ρ field must be dynamical, but as yet no acceptable quantum dynamics for ρ has been formulated for $2 \leq d \leq 25$.

In the critical dimension $d = 26$, the vanishing of the conformal anomaly means that the x^μ, b, c quantum field theory depends only on the conformal class of the surface, and its partition function transforms as a density on moduli space, so that the integral 1.1.2 over moduli makes sense (locally in moduli space).

1.2 Scattering amplitudes

To calculate a Greens function of N strings, let the sum over topologies in 1.1.2 range over surfaces with N boundary components and fixed wave functionals on the boundary values, representing N external strings. The boundaries can be pictured as holes in a compact Riemann surface without boundary. The radii of the holes are N of the (real) moduli of the original surface. The integrals over radii near zero produce poles in the external spacetime momenta, and the N point scattering amplitudes are the residues at these poles. The amplitudes can thus be calculated as functional integrals over surfaces with N infinitesmal holes, and particular boundary conditions at the holes. The locations of the holes are the remaining moduli for the boundaries. The infinitesmal holes can be represented as local quantum fields on the world surface, called *vertex operators.*

The scattering amplitudes have the form

$$G(p_1, \ldots, p_B) = \sum_{topologies} \int_{moduli} dm \int d^2z_1 \cdots d^2z_N \; Z(m) \left\langle V_1(p_1, z_1) \cdots V_N(p_N, z_N) \right\rangle_m \quad (1.2.1)$$

where $Z(m)$ is the partition function of the x^μ, b, c system (including the string coupling) on the compact Riemann surface without boundary whose moduli are m, and $\langle \cdots \rangle_m$ is the correlation function on the surface. The contribution from the simplest topology, the two sphere, gives the tree amplitudes.

The reparametrization invariance of the original functional integral means that the integrals over the z_i should be conformally invariant, which implies that the vertex operators should have quantum dimension 1 in z and also 1 in $\bar{z}$. The simplest examples are the exponentials $e^{ik\cdot x}$, $\frac{1}{2}k^2 = 1$, which are the vertex operators for the tachyonic states of the bosonic string. The duality properties of the string amplitudes are manifest in 1.2.1; the factorization of amplitudes is expressed by the operator product expansion of vertex operators. For example, the leading singularity in the operator product of tachyon vertex operators,

$$e^{ik\cdot x}(z_1)\, e^{ik'\cdot x}(z_2) \sim (z_1 - z_2)^{k\cdot k'}\, e^{i(k+k')\cdot x}(z_2)\,, \quad (1.2.2)$$

yields a tachyon pole in the intermediate momentum $k + k'$ at $\frac{1}{2}(k + k')^2 = 1$, coming from the integration over z_1 near z_2.

1.3 Unitarity

Scattering amplitudes calculated by this prescription are manifestly Lorentz covariant, but not obviously unitary. The demonstration of unitarity has two parts. First, the tree amplitudes must be shown free of ghosts; and, second, the sum over surfaces of nontrivial topology must be shown to produce loop corrections consistent with the tree amplitudes.

In the covariant first quantization, the states of the two dimensional field theory on the cylinder, subject to the residual constraints of reparametrization invariance, are the physical states of a single string. This identification of the states is formally apparent the Schrödinger picture of the two dimensional field theory on the cylinder, where the states are wave functionals on circles in spacetime. The Hilbert space of the two dimensional field theory has an indefinite metric, because of the Lorentz signature of spacetime and the Fermi statistics of the ghosts.

The original covariant approach worked only with the matter fields x^μ. The physical states are defined by the gauge conditions $L^x_{+n}\,|phys\rangle = 0$ and the mass-shell condition $(L^x_0 - 1)\,|phys\rangle = 0$, in terms of the Virasoro operators L^x_n generating the residual gauge algebra of conformal transformations. The gauge and mass-shell conditions are equivalent to the conformal invariance of the integral over locations of vertex operators in 1.1.2. The metric on the physical states can be shown nonnegative, the unphysical states can be shown to decouple from the physical states in tree amplitudes The problem with the classical approach is that the matter sector by itself is conformally anomalous, so its partition function must be corrected to become a density on moduli space, and the unphysical states do not manifestly decouple in the loop corrections.

1.4 The BRST current

An alternative approach uses the BRST quantization of the world surface[9]. The fermionic BRST charge is defined in the combined matter-ghost system, satisfying $Q^\dagger_{BRST} = Q_{BRST}$, $Q^2_{BRST} = 0$. The physical states are the invariant states, $Q_{BRST}\,|phys\rangle = 0$, modulo the null invariant states $Q_{BRST}\,|state\rangle$. The metric on physical states can be shown to be positive, if only by showing that the two definitions of the physical states are equivalent.

The decoupling of physical states can be shown by writing the BRST variations of fields as contour integrals of a conformal (chiral) current:

$$(\delta_{BRST}\Phi)(w) = \frac{1}{2\pi i}\oint_{C_w} dz\; j_{BRST}(z)\;\Phi(w) \tag{1.4.1}$$

where C_w is a simple contour surrounding w. The physical vertex operators $V(z)$ are the BRST invariant fields, where invariance means that $(\delta_{BRST}V)(z)$ is a total derivative, so that the integral over z vanishes. Equation 1.4.1 is equivalent to an operator product formula. This is a local property of the conformal field theory which obtains inside all correlation functions, on all surfaces.

The decoupling of null states can be shown by considering the correlation function of $N-1$ physical vertex operators and one $BRST$-null vertex operator. Write the null vertex in the form 1.4.1. Deform the contour to surround the each of the $N-1$ physical vertices, giving a sum of total derivatives, each of which vanishes after integration over the locations of the vertices. This argument depends on j_{BRST} being conserved on any surface, which means that it must be a conformal current, a conformal field of weight $(1,0)$, i.e., of weight 1 in z and 0 in $\bar{z}$. The expectation values of vertex operators on arbitrary surfaces must be $BRST$ invariant, which means checking that contour integrals of j_{BRST} vanish up to total derivatives, as long as the contour surrounds no vertex operators.

One way to show that the loop corrections are consistent with the tree amplitudes is to use a representation of the Riemann surfaces in which all the curvature of the intrinsic metric $g^{(m)}_{ab}$ is concentrated at isolated points[10]. The functional integrals are explicit sums over states, and the moduli play the role of Schwinger parameters. Since the conformal symmetry is anomaly-free, any representation of the surfaces is equivalent to any other, so a more suitable representation can be used for calculation (for example, letting the $g^{(m)}_{ab}$ be the constant curvature metrics). It would be good to have a less mechanical method for showing unitarity in the BRST formalism, perhaps employing relations between functional integrals over Riemann surfaces of different topologies.

1.5 Supersymmetric strings

The covariant first quantisation of supersymmetric strings[11] follows the pattern of the bosonic theory. The main difference is that the gauge symmetry of the fermionic world surface is two dimensional superconformal invariance. The integral over Riemann surfaces turns into an integral over super Riemann surfaces (see section 2).

These notes concentrate on the local properties of the superconformal field theory of the superstring world surface and on the calculation of tree amplitudes. The main problem is to construct the superconformal BRST current and the algebra of BRST invariant vertex operators for the bosonic and fermionic modes of the string[4,5]. Given the superconformal covariance of the BRST current and the vertex operators, the remaining issues in the calculation of loop corrections are construction of BRST invariant expectation values for arbitrary surfaces, and proof of the finiteness of the integrals over moduli[12].

This covariant approach is not manifestly supersymmetric. The supersymmetry of the amplitudes is demonstrated by constructing conformal currents $j_\alpha(z)$ whose charges generate the spacetime supersymmetry algebra[4,5]. Contour integrals of $j_\alpha(z)$ give the supersymmetry variations of vertex operators:

$$(\delta_\alpha V)(w) = \frac{1}{2\pi i}\oint_{C_w} dz\, j_\alpha(z)\, V(w)\,. \tag{1.5.1}$$

Integrating j_α around a contractible contour gives zero. On the other hand, a trivial contour can be deformed into a sum of contours, one surrounding each vertex operator. Thus the supersymmetry variation of any correlation function vanishes, even before the integral over the positions of vertex operators and over moduli. Therefore the amplitudes are supersymmetric. In calculating loop corrections, the crucial issue is whether contour integrals of j_α vanish for contractible contours on non-simply connected surfaces.

2. Super Riemann Surfaces

This section is about tensor analysis on superconformal manifolds of one complex dimension, the super Riemann surfaces. In the covariant first quantization, the world surfaces of fermionic strings are super Riemann surfaces. The point of view taken in this section has evolved from references 13 and 2-5.

2.1 Super coordinates

A one dimensional complex supermanifold is locally described by an ordinary complex coordinate z and an anticommuting coordinate θ, $\theta^2 = 0$, making a complex super coordinate $\mathbf{z} = (z, \theta)$. The superderivative is the square root of the ordinary derivative:

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z} \qquad D^2 = \frac{\partial}{\partial z}\,. \tag{2.1.1}$$

A super analytic function is a solution of $\overline{D}f = 0$, and consists of two ordinary analytic functions: $f(\mathbf{z}) = f_0(z) + \theta f_1(z)$, with f_0 commuting with θ and f_1 anticommuting with θ.

2.2 Superconformal transformations

A super analytic map $\mathbf{z} \to \tilde{\mathbf{z}}(\mathbf{z}) = (\tilde{z}(z,\theta), \tilde{\theta}(z,\theta))$ transforms the superderivative according to

$$D = (D\tilde{\theta})\widetilde{D} + (D\tilde{z} - \tilde{\theta}D\tilde{\theta})\widetilde{D}^2\,. \tag{2.2.1}$$

A super analytic map is called a superconformal transformation when the superderivative transforms homogeneously:

$$D = (D\tilde{\theta})\,\widetilde{D}\,, \tag{2.2.2}$$

i.e., if

$$D\tilde{z} - \tilde{\theta}D\tilde{\theta} = 0\,. \tag{2.2.3}$$

2.3 Super Riemann surfaces

It follows from 2.2.2 that a composition of superconformal transformations, $\mathbf{s} \to \tilde{\mathbf{s}} \to \tilde{\tilde{\mathbf{s}}}$, is also a superconformal transformation. A super Riemann surface can thus be defined as a collection of superconformal coordinate patches, that is, as a set of super coordinate neighborhoods patched together by superconformal transformations. A super coordinate neighborhood is just an ordinary neighborhood in z.

Consider special super Riemann surfaces for which the patching transformations are all of the form $\mathbf{s} \to (\tilde{z}, \tilde{\theta}) = (\tilde{z}(z), \mu(z)\,\theta)$, so that the ordinary patching transformations $z \to \tilde{z}$ make an ordinary Riemann surface, and θ behaves like an ordinary tensor. The superconformal condition 2.2.2 becomes $\partial\tilde{z} = \mu(z)^2$. Thus the $\mu(z)$ are transition functions for a line bundle over the ordinary Riemann surface whose square is the canonical line bundle. The canonical line bundle is the bundle of $(1,0)$ forms; its transition functions are $\partial\tilde{z}$. A square root of the canonical bundle is called a bundle of half-forms. On Riemann surfaces half-forms are spinors. Thus ordinary Riemann surfaces with spin structures are special cases of super Riemann surfaces.

2.4 Superconformal tensor fields

In a composition of superconformal transformations, $\mathbf{s} \to \tilde{\mathbf{s}} \to \tilde{\tilde{\mathbf{s}}}$, the super jacobians obey

$$D\tilde{\tilde{\theta}} = (D\tilde{\theta})\,(\widetilde{D}\tilde{\tilde{\theta}})\,. \tag{2.4.1}$$

This composition law allows a super differential $\mathbf{ds}$ to be defined by the transformation law

$$\mathbf{d\tilde{s}} = (D\tilde{\theta})\,\mathbf{ds} \qquad \frac{\mathbf{d\tilde{s}}}{\mathbf{ds}} = D\tilde{\theta}\,. \tag{2.4.2}$$

Then superconformal tensor fields $\phi(\mathbf{s})$, can be defined by the condition that $\phi(\mathbf{s})\mathbf{ds}^{2h}$ be superconformally covariant, where h is called the *weight* or *dimension* of ϕ. This means that

$$\phi(\mathbf{s})\mathbf{ds} = \tilde{\phi}(\tilde{\mathbf{s}})\mathbf{d\tilde{s}}, \qquad \phi(\mathbf{s}) = \tilde{\phi}(\tilde{\mathbf{s}})(D\tilde{\theta})^{2h}\,. \tag{2.4.3}$$

The superconformal tensor fields are the analogues of ordinary conformal tensor fields $\phi(z)$, of weight or dimension h, for which $\phi(z)dz^h$ is conformally covariant, i.e., for which $\phi(z) = \tilde{\phi}(\tilde{z})(d\tilde{z}/dz)^h$.

The component fields of $\phi(\mathbf{s}) = \phi_0(z) + \theta\phi_1(z)$ consist of an ordinary conformal field of weight h, ϕ_0, and an ordinary conformal field of weight $h+1/2$, ϕ_1. When ϕ is a quantum field, its Fermi/Bose statistics are the statistics of ϕ_0, opposite to the statistics of ϕ_1.

Globally defined superconformal tensor fields have weights which are either integer or half-integer. But in what follows it will be useful to manipulate quantum tensor fields whose weights are not integer or half-integer, and so can only be defined locally. Globally defined fields can be constructed as products of locally defined fields.

2.5 Superconformal vector fields

Infinitesimal superconformal transformations $\mathbf{s} \to \tilde{\mathbf{s}} = \mathbf{s} + \delta\mathbf{s}(\mathbf{s})$, transform superconformal tensor fields by the infinitesimal version of 2.4.3,

$$\phi = \tilde{\phi} + \delta_v\phi \qquad \delta_v\phi = (v\partial + \tfrac{1}{2}DvD + h\partial v)\,\phi \qquad v(\mathbf{s}) = \delta z + \theta\,\delta\theta\,, \tag{2.5.1}$$

written in terms of the superconformal vector field $v(\mathbf{s})$, which is itself a superconformal tensor field of weight -1. The commutation relations of the Lie algebra of infinitesimal superconformal transformations is the same as the commutation relations of the Lie derivatives δ_v:

$$\delta_{[v,w]} = [\delta_v,\, \delta_w]\,, \qquad [v,\, w] = v\partial w - w\partial v + \tfrac{1}{2}DvDw\,. \tag{2.5.2}$$

2.6 Super contour integrals

Integration over the anticommuting coordinate θ is given by

$$\int d\theta\, \theta = 1\,, \qquad \int d\theta\, 1 = 0\,. \tag{2.6.1}$$

The super contour integral is the ordinary contour integral over z combined with the integral over θ , that is,

$$\oint_C d\mathbf{z}\, \omega(\mathbf{z}) = \oint_C dz \int d\theta\, \omega(\mathbf{z}) = \oint_C dz\, \omega_1(z)\,. \tag{2.6.2}$$

The use of the super differential $d\mathbf{z}$ is justified by considering the behavior of the super contour integral under superconformal transformations:

$$\oint_{\tilde{C}} d\tilde{\mathbf{z}}\, \tilde{\omega}(\tilde{\mathbf{z}}) = \oint_C d\mathbf{z}\, \tilde{\omega}(\tilde{\mathbf{z}}(\mathbf{z}))\,. \tag{2.6.3}$$

A dimension $\frac{1}{2}$ superconformal tensor field is called a ***superconformal current***. By 2.6.3, the super contour integrals of superconformal currents are invariant under superconformal transformations. Also, if $f(\mathbf{z})$ is a regular super analytic function in a domain bounded by C, then $\oint_C d\mathbf{z}\, Df = 0$.

2.7 Indefinite integrals and Cauchy formulas

Define the indefinite integral

$$f(\mathbf{z}_1, \mathbf{z}_2) = \int_{\mathbf{z}_2}^{\mathbf{z}_1} d\mathbf{z}\, \omega(\mathbf{z}) \tag{2.7.1}$$

by

$$f(\mathbf{z}_2, \mathbf{z}_2) = 0\,, \qquad D_1 f(\mathbf{z}_1, \mathbf{z}_2) = \omega(\mathbf{z}_1)\,. \tag{2.7.2}$$

The natural coordinates for super translation invariant functions on the plane are

$$\begin{aligned} &\theta_{12} = \theta_1 - \theta_2 = \int_{\mathbf{z}_2}^{\mathbf{z}_1} d\mathbf{z} \qquad && z_{12} = z_1 - z_2 - \theta_1\theta_2 = \int_{\mathbf{z}_2}^{\mathbf{z}_1} d\mathbf{z} \int_{\mathbf{z}_2}^{\mathbf{z}} d\mathbf{z}' \\ &D_1 z_{12} = \theta_{12} = D_2 z_{12} && D_1\theta_{12} = 1 = -D_2\theta_{12}\,. \end{aligned} \tag{2.7.3}$$

A super analytic function $f(\mathbf{z})$ can be expanded in a power series around $\mathbf{z}_2$:

$$\begin{aligned} f(\mathbf{z}_1) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_{12})^n\, \partial_2^n\, (1 + \theta_{12} D_2)\, f(\mathbf{z}_2) \\ &= f(\mathbf{z}_2) + \theta_{12} D_2 f + z_{12}\partial_2 f + \cdots\,. \end{aligned} \tag{2.7.4}$$

The super Cauchy formulas are

$$\frac{1}{2\pi i}\oint_{C_2} d\mathbf{z}_1\, z_{12}^{-n-1} = 0 \qquad \frac{1}{2\pi i}\oint_{C_2} d\mathbf{z}_1\, \theta_{12}\, z_{12}^{-n-1} = \delta_{n,0} \tag{2.7.5}$$

where C_2 is a simple contour winding once around z_2. Combining 2.7.4 and 2.7.5,

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_2} d\mathbf{z}_1\, f(\mathbf{z}_1)\, \theta_{12}\, z_{12}^{-n-1} &= \frac{1}{n!}\, \partial_2^n\, f(\mathbf{z}_2) \\ \frac{1}{2\pi i}\oint_{C_2} d\mathbf{z}_1\, f(\mathbf{z}_1)\, z_{12}^{-n-1} &= \frac{1}{n!}\, \partial_2^n\, D_2\, f(\mathbf{z}_2)\,. \end{aligned} \tag{2.7.6}$$

2.8 Periods and moduli

On a topologically nontrivial super Riemann surface, the indefinite integral 2.7.1 is defined only up to the super periods of ω, $\int_{\mathbf{z}}^{\mathbf{z}} d\mathbf{z}'\, \omega(\mathbf{z}')$. The theory of super Jacobian varieties and super theta functions should parallel and organize the theory of ordinary theta functions.

The super moduli of super Riemann surfaces are the variations of the patching transformations which define the super Riemann surface, modulo superconformal transformations of the coordinate neighborhoods[14]. Infinitesimally the moduli are the superconformal vector fields on the overlaps of coordinate neighborhoods, modulo differences of superconformal vector fields on the neighborhoods themselves. This describes the first cohomology group of the super Riemann surface with coefficients in the superconformal vector fields. This cohomology group is realized as the $(-\frac{1}{2},\frac{1}{2})$ forms modulo the image of $\bar{D}$ acting on the $(-\frac{1}{2},0)$ forms. Super integration on the super Riemann surface identifies the dual space of the infinitesimal moduli space with space of superconformal tensors of weight $\frac{3}{2}$ (see equation 3.2.5 below). For genus $g>1$, the Riemann-Roch theorem (see section 5.8 below) and the vanishing theorem for superconformal fields of negative weight, give $2(g-1)$ as the number of weight $\frac{3}{2}$ conformal fields and $3(g-1)$ as the number of weight 2 conformal tensor fields. Therefore the super moduli space has $3(g-1)$ ordinary complex dimensions and $2(g-1)$ anticommuting complex dimensions.

The bosonic dimension of the super moduli space is exactly the dimension of the moduli space of ordinary Riemann surfaces with spin structure. Therefore the super moduli space consists of the ordinary moduli space of Riemann surfaces with spin structures, plus $2(g-1)$ fermionic coordinates lying in a vector bundle over the ordinary moduli space.

For the generic compact Riemann surface of genus g there are 2^{2g} spin structures, corresponding to all the possible sign changes of half-forms transported around the $2g$ non-trivial cycles on the surface. When the corresponding Riemann surface has nontrivial automorphisms, there are fewer spin structures. The space of spin structures is thus a 2^{2g}-sheeted covering of ordinary moduli space, branched at the singular points. This covering can be realized by the 2^{2g} first order theta functions of integer characteristic which are the partition functions of free Majorana-Weyl fermions in the various spin structures.

3. Superconformal Field Theory

3.1 Conformal fields and operator product expansions

In a conformal field theory[8,7,15,16] the *primary* or *conformal* fields are conformal tensors of weight $(h,\bar{h})$, or, equivalently, scaling dimension $h+\bar{h}$ and spin $h-\bar{h}$. Chiral fields have $\bar{h}=0$ (or $h=0$), so they are analytic fields (or antianalytic fields). The operator product expansions of conformal fields,

$$\phi_i(z_1)\,\phi_j(z_2)\sim\sum_k(z_1-z_2)^{h_k-h_i-h_j}\,C_{ijk}\,\phi_k(z_2)\,, \tag{3.1.1}$$

stand for identities which obtain in every correlation function of ϕ_i and ϕ_j:

$$\langle\cdots\phi_i(z_1)\,\phi_j(z_2)\cdots\rangle=\sum_{k,n}(z_1-z_2)^{h_k-h_i-h_j+n}C_{ijk}\,\langle\cdots[\phi_k]_n(z_2)\cdots\rangle\;, \tag{3.1.2}$$

where the notation $[\phi_k]_n$, $[\phi_k]_0=\phi_k$, stands for a sum over the descendent fields of ϕ_k on level n with coefficients which depend only on the weights $h_{i,j,k}$ (see Shenker's lectures[1] for an explanation of descendent fields). The identity 3.1.2 holds for z_1 near z_2 and, by analytic continuation, for all z_1 and z_2. For nonchiral fields the sum in 3.1.2 should include factors $(\bar{z}_1-\bar{z}_2)^{h_k-h_i-h_j+m}$.

There is an expectation value $\langle\cdots\rangle_m$ for each Riemann surface, but the operator product identities are independent of the surface. The operator product expansions are local properties of the conformal field theory. They can be regarded as defining the quantum field theory, since 3.1.2 can be used to reconstruct the correlation functions from the operator product expansions.

In a superconformal field theory the primary superfields, the superconformal fields, are superconformal tensor fields on super Riemann surfaces. A two dimensional quantum field theory can be invariant under superconformal transformations in z alone or in both z and $\bar{z}$. It suffices to consider z alone, since the discussion of local superconformal invariance in $\bar{z}$ is parallel and independent.

The partition function $Z(m)$ depends on the super moduli parametrizing the super Riemann surfaces. The superconformal fields obey operator product expansions analogous to 3.1.1, with z_{12} and θ_{12} taking the place of $z_1 - z_2$. In power counting, θ_{12} counts as $z_{12}^{1/2}$.

3.2 The super stress-energy tensor

The fundamental quantum field in a superconformal field theory is the super stress-energy tensor

$$T(\mathbf{z}) = T_F(z) + \theta T_B(z). \tag{3.2.1}$$

$T(\mathbf{z})$ is a chiral superfield of dimension 3/2. T_B is the ordinary stress-energy tensor (dimension $(2,0)$); T_F is its super partner (dimension $(3/2,0)$). $T(\mathbf{z})$ generates the superconformal transformations 2.5.1 by

$$\delta_v \phi(\mathbf{z}_2) = \frac{1}{2\pi i} \oint_{C_2} d\mathbf{z}_1\, v(\mathbf{z}_1)\, T(\mathbf{z}_1)\, \phi(\mathbf{z}_2)\,, \tag{3.2.2}$$

where C_2 is a simple contour winding once around z_2. As usual, this identity holds within correlation functions.

By the super Cauchy formulas 2.7.6, the transformation law 3.2.2 is equivalent to the operator product expansion

$$T(\mathbf{z}_1)\, \phi(\mathbf{z}_2) \sim \frac{\theta_{12}}{z_{12}^2}\, h\, \phi(\mathbf{z}_2) + \frac{1/2}{z_{12}} D_2\phi + \frac{\theta_{12}}{z_{12}} \partial_2\phi + \cdots \tag{3.2.3}$$

where the ommitted terms are nonsingular. Only the singular part of the operator product expansion contributes to the contour integral.

The super stress-energy tensor is itself an anomalous superconformal field of weight 3/2:

$$\begin{aligned} T(\mathbf{z}_1)\, T(\mathbf{z}_2) \;&\sim\; \frac{\hat c}{4}\frac{1}{z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{z_{12}^2} T(\mathbf{z}_2) + \frac{1}{2}\frac{1}{z_{12}} D_2 T + \frac{\theta_{12}}{z_{12}} \partial_2 T \\ \delta_v T \;&=\; \left(v\partial + \tfrac{1}{2}(Dv)D + \tfrac{3}{2}\partial v\right) T + \tfrac{1}{8}\hat c\, \partial^2 D v\,. \end{aligned} \tag{3.2.4}$$

There are at least two approaches to deriving 3.2.4. In the first approach, the subleading singularities in $T(\mathbf{z}_1)\,T(\mathbf{z}_2)$ are determined by 3.2.2, which fixes the dimension of $T(\mathbf{z})$ to be 3/2, which fixes the two point function, which determines the leading singularity, up to a constant $\hat c$. The second approach is to determine the form of 3.2.4, up to the arbitrary number $\hat c$, by symmetries of the quantum field theory on the plane: Euclidean invariance, supersymmetry and scale invariance. The second argument applies especially to superymmetric critical phenomena[16,3]. The coefficient $\hat c$ of the anomaly, the central term in the operator product, is the fundamental characteristic number of a two dimensional superconformal field theory.

The super stress-energy tensor represents the infinitesimal variations of the super moduli:

$$\frac{\partial}{\partial m^i} \log Z(m) = \int d^2\mathbf{z}\; \bar f_i(z,\bar z)\; \langle T(\mathbf{z})\rangle_m + \text{c.c.} \tag{3.2.5}$$

where $\bar f_i$ is a $(-1,\frac{1}{2})$ form representing the infinitesmal variation of m. One way to calculate $\langle T(\mathbf{z})\rangle_m$ is to take the expectation value of 3.2.4:

$$\langle T(\mathbf{z}_1)\, T(\mathbf{z}_2)\rangle_m \sim \tfrac{1}{4}\hat c\, z_{12}^{-3} + \tfrac{3}{2}\theta_{12} z_{12}^{-2}\, \langle T(\mathbf{z}_2)\rangle_m\,. \tag{3.2.6}$$

3.3 The global superconformal group $\widehat{\mathrm{SL}}_2$

The effect of a finite superconformal transformation $\mathbf{z} \to \tilde{\mathbf{z}}$ on the super stress-energy tensor is computed by requiring the operator product expansion 3.2.4 to hold in both $\mathbf{z}$ and $\tilde{\mathbf{z}}$:

$$T(\mathbf{z}) = \tilde T(\tilde{\mathbf{z}})(D\tilde\theta)^3 + \tfrac{1}{4}\hat c S(\mathbf{z},\tilde{\mathbf{z}}) \tag{3.3.1}$$

where $S(\mathbf{z},\tilde{\mathbf{z}})$ is the super Schwarzian derivative

$$S(\mathbf{z},\tilde{\mathbf{z}}) = \frac{D^4\tilde\theta}{D\tilde\theta} - 2\frac{D^3\tilde\theta}{D\tilde\theta}\frac{D^2\tilde\theta}{D\tilde\theta} = \partial(\log D\tilde\theta)\, D(\log \partial(-1/D\tilde\theta))\,. \tag{3.3.2}$$

On the sphere, the globally defined superconformal vector fields are of the form

$$v(\mathbf{z}) = (v_{-1} + v_0 z + v_1 z^2) + \theta(\hat{v}_{-1/2} + \hat{v}_{1/2} z)\,, \tag{3.3.3}$$

forming the super Lie algebra Osp(2, 1). The correlation functions of superconformal fields are Osp(2, 1) invariant, because at large distance the correlation functions of the super stress-energy tensor are of the form of its two-point function,

$$\langle T(\mathbf{z})\cdots\rangle \underset{z\to\infty}{\sim} O(z^{-3} + \theta z^{-4})\,, \tag{3.3.4}$$

which implies that contour integrals of $v(\mathbf{z})T(\mathbf{z})$ vanish at infinity for the vector fields 3.3.3.

The vector fields 3.3.3 are exactly the solutions of $\partial^2 D v = 0$, which is the infinitesmal form of the super Schwarzian derivative 3.3.2. By equation 3.3.1, the super Schwarzian derivative obeys the composition law

$$S(\mathbf{z}, \tilde{\tilde{\mathbf{z}}}) = S(\mathbf{z}, \tilde{\mathbf{z}}) + (D\tilde{\theta})^3 S(\tilde{\mathbf{z}}, \tilde{\tilde{\mathbf{z}}}) \tag{3.3.5}$$

implying that $S(\mathbf{z}, \tilde{\mathbf{z}}) = 0$ for all the global superconformal transformations which can be made from successive infinitesimal transformations, i.e., the connected component of the identity in the global superconformal super group. The solutions of $S(\mathbf{z}_1, \tilde{\mathbf{z}}_1) = 0$ are

$$\tilde{\theta}_1 = \theta_0 + \theta_{12}/z_{12} \qquad \tilde{z}_1 = z_0 + (\alpha + \theta_1\theta_0)/z_{12} \tag{3.3.6}$$

where the parameters of the transformation are $\mathbf{z}_0 = (z_0, \theta_0)$, $\mathbf{z}_2 = (z_2, \theta_2)$ and α. This group is $\widehat{SL}_2$, a supersymmetric extension of the ordinary global conformal group SL_2 of fractional linear maps $z \to (az+b)/(cz+d)$.

The Lie algebra of ordinary conformal vector fields on the cylinder, or the punctured plane, is the the complexification of the Lie algebra of $\mathrm{Diff}(S^1)$, which is the group of diffeomorphisms of the circle. $\mathrm{Diff}(S^1)$ can be identified with

the conformal transformations of the punctured plane which satisfy the reality condition $\tilde{z}(1/z^*) = 1/\tilde{z}(z)^*$.

The superconformal vector fields on cylinder or the punctured plane form two super Lie algebras, the Ramond and Neveu-Schwarz algebras, corresponding to the two spin structures (boundary conditions) on the cylinder. The superconformal algebras are the complexifications of the super Lie algebras of the two groups of super diffeomorphisms of the circle, $\widehat{\mathrm{Diff}}_{\pm}(S^1)$. The distinction between the two super groups only appears when there is a strong enough topology on the group to distinguish the two boundary conditions on the circle which define trivial and nontrivial $O(1)$ spinors. The super Schwarzian derivative is the globally invariant generator of the second cohomology group of the super group $\widehat{\mathrm{Diff}}_{\pm}(S^1)$, just as the ordinary Schwarzian derivative is the SL_2 invariant generator for the two cohomology of $\mathrm{Diff}(S^1)$[17].

3.4 Operator interpretation

The simplest operator interpretation of a conformal field theory is given by the radial quantization. It is constructed from the correlation functions on the sphere or, equivalently, on the plane or the infinite cylinder. If $z = e^w = e^{\tau + i\sigma}$ is the standard complex coordinate for the plane, so that w is the standard coordinate on the cylinder, then correlation functions on the sphere or the plane or the cylinder are interpreted as vacuum expectation values of τ-ordered products

$$\langle \phi(z_1)\cdots\rangle = \langle 0|\,\tau\{\phi(z_1)\cdots\}\,|0\rangle \tag{3.4.1}$$

where the operator valued fields are τ-ordered by putting fields of large $|z|$ to the left and fields of small $|z|$ to the right.

On the cylinder there are two spin structures, given by periodic or antiperiodic boundary conditions in the σ direction. Thus the Hilbert space of the radial quantization divides into two sectors; the Neveu-Schwarz (NS) sector, in which the spinor fields are single valued on the plane (but double valued on the cylinder because of the factor $dz^{1/2}$) and the Ramond (R) sector in which the spinor fields are single valued on the cylinder (but double valued on the plane). The superconformal fields are block diagonal in the NS $\oplus$ R decomposition of the Hilbert space. Vacuum expectation values are single valued in the plane, so the vacuum state $|0\rangle$ is in the NS sector.

A highest weight state $|S\rangle$ in the R sector is an ordinary conformal highest weight state, so it corresponds to some ordinary conformal field $S(z)$. This conformal field is called a *spin* field. It is block off-diagonal in the NS $\oplus$ R decomposition. A spin field $S(z)$, acting on the vacuum in the NS sector, creates the highest weight state $|S\rangle = S(0)\,|0\rangle$ in the R sector.

Integrating over all super Riemann surfaces includes summing over all spin structures. On the torus there are four spin structures, two boundary conditions in each of two directions. The partition function is a Hilbert space trace. Picture one direction as euclidean time and the other as space. The sum over spatial boundary conditions is the sum over NS and R sectors in the trace. Summing over boundary conditions for the spinor fields introduces a projection operator $\frac{1}{2} + \frac{1}{2}\Gamma$ in the trace, where the chirality operator Γ commutes with integer spin fields and anticommutes with half-integer spin fields. On surfaces of genus > 1 the sum over spin structures provides in each loop a sum over R and NS sectors and a chiral projection.

3.5 Superconformal generators

An infinitesmal superconformal transformation of the punctured plane corresponds to a superconformal vector field $v(\mathbf{z})$, analytic away from the origin. The transformation is generated by the operator

$$T_{[v]} = \frac{1}{2\pi i} \oint_{C_0} d\mathbf{z}\; v(\mathbf{z}) T(\mathbf{z}) \tag{3.5.1}$$

where C_0 winds once around the origin, making a "space-like hypersurface" in the radial quantization. Commutation relations of operators can be represented in terms of τ-ordered products,

$$\begin{aligned} [T_{[v]},\, \phi(\mathbf{z}_2)] &= \frac{1}{2\pi i} \oint_{C_{0,2}-C_0} d\mathbf{z}_1\; v(\mathbf{z}_1)\, T(\mathbf{z}_1)\, \phi(\mathbf{z}_2) \\ &= \frac{1}{2\pi i} \oint_{C_2} d\mathbf{z}_1\; v(\mathbf{z}_1)\, T(\mathbf{z}_1)\, \phi(\mathbf{z}_2) \\ &= \delta_v \phi(\mathbf{z}_2) \end{aligned} \tag{3.5.2}$$

where $C_{0,2}$ winds around z_2 and the origin, e.g. $|z| > |z_2|$, and C_0 winds around the origin but not z_2, e.g. $|z_2| > |z| > 0$. The deformation of contours is justified by the analyticity of the τ-ordered products for $z_2 \neq 0, z_1$. The contours and deformations can always be chosen so as to miss any other quantum fields which might be present in the correlation function.

The contour integral argument shows that the commutation relations of the generators $L_{[v]}$ are encoded in the singular part of the operator products of $T(\mathbf{z})$. This argument is quite general. The singular parts of the operator product expansions of analytic (chiral) fields, with themselves and with other quantum fields, are equivalent to their commutation relations. The equivalence is realized by the contour argument. Even for nonchiral fields, the singular operator product expansions are equivalent to commutation relations, but in the absence of analyticity the contour integral must be replaced by a principal part interpretation of the singularity. The advantages of operator products is that they are independent of the Hilbert space interpretation, they obtain on arbitrary two dimensional surfaces, and they are easily calculated from functional integral representations of correlation functions.

3.6 Operator products of component fields

Expanding 3.2.3, 3.2.4 in component fields gives

$$
\begin{aligned}
T_B(z_1)\,T_B(z_2) &\sim \frac{3\hat{c}/4}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}T_B(z_2) + \frac{1}{z_1-z_2}\partial_2 T_B \\
T_B(z_1)\,T_F(z_2) &\sim \frac{3/2}{(z_1-z_2)^2}T_F(z_2) + \frac{1}{z_1-z_2}\partial_2 T_F \\
T_F(z_1)\,T_F(z_2) &\sim \frac{\hat{c}/4}{(z_1-z_2)^3} + \frac{1/2}{z_1-z_2}T_B(z_2) \\
T_B(z_1)\,\phi_0(z_2) &\sim \frac{h}{(z_1-z_2)^2}\phi_0(z_2) + \frac{1}{z_1-z_2}\partial_2\phi_0 \\
T_B(z_1)\,\phi_1(z_2) &\sim \frac{h+1/2}{(z_1-z_2)^2}\phi_1(z_2) + \frac{1}{z_1-z_2}\partial_2\phi_1 \\
T_F(z_1)\,\phi_0(z_2) &\sim \frac{1/2}{z_1-z_2}\phi_1(z_2) \\
T_F(z_1)\,\phi_1(z_2) &\sim \frac{h}{(z_1-z_2)^2}\phi_0(z_2) + \frac{1/2}{z_1-z_2}\partial_2\phi_0\,.
\end{aligned}
\qquad (3.6.1)
$$

3.7 Mode expansions

The fields expand in Laurent (Fourier) series:

$$
\begin{aligned}
T_F(z) &= \sum_n z^{-n-3/2}\,\tfrac{1}{2}G_n & \phi_0(z) &= \sum_n z^{-n-h}\,\phi_{0,n} \\
T_B(z) &= \sum_n z^{-n-2}\,L_n & \phi_1(z) &= \sum_n z^{-n-h-1/2}\,\phi_{1,n}\,.
\end{aligned}
\qquad (3.7.1)
$$

The powers of z are such that, when $z \to \log z$ takes the plane to the cylinder, the covariance of, for example, $\phi_0(z)\,(dz)^h$, implies that the $\phi_{0,n}$ are the Fourier coefficients of ϕ_0 on the cylinder. A component field of integer weight h is always indexed by integers n. A field of half-integer weight h is indexed by integers n in the R sector and by half-integers n in the NS sector. Euclidean time reversal on the cylinder corresponds to $z \to \bar{z} = 1/\bar{z}$. The adjoint of a field is given by $(\phi(z)dz^h)^\dagger = \phi^\dagger(\bar{z})d\bar{z}^h$. The reality of the super stress-energy tensor implies

$$
L_m^\dagger = L_{-m} \qquad G_m^\dagger = G_{-m}\,. \qquad (3.7.2)
$$

3.8 Commutation relations of normal modes

Commutation relations are derived from the operator product expansions by representing modes as contour integrals, then deforming contours. An anticommuting parameter ϵ is introduced in order to express anticommutation relations as commutation relations, whatever the statistics of the field ϕ. The commutation relations are

$$
\begin{aligned}
[L_m,\ \phi_0(z)] &= z^{m+1}\partial\phi_0 + h(m+1)z^m\phi_0(z) \\
[L_m,\ \phi_1(z)] &= z^{m+1}\partial\phi_0 + (h+\tfrac{1}{2})(m+1)z^m\phi_0(z) \\
[\epsilon G_m, \phi_0(z)] &= \epsilon\, z^{m+1/2}\phi_1(z) \\
[\epsilon G_m, \phi_1(z)] &= \epsilon[z^{m+1/2}\partial\phi_0 + 2(m+\tfrac{1}{2})h\,z^{m-1/2}\phi_0(z)]
\end{aligned}
\qquad (3.8.1)
$$

$$
\begin{aligned}
[L_m,\ T_F(z)] &= z^{m+1}\partial T_F + \tfrac{3}{2}(m+1)z^m T_F(z) \\
[L_m,\ T_B(z)] &= z^{m+1}\partial T_B + 2(m+1)z^m T_B(z) + \tfrac{1}{8}\hat{c}(m^3-m)z^{m-2} \\
[G_m,\ T_F(z)]_+ &= z^{m+1/2}T_B(z) + \tfrac{1}{4}\hat{c}(m^2-\tfrac{1}{4})z^{m-3/2} \\
[G_m,\ T_B(z)] &= z^{m+1/2}\partial T_F + 3(m+\tfrac{1}{2})z^{m-1/2}T_F(z)
\end{aligned}
\qquad (3.8.2)
$$

$$
\begin{aligned}
[L_m,\ \phi_{0,n}] &= [(h-1)m-n]\phi_{0,m+n} \\
[\epsilon G_m, \phi_{0,n}] &= \epsilon\phi_{1,m+n} \\
[\epsilon G_m, \phi_{1,n}] &= \epsilon[(2h-1)m-n]\phi_{0,m+n}
\end{aligned}
\qquad (3.8.3)
$$

$$
\begin{aligned}
[L_m,\ L_n] &= (m-n)L_{m+n} + \tfrac{1}{8}\hat{c}(m^3-m)\delta_{m+n,0} \\
[L_m,\ G_n] &= (\tfrac{1}{2}m-n)G_{m+n} \\
[G_m,\ G_n]_+ &= 2L_{m+n} + \tfrac{1}{2}\hat{c}(m^2-\tfrac{1}{4})\delta_{m+n,0}
\end{aligned}
\qquad (3.8.4)
$$

The conformal generators L_n form the Virasoro algebra; the superconformal generators G_n, L_n form the Ramond algebra (integer n) and the Neveu-Schwarz algebra (half-integer n). $\widehat{SL}_2$ is generated by $G_{-1/2}$, $G_{1/2}$, L_{-1}, L_0, L_1. In particular, L_0 $(+\bar{L}_0)$ is the generator of dilations, which is the hamiltonian in radial quantization. The mode expansions are arranged so that

$$[L_0,\ \phi_n] = -n\,\phi_n \tag{3.8.5}$$

as long as the form 3.7.1 is used for the mode expansion of $\phi(z)$.

3.9 Highest weight states and conformal fields

A ground state for the superconformal algebra is a state $|h\rangle$ which is annihilated by all the lowering operators L_{+n}, G_{+n} and has eigenvalue h for L_0. In mathematical terminology the ground states are called highest weight states, because mathematicians usually call $-h$ the weight of the state.

In the Ramond sector G_0 commutes with L_0 and therefore acts on the ground states. G_0 anticommutes with the chirality operator Γ. If $h \neq \hat{c}/16$ then $G_0^2 \neq 0$ and the ground states come in pairs of opposite chirality. If $h = \hat{c}/16$ then $G_0\,|h\rangle = 0$ (this is a bit subtle in a nonunitary theory). Then $|h\rangle$ is a supersymmetric ground state for the Ramond system on the cylinder with supersymmetry generator G_0, and is not necessarily paired with a state of opposite chirality. The Witten index of the Ramond sector is the net chirality of the $h = \hat{c}/16$ states. In a unitary system $G_0^2 \geq 0$ so all $h = \hat{c}/16$ states are heighest weight states.

In an ordinary conformal field theory there is a one to one correspondence between conformal fields $\phi(z)$ of conformal weight h and highest weight states $|h\rangle$ (for the L_n) with eigenvalue $L_0 = h$. The correspondence is $|h\rangle = \phi(0)\,|0\rangle$. Given the conformal field ϕ, the state $\phi(0)\,|0\rangle$ exists and is nonzero, because correlation functions of $\phi(z)$ are finite at $z = 0$, and because no quantum field can annihilate the vacuum (in the unitary two dimensional field theory associated with euclidean spacetime). The highest weight condition on $|h\rangle$ then follows from the commutation relations of $\phi(z)$ with the conformal generators. Conversely, given the highest weight state $|h\rangle$, completeness of the field algebra implies that there is a quantum field $\phi(z)$ which has a matrix element between the vacuum and

$|h\rangle$. By subtracting other quantum fields it can be assured that ϕ creates no states of energy less than h. Then $|h\rangle = \phi(0)\,|0\rangle$, because $z = 0$ corresponds to $\tau = -\infty$. The highest weight condition on $|h\rangle$ implies the operator product 3.6.1, first only for $z_2 = 0$ and acting on the vacuum, then as an operator statement because no fields annihilate the vacuum, and then for all z_2 because of two dimensional translation invariance.

In superconformal field theories, the NS heighest weight states $|h\rangle$ correspond to superconformal fields $\phi(\mathbf{z}) = \phi_0(z) + \theta\phi_1(z)$, where $\phi_0(0)\,|0\rangle = |h\rangle$ and $\phi_1(0)\,|0\rangle = G_{-1/2}\,|h\rangle$. The highest weight conditions for the superconformal algebra correspond to the operator product 3.2.3 between T and ϕ. The highest weight states $|h\rangle$ in the R sector correspond to the spin fields. These are pairs $S_\pm(z)$ of conformal fields such that

$$\begin{aligned} &S_+(0)\,|0\rangle = |h\rangle \qquad S_-(0)\,|0\rangle = G_0\,|h\rangle \\ &T_F(z_1)\,S_+(z_2) \ \sim\ \tfrac{1}{2}(z_1 - z_2)^{-3/2}\,S_-(z_2) \\ &T_F(z_1)\,S_-(z_2) \ \sim\ \tfrac{1}{2}(h - \tfrac{1}{16}\hat{c})(z_1 - z_2)^{-3/2}\,S_+(z_2)\,. \end{aligned} \tag{3.9.1}$$

The Ramond supersymmetry is unbroken only if $h = \hat{c}/16$. Then it is possible to have $G_0\,|h\rangle = 0$, $S_-(z) = 0$ and a nonzero Witten index.

Note that T_F and $S_\pm$ are not mutually local. The field theory containing spin fields $S_\pm$ becomes local only in combination with the chiral projection $\Gamma = 1$. Both the sum over sectors and the chiral projection are accomplished by the sum over spin structures. The projection eliminates T_F and the other half-integer fields, and eliminates one spin field in every chiral pair.

4. THE FERMIONIC STRING

The world surface of fermionic string is described by a two dimensional superconformal field theory, consising of a matter superfield, X^μ, $\mu = 1, \ldots, d$ which gives the location of the world surface in spacetime; and ghost superfields B, C which arise from fixing the superconformal gauge on the world surface[11].

The radial quantisation of this superconformal field theory gives the one string Hilbert space. Both matter and ghost systems are indefinite metric field theories (although the matter sector becomes unitary when the spacetime metric has euclidean signature); the positive metric of the string Hilbert space only appears after the BRST condition is imposed on the states of the two dimensional field theory.

4.1 Matter fields

The action and equation of motion of the matter superfield (in flat spacetime) are

$$S_{matter} = \frac{1}{2\pi}\int d^2z\, d\theta d\bar\theta\, \bar D X^\mu D X_\mu\,, \qquad \bar D D\, X^\mu = 0\,, \tag{4.1.1}$$

$$X^\mu(\mathbf{z},\bar{\mathbf{z}}) = X^\mu(\mathbf{z}) + X^\mu(\bar{\mathbf{z}})\,, \qquad X^\mu(\mathbf{z}) = x^\mu(z) + \theta\psi^\mu(z)\,, \tag{4.1.2}$$

after eliminating the auxiliary field $F^\mu = \partial_\theta \partial_{\bar\theta} X^\mu$ by its equation of motion. The action of the component fields is

$$S_{matter} = \frac{1}{2\pi}\int d^2z\, \left(\bar\partial x^\mu \partial x_\mu - \psi\bar\partial\psi - \bar\psi\partial\bar\psi\right)\,. \tag{4.1.3}$$

This is the model appropriate to the type II superstrings. For heterotic strings there is no $\bar\theta$ and an additional $E_8 \times E_8$ or $SO(32)$ chiral current algebra in $\bar{\mathbf{z}}$. For type I superstrings the world surfaces include the nonorientable surfaces (which are not globally complex), and have boundaries; and there are gauge degrees of freedom on the boundaries. In any case, the present discussion is only concerned

with the superconformal aspect of the world surface, so only the (z,θ) sector is discussed.

The matter chirality operator Γ is defined by $[\Gamma,\, x^\mu] = 0$, $[\Gamma,\, \psi^\mu]_+ = 0$. In the type II theory there are two separate chirality operators, Γ and $\bar\Gamma$, and two projections.

4.2 Superconformal ghosts

The field C is the ghost for infinitesimal transformations of the super world surface; it has weight -1 and Fermi statistics. The field B is the ghost for infinitesimal variations of the superconformal gauge condition and is conjugate to C; it has weight 3/2 and Bose statistics. The action and equations of motion are

$$S_{ghost} = \frac{1}{\pi}\int d^2z\, d\theta d\bar\theta\, B\bar D C\,, \qquad \bar D B = 0 = \bar D C\,. \tag{4.2.1}$$

The dimensions and statistics of the component fields are

$$\begin{array}{llll} B(\mathbf{z}){=}\beta(z) + \theta b(z), & \beta:\ h{=}3/2\ \text{(Bose)}, & b:\ h{=}2 & \text{(Fermi)} \\ C(\mathbf{z}){=}c(z) + \theta\gamma(z), & c:\ h{=}{-1}\ \text{(Fermi)}, & \gamma:\ h{=}{-1/2} & \text{(Bose)} \end{array} \tag{4.2.2}$$

The action for the component fields is

$$S_{ghost} = \frac{1}{\pi}\int d^2z\, \left(b\bar\partial c + \beta\bar\partial\gamma\right)\,. \tag{4.2.3}$$

The chirality operator Γ commutes with b and c and anticommutes with β and γ.

4.3 Two-point functions

All correlation functions of free fields are determined by the two-point functions. For the time being the following expressions should be regarded as the singular parts of the two point functions on any surface. Later it will become apparent that these are also the exact two point functions on the sphere or plane. They are derived from the action using the identities $\partial\bar\partial \ln|z|^2 = \pi\delta^2(z)$, $\bar\partial\, z^{-1} = \pi\delta^2(z)$.

$$
\begin{array}{lll}
X^\mu(\mathbf{z}_1)\, X^\nu(\mathbf{z}_2) \sim -g^{\mu\nu} \ln z_{12} & B(\mathbf{z}_1)\, C(\mathbf{z}_2) \sim \quad \theta_{12} z_{12}^{-1} & \sim C(\mathbf{z}_1)\, B(\mathbf{z}_2) \\
x^\mu(z_1)\, x^\nu(z_2) \sim -g^{\mu\nu} \ln(z_1 - z_2) & c(z_1)\, b(z_2) \sim (z_1 - z_2)^{-1} & \sim b(z_1)\, c(z_2) \\
\psi^\mu(z_1)\, \psi^\nu(z_2) \sim -g^{\mu\nu} (z_1 - z_2)^{-1} & \gamma(z_1)\, \beta(z_2) \sim (z_1 - z_2)^{-1} & \sim -\beta(z_1)\, \gamma(z_2)
\end{array}
\tag{4.3.1}
$$

4.4 Stress-energy tensors

The stress-energy tensor (and central charge) has contributions from the matter fields and from the ghosts: $T = T^X + T^{gh}$, $\hat c = \hat c^X + \hat c^{gh}$. One way to find the stress-energy tensor is to use the known form of the operator product expansions with the free fields. Write T as the most general superfield of dimension 3/2 bilinear in the fields and neutral in all conserved charges, and then fix the unknown numerical coefficients by the operator products. For example, write $T^{gh} = a_1 C\partial B + a_2 DCDB + a_3 \partial CB$ and then calculate operator products with B, C by making partial contractions.

Products of fields at coincident points usually need renormalization. In free field theory all divergences come from self-contractions, so bilinears in the free fields have finite connected correlation functions, thus finite singular operator product expansions. For calculations of finite parts of operator products a simple systematic regularization of bilinears is simply to subtract the singular part of the operator product of the free fields. On the sphere this means simply omitting all self-contractions. Here is a sample calculation of one contribution to the $T\,C$

operator product:

$$
\begin{aligned}
C\partial_1 B\;\; C(\mathbf{z}_2) &\sim -\langle \partial_1 B\, C(\mathbf{z}_2)\rangle\; C(\mathbf{z}_1) \\
&\sim \theta_{12} z_{12}^{-2}\, C(\mathbf{z}_1) \\
&\sim \theta_{12} z_{12}^{-2}\, C(\mathbf{z}_2) + \theta_{12} z_{12}^{-1}\, \partial_2 C\,.
\end{aligned}
\tag{4.4.1}
$$

For the rest of the calculations, keep in mind that C, DB and DX obey Fermi statistics, and take advantage of the identity $\theta_{12}^2 = 0$. The results are

$$
\begin{aligned}
T^X &= -\frac{1}{2} DX^\mu \partial X_\mu \\
T^{gh} &= -C\,\partial B + \frac{1}{2} DC\, DB - \frac{3}{2} \partial C\, B\,.
\end{aligned}
\tag{4.4.2}
$$

$$
\begin{array}{ll}
T_F^X = -\tfrac{1}{2}\psi_\mu \partial x^\mu & T_B^X = -\tfrac{1}{2}\partial x^\mu \partial x_\mu - \tfrac{1}{2}\partial\psi^\mu\psi_\mu \\
T_F^{gh} = -c\partial\beta - \tfrac{3}{2}\partial c\beta + \tfrac{1}{2}\gamma\, b & T_B^{gh} = c\partial b + 2\partial c b - \tfrac{1}{2}\gamma\partial\beta - \tfrac{3}{2}\partial\gamma\beta
\end{array}
\tag{4.4.3}
$$

Once the coefficients in T are fixed, the $T\,T$ operator products are calculated by partial contractions; the central terms are the double contractions. For example,

$$
\begin{aligned}
&T^X(\mathbf{z}_1)\, T^X(\mathbf{z}_2) \sim \tfrac{1}{4}\left(\langle D_1 X^\mu\, D_2 X^\nu\rangle \langle \partial_1 X_\mu\, \partial_2 X_\nu\rangle + \langle D_1 X^\mu\, \partial_2 X_\nu\rangle \langle \partial_1 X_\mu\, D_2 X^\nu\rangle \right) \\
&T^{gh}(\mathbf{z}_1)\, T^{gh}(\mathbf{z}_2) \sim \langle \partial_1 B\, C(\mathbf{z}_2)\rangle \langle C(\mathbf{z}_1)\, \partial_2 B\rangle + \cdots\,.
\end{aligned}
\tag{4.4.4}
$$

The results are

$$
\hat c^X = d \qquad \hat c^{gh} = -10 \qquad \hat c = d - 10\,. \tag{4.4.5}
$$

The critical dimension $d = 10$ is determined by the condition that the combined matter ghost system be free of conformal anomaly. The combined two dimensional quantum field theory then depends only on the super conformal class of the world surface.

4.5 Mode expansions

The mode expansions of the free fields are

$$
\begin{aligned}
\partial x^\mu &= \textstyle\sum_n z^{-n-1} a_n^\mu & x^\mu(z) &= \tfrac{1}{2} q^\mu + a_0^\mu \ln z + \textstyle\sum_{n\neq 0} \frac{z^{-n}}{-n}\, a_n^\mu \\
\psi^\mu(z) &= \textstyle\sum_n z^{-n-1/2}\, \psi_n^\mu & & \\
b(z) &= \textstyle\sum_n z^{-n-2}\, b_n & \beta(z) &= \textstyle\sum_n z^{-n-3/2}\, \beta_n \\
c(z) &= \textstyle\sum_n z^{-n+1}\, c_n & \gamma(z) &= \textstyle\sum_n z^{-n+1/2}\, \gamma_n
\end{aligned}
\tag{4.5.1}
$$

The superconformal generators are

$$
\begin{aligned}
G_n^X &= \textstyle\sum_k -\psi_{n-k}^\mu a_k^\mu \\
L_n^X &= \textstyle\sum_k -\tfrac{1}{2} a_{n-k}^\mu a_k^\mu + \tfrac{1}{2}(\tfrac{1}{2}n - k)\psi_{n-k}^\mu \psi_k^\mu \\
G_n^{gh} &= \textstyle\sum_k (3n-k) c_{n-k}\beta_k + \gamma_{n-k} b_k \\
L_n^{gh} &= \textstyle\sum_k (k-2n) c_{n-k} b_k + (\tfrac{3}{2}n - k)\gamma_{n-k}\beta_k \,.
\end{aligned}
\tag{4.5.2}
$$

There is implicit normal ordering of the quadratic expressions for L_0 in 4.5.2 (see the discussion of renormalization in section 4.4 and of ground state energies in section 4.7). The indices n of the modes L_n, a_n, b_n and c_n are integers. In the NS sector the indices n of the modes G_n, ψ_n^μ, γ_n, and β_n are half-integers ($n \in \mathbf{Z} + \frac{1}{2}$). In the R sector they are integers ($n \in \mathbf{Z}$). The behavior of the fields under $z \to 1/\bar{z}$ and hermitian conjugation gives

$$
\begin{aligned}
(a_n^\mu)^\dagger &= -a_n^\mu & b_n^\dagger &= b_n & c_n^\dagger &= c_n \\
(\psi_n^\mu)^\dagger &= \psi_n^\mu & \beta_n^\dagger &= -\beta_n & \gamma_n^\dagger &= \gamma_n \,.
\end{aligned}
\tag{4.5.3}
$$

4.6 (Anti-)commutation relations

The (anti-)commutation relations of the modes are calculated from contour integrals of the two point functions, and depend only on the singular parts. Only

the nonzero relations are written here:

$$
\begin{aligned}
[a_m^\mu,\, a_n^\nu] &= -g^{\mu\nu}\, m\, \delta_{m+n} & [a_0^\mu,\, q^\nu] &= -g^{\mu\nu} \qquad p^\mu = i a_0^\mu \\
[\psi_m^\mu,\, \psi_n^\nu]_+ &= -g^{\mu\nu}\delta_{m+n} & [c_m,\, b_n]_+ &= \delta_{m+n} = [\gamma_m,\, \beta_n]
\end{aligned}
\tag{4.6.1}
$$

$$
\begin{aligned}
[G_m,\, a_n^\mu] &= -n\psi_{m+n}^\mu & [L_m,\, a_n^\mu] &= -n a_{m+n}^\mu \\
[G_m,\, \psi_n^\mu]_+ &= a_{m+n}^\mu & [L_m,\, \psi_n^\mu] &= (-\tfrac{1}{2}m - n)\psi_{m+n}^\mu \\
[G_m,\, c_n]_+ &= \gamma_{m+n} & [L_m,\, c_n] &= (-2m-n)c_{m+n} \\
[G_m,\, b_n]_+ &= (2m-n)\beta_{m+n} & [L_m,\, b_n] &= (m-n)b_{m+n} \\
[G_m,\, \gamma_n] &= (-3m-n)c_{m+n} & [L_m,\, \gamma_n] &= (-\tfrac{3}{2}m - n)\gamma_{m+n} \\
[G_m,\, \beta_n] &= b_{m+n} & [L_m,\, \beta_n] &= (\tfrac{1}{2}m - n)\beta_{m+n}
\end{aligned}
\tag{4.6.2}
$$

4.7 Matter ground states and zero modes

The zero mode algebra of the matter system is generated by the total spacetime momentum operator $p^\mu = i a_0^\mu$ and, in the R sector, the fermionic zero modes ψ_0^μ. From 4.6.1, the ψ_0^μ zero modes satisfy the anticommutation relations of the spacetime γ-matrices:

$$
[\psi_0^\mu,\, \psi_0^\nu]_+ = -g^{\mu\nu}\,. \tag{4.7.1}
$$

The matter ground states are the states annihilated by all the lowering operators a_{+n}^μ, ψ_{+n}^μ. In the NS sector there is one ground state $|k\rangle$ for each momentum eigenvalue $p^\mu = k^\mu$. In the R sector the ground states can be written $|k,\alpha\rangle = |k\rangle \otimes |\alpha\rangle$ where $|k\rangle$ is the ground state of x^μ and $|\alpha\rangle$ is a ground state of ψ^μ. By 4.7.1, the states $|\alpha\rangle$ form a Dirac spinor, with indices $\alpha = 1 \cdots 2^{d/2}$. Summarizing,

$$
\begin{aligned}
&a_0^\mu\,|k\rangle = i k^\mu\,|k\rangle \qquad && \psi_0^\mu\,|\alpha\rangle = \gamma^{\mu\alpha}{}_\beta\,|\beta\rangle \qquad && G_0^X\,|k,\alpha\rangle = -i\not{k}\,|k,\alpha\rangle \\
&L_0^x\,|k\rangle = \tfrac{1}{2}k^2\,|k\rangle && L_0^\psi\,|\alpha\rangle = \tfrac{1}{16}d\,|\alpha\rangle && L_0^X\,|k,\alpha\rangle = (\tfrac{1}{2}k^2 + \tfrac{1}{16}d)\,|k,\alpha\rangle \,.
\end{aligned}
\tag{4.7.2}
$$

The L_0^ψ eigenvalue follows from the Ramond supersymmetry algebra $L_0 = G_0^2 + d/16$ and the supersymmetry of the Ramond ground states $|0,\alpha\rangle$.

The $\widehat{SL}_2$ invariant matter vacuum $|0\rangle$ is the zero momentum ground state of the NS sector. The full Hilbert space of the two dimensional field theory is generated from the ground states by the raising operators a^μ_{-n}, ψ^μ_{-n}. Thus the NS sector contains only states which transform as Lorentz vectors, i.e., are spacetime bosons; and the R sector contains only states which transform as Lorentz spinors, and so are spacetime fermions.

The chirality operator Γ is normalised to be $+1$ on the vacuum, thus $\Gamma = 1$ on the NS ground states (since $[\Gamma, q] = 0$), and on the excited states $\Gamma = (-1)^F$, the fermion parity. Γ acts on the R ground states $|\alpha\rangle$ as $\pm\gamma_{d+1}$, because it anticommutes with the ψ^μ_0. The choice of sign is conventional and immaterial except in the type II theories where $\Gamma = \gamma_{d+1} = \pm\overline{\Gamma}$ are the two inequivalent possibities.

If the projection $\Gamma = 1$ were made only on the matter sector, it would produce a theory with spacetime chirality (except for the $\Gamma = -\overline{\Gamma}$ type II theory). But the Γ projection acts on the combined matter ghost system, so there will only be spacetime chirality after projection if the ghost states do *not* come in chiral pairs, i.e., if the Witten index of the ghost Ramond sector is nonzero.

The ground states $|k\rangle$ of the NS sector are created from the vacuum by the superfields $e^{ik\cdot X}$, because they are superconformal fields which create momentum k^μ and have weight $\frac{1}{2}k^2$:

$$\begin{aligned} DX^\mu(\mathbf{z}_1)\, e^{ik\cdot X}(\mathbf{z}_2) &\sim \langle DX^\mu(\mathbf{z}_1)\, X^\nu(\mathbf{z}_2)\rangle\, (ik_\nu)\, e^{ik\cdot X} \sim \theta_{12} z_{12}^{-1}\, (-ik^\mu)\, e^{ik\cdot X} \\ T^X(\mathbf{z}_1)\, e^{ik\cdot X}(\mathbf{z}_2) &\sim \tfrac{1}{2}k_\sigma k_\tau\, \langle D_1 X^\mu\, X^\sigma(\mathbf{z}_2)\rangle \langle \partial_1 X_\mu\, X^\tau(\mathbf{z}_2)\rangle\, e^{ik\cdot X}(\mathbf{z}_2) + \cdots \\ &\sim \left(\tfrac{1}{2}k^2\, \theta_{12} z_{12}^{-2} + \tfrac{1}{2} z_{12}^{-1} D_2 + \theta_{12} z_{12}^{-1}\, \partial_2\right) e^{ik\cdot X}\,. \end{aligned} \tag{4.7.3}$$

The spin fields $S_\alpha(z)$ are the conformal fields which create the ground states $|\alpha\rangle$ of ψ in the R sector:

$$|\alpha\rangle = S_\alpha(0)\ |0\rangle \qquad |k,\alpha\rangle = S_\alpha e^{ik\cdot X}(0)\ |0\rangle\,. \tag{4.7.4}$$

The Ramond supersymmetry of $|\alpha\rangle$, equation 4.7.2, implies that $S_\alpha(z)$ has weight $d/16 = 5/8$.

4.8 SO(10) current algebra

The spin field $S_\alpha(z)$ can be constructed from the $SO(10)$ chiral current algebra of the ψ system[6]:

$$\begin{aligned} j^{\mu\nu}(z) &= \psi^\mu\psi^\nu(z) \\ j^{\mu\nu}(z) j^{\sigma\tau}(w) &\sim (z-w)^{-2}(g^{\mu\tau}g^{\nu\sigma} - \mu\leftrightarrow\nu) + (z-w)^{-1} \\ &\qquad \times g^{\mu\sigma} j^{\nu\tau}(w)\, (1-\mu\leftrightarrow\nu)(1-\sigma\leftrightarrow\tau)\,. \end{aligned} \tag{4.8.1}$$

The current algebra determines the entire theory because the stress-energy tensor T^ψ_B is generated by the currents[18]:

$$j^{\mu\nu}(z)\ j_{\mu\nu}(w) \sim \frac{d-d^2}{(z-w)^2} + 2(d-1)(\partial\psi^\mu)\psi^\mu(w) \tag{4.8.2}$$

$$T^\psi_B(z) = \frac{-1/4}{d-1}\, j^{\mu\nu} j_{\mu\nu}(z)\,. \tag{4.8.3}$$

The Sugawara stress-energy tensor 4.8.3 is renormalized by subtracting the leading singularity in the operator product 4.8.2.

Adopt the spinor conventions

$$\begin{array}{lll} [\gamma_\mu,\ \gamma_\nu]_+ = -g_{\mu\nu} & \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} & \epsilon_{\alpha\beta}\,\epsilon^{\beta\gamma} = \delta^\gamma_\alpha \\ u^\alpha = \epsilon^{\alpha\beta}u_\beta & u_\alpha = \epsilon_{\alpha\beta}u^\beta & \bar{A}^\alpha_\beta = \epsilon^{\alpha\gamma}A^\delta_\gamma\epsilon_{\delta\beta} \\ \gamma^{\alpha\beta}_\mu = \gamma^{\beta\alpha}_\mu & \gamma_{[\mu}\gamma_{\nu]}{}^{\alpha\beta} = \gamma_{[\mu}\gamma_{\nu]}{}^{\beta\alpha} & \bar{\gamma}_\mu = -\gamma_\mu\,. \end{array} \tag{4.8.4}$$

The fermion field ψ^μ is completely determined from the current algebra by the operator product

$$j^{\mu\nu}(z)\ \psi^\sigma(w) \sim \frac{1}{z-w}(g^{\mu\sigma}\psi^\nu - g^{\nu\sigma}\psi^\mu)(w) \tag{4.8.5}$$

and S_α is determined by

$$j^{\mu\nu}(z)\,S_\alpha(w) \sim \frac{1}{z-w}\,\tfrac{1}{2}\,\gamma^{[\mu}\gamma^{\nu]}{}^{\beta}_{\alpha}\,S_\beta(w)\,, \tag{4.8.6}$$

in the sense that all of their operator products are determined by the currents.

The operator products (for $d = 10$) are (writing S_α for $S_\alpha(0)$, ψ^μ for $\psi^\mu(0)$):

$$\begin{aligned}
\psi^\mu(z)\;\psi^\nu &\sim -g^{\mu\nu}\,(z-w)^{-1} \\
\psi^\mu(z)\;S_\alpha &\sim (z-w)^{-\frac{1}{2}}\gamma^{\mu}{}^{\beta}_{\alpha}\,S_\beta \\
\psi^\mu(z)\;S^\alpha &\sim (z-w)^{-1/2}(-\gamma^{\mu}{}^{\alpha}_{\beta})\,S^\beta \\
\psi^\mu\psi^\nu(z)\;S_\alpha &\sim (z-w)^{-1}\,\tfrac{1}{2}\gamma^{[\mu}\gamma^{\nu]}{}^{\beta}_{\alpha}\,S_\beta \\
\psi^\mu\psi^\nu(z)\;S^\alpha &\sim (z-w)^{-1}(-\tfrac{1}{2})\gamma^{[\mu}\gamma^{\nu]}{}^{\alpha}_{\beta}\,S^\beta \\
S_\alpha(z)\;S_\beta &\sim z^{-5/4}\epsilon_{\alpha\beta} + z^{-3/4}\gamma^{\mu}_{\alpha\beta}\,\psi_\mu + z^{-1/4}\tfrac{1}{2}\gamma^\mu\gamma^\nu{}_{\alpha\beta}\,j_{\mu\nu} \\
S^\alpha(z)\;S^\beta &\sim z^{-5/4}(-\epsilon^{\alpha\beta}) + z^{-3/4}\gamma^{\alpha\beta}_{\mu}\psi^\mu + z^{-1/4}\tfrac{1}{2}\gamma_\mu\gamma_\nu{}^{\alpha\beta}\,j^{\mu\nu} \\
S^\alpha(z)\;S_\beta &\sim z^{-5/4}\delta^{\alpha}_{\beta} + z^{-3/4}\gamma_{\mu}{}^{\beta}_{\alpha}\psi^\mu + z^{-1/4}\tfrac{1}{2}\gamma_\mu\gamma_\nu{}^{\alpha}_{\beta}\,j^{\mu\nu} \\
S_\alpha(z)\;S^\beta &\sim z^{-5/4}(-\delta^{\alpha}_{\beta}) + z^{-3/4}\gamma_{\mu}{}^{\beta}_{\alpha}\psi^\mu + z^{-1/4}\tfrac{1}{2}\gamma_\mu\gamma_\nu{}^{\beta}_{\alpha}j^{\mu\nu}\,.
\end{aligned} \tag{4.8.7}$$

The coefficients are given by the following arguments. The $\psi^\mu\,\psi^\nu$ operator product and the leading term in the $S_\alpha\,S_\beta$ operator product are fixed by $SO(10)$ invariance up to normalizations. The $\psi\;S$ operator product is obtained by requiring consistency of $j^{\mu\nu}\,S_\alpha$ with $\psi^\mu\,\psi^\nu\,S_\alpha$. The ψ contribution to $S\,S$ is found by evaluating $\langle\psi(z)\,S(w)\,S\rangle \propto w^{-1/2}z^{-3/4}(z-w)^{-1/2}$ in the two limits $z\to w$ and $w\to 0$. Similarly, the $j^{\mu\nu}$ contribution is determined by evaluating $\langle j\;S\;S\rangle$.

Note the fractional powers of z in the operator products 4.8.7, appropriate to the fractional dimension of S_α. Note also that $\Gamma\,|\alpha\rangle = \gamma_{11}\,|\alpha\rangle$ implies $\Gamma S_\alpha\Gamma^{-1} = (\gamma_{11}S)_\alpha$, which is inconsistent with the $S\,S\sim\gamma\psi$ operator product. Thus the spin fields of the matter sector do not by themselves form a local quantum field

theory, and the chirality operator Γ acting in the matter sector alone is not an automorphism of the local algebra of spin fields. These difficulties are resolved by combining S_α with the spin fields of the superconformal ghosts.

The advantage of the current algebra approach is its manifest Lorentz invariance. The spin fields S_α can also be realized explicitly as ordinary vertex operators, that is, as exponentials of free chiral scalar fields[1-6]. The vertex operator construction is not manifestly Lorentz invariant, but it allows explicit calculation of correlation functions, on any surface. On the other hand, the current algebra is useful for obtaining the first few coefficients in operator product expansions. It can be used to find correlation functions[18,6], but not easily, except in the simplest situations.

It will be useful to know some subleading terms in the operator products when leading terms vanish. Assume that the indices in the following operator products are contracted with spinors u^α, v^β and a vector k_μ satisfying $\bar{u}\not{k}v = 0$, and use the shorthand $\gamma^{\mu}_{\alpha\beta} = 0$ for this situation. Then

$$\psi^\mu(z)\,S_\alpha \sim z^{1/2}\,\psi^\mu_{-1}S_\alpha \tag{4.8.8}$$

where $\psi^\mu_{-1}S_\alpha(z)$ is the conformal field corresponding to the state $\psi^\mu_{-1}\,|\alpha\rangle$. In general,

$$\psi^\mu\psi^\nu(z)\,S^\beta \sim z^{-1}\tfrac{1}{2}\gamma^{[\nu\beta\gamma}\gamma^{\mu]}_{\gamma\delta}S^\delta + \gamma^{[\nu\beta\gamma}\psi^{\mu]}_{-1}S_\gamma \tag{4.8.9}$$

Thus, when $\gamma^{\mu}_{\alpha\beta} = 0$, the following is a finite product,

$$\gamma_{\nu\alpha\beta}\psi^\mu\psi^\nu S^\beta = (1-\tfrac{1}{2}d)\psi^\mu_{-1}S_\alpha\,, \tag{4.8.10}$$

and

$$\psi^\mu(z)\,S_\alpha \sim (z-w)^{\frac{1}{2}}(1-\tfrac{1}{2}d)^{-1}\gamma_{\nu\alpha\beta}\psi^\mu\psi^\nu S^\beta\,. \tag{4.8.11}$$

4.9 N=2 supersymmetry of the ghosts

As an aside, it might be interesting that the superconformal ghost system has an additional supersymmetry. Combined with the manifest $N = 1$ superconformal invariance, this gives $O(2)$ extended superconformal symmetry. The fundamental $O(2)$ superconformal multiplet in two dimensions consists of the stress-energy tensor, two dimension 3/2 conformal fields, and the dimension 1 current of the $O(2)$ symmetry. Under an $N = 1$ superconformal subalgebra these fields split into the $N = 1$ super stress-energy tensor $T(\mathbf{z})$ and a dimension 1 superconformal field $J(\mathbf{z})$. The condition on $J(\mathbf{z})$ which gives the closure of the $N = 2$ algebra is

$$J(\mathbf{z}_1)\, J(\mathbf{z}_2) \sim \tfrac{1}{2}\hat{c}\, z_{12}^{-1} + 2\theta_{12}\, z_{12}^{-1}\, T(\mathbf{z}_2)\,. \tag{4.9.1}$$

In the superconformal ghost system,

$$J = 2(DB)C + 3B(DC) \tag{4.9.2}$$

is a dimension 1 superconformal field which satisfies 4.9.1. Thus the ghost system has an $N = 2$ superconformal symmetry.

5. First Order Free Fields

The component fields b, c and β, γ of the superconformal ghosts are special cases of free fields satisfying first order equations of motion. This section discusses the general case[8,11,19,2,4,5].

5.1 Fields, action, modes, two-point functions

Let $\mathbf{b}(z)$ and $\mathbf{c}(z)$ be conjugate conformal fields:

$$S = \frac{1}{\pi}\int d^2z\, \mathbf{b}\bar\partial\mathbf{c} \qquad \bar\partial\mathbf{b} = \bar\partial\mathbf{c} = 0$$
$$\text{weight}(\mathbf{b}) = \lambda \qquad \text{weight}(\mathbf{c}) = 1-\lambda\,. \tag{5.1.1}$$

The conformal ghosts b, c have $\lambda = 2$; their superpartners β, γ have $\lambda = 3/2$. The basic facts are

$$\epsilon = \begin{cases} +1 & \text{Fermi statistics} \\ -1 & \text{Bose statistics}\end{cases} \qquad \delta = \begin{cases} 0 & \text{NS sector} \\ \frac{1}{2} & \text{R sector}\end{cases}$$

$$\mathbf{c}(z)\,\mathbf{b}(w) \sim \frac{1}{z-w} \qquad \mathbf{b}(z)\,\mathbf{c}(w) \sim \frac{\epsilon}{z-w}$$
$$\mathbf{b}(z) = \sum_{n\in\delta-\lambda+\mathbf{Z}} z^{-n-\lambda}\,\mathbf{b}_n \qquad \mathbf{c}(z) = \sum_{n\in\delta+\lambda+\mathbf{Z}} z^{-n-(1-\lambda)}\,\mathbf{c}_n \tag{5.1.2}$$
$$\mathbf{c}_m\mathbf{b}_n + \epsilon\,\mathbf{c}_m\mathbf{b}_n = \delta_{m+n} \qquad \mathbf{b}_n^{\dagger} = \epsilon\mathbf{b}_{-n} \qquad \mathbf{c}_n^{\dagger} = \mathbf{c}_{-n}\,.$$
$$Q = \epsilon(1-2\lambda) \qquad \lambda = \tfrac{1}{2}(1-\epsilon Q)\,.$$

In the NS sector the fields are single valued on the plane; in the R sector they are double valued. Strictly speaking, the R sector should be present only for $\lambda \in \frac{1}{2} + \mathbf{Z}$. When $\lambda \in \mathbf{Z}$ the case $\delta = \frac{1}{2}$ is a *twisted* sector. In 5.1.2 the fields are operators on an indefinite metric Hilbert space.

5.2 The stress-energy tensor

The stress-energy tensor is determined by the weights of $\mathbf{b}$ and $\mathbf{c}$:

$$T_B^{\mathbf{bc}} = -\lambda\mathbf{b}\partial\mathbf{c} + (1-\lambda)\,\partial\mathbf{b}\,\mathbf{c} = \tfrac{1}{2}(\partial\mathbf{b}\mathbf{c} - \mathbf{b}\partial\mathbf{c}) + \tfrac{1}{2}\epsilon Q\partial(\mathbf{b}\mathbf{c})\,. \tag{5.2.1}$$

As usual, $T_B^{\mathbf{bc}}$ is renormalized by subtracting the singular part of the $\mathbf{b}$, $\mathbf{c}$ operator products. Double contractions give the conformal anomaly

$$T_B^{\mathbf{bc}}(z)\,T_B^{\mathbf{bc}}(w) \sim \tfrac{1}{2}c^{\mathbf{bc}}(z-w)^{-4}$$
$$c^{\mathbf{bc}} = -\epsilon(12\lambda^2 - 12\lambda + 2) = \epsilon(1-3Q^2)\,. \tag{5.2.2}$$

The mode expansions are:

$$L_m^{\mathbf{bc}} = \textstyle\sum_k\, [k-(1-\lambda)m]\,\mathbf{b}_{m-k}\mathbf{c}_k = \sum_k\, \epsilon(k-\lambda m)\,\mathbf{c}_{m-k}\mathbf{b}_k$$
$$[L_m^{\mathbf{bc}}, \mathbf{b}_n] = (-(1-\lambda)m - n)\mathbf{b}_{m+n} \qquad [L_m^{\mathbf{bc}}, \mathbf{c}_n] = (-\lambda m - n)\mathbf{c}_{m+n} \tag{5.2.3}$$

For the superconformal ghosts,

$$\begin{array}{llllll} b,\ c: & \epsilon = 1, & \lambda = 2, & Q = -3, & c^{bc} = -26 \\ \beta,\ \gamma: & \epsilon = -1, & \lambda = \frac{3}{2}, & Q = 2, & c^{\beta\gamma} = 11 \end{array} \qquad (5.2.4)$$

5.3 The U(1)-current

The action 5.1.1 has a chiral $U(1)$ symmetry whose chiral current is

$$\mathbf{j}(z) = -\mathbf{b}\mathbf{c} = \epsilon\,\mathbf{c}\mathbf{b} = \sum_n z^{-n-1}\mathbf{j}_n \qquad \mathbf{j}_n = \sum_k \epsilon\,\mathbf{c}_{n-k}\mathbf{b}_k$$

$$\begin{array}{lll} \mathbf{j}(z)\,\mathbf{b}(w) \sim (-1)\,(z-w)^{-1}\,\mathbf{b}(w) & \quad & \mathbf{j}(z)\,\mathbf{c}(w) \sim (+1)\,(z-w)^{-1}\,\mathbf{c}(w) \\ {[\mathbf{j}_m, \mathbf{b}_n]} = -\mathbf{b}_{m+n} & & {[\mathbf{j}_m, \mathbf{c}_n]} = +\mathbf{c}_{m+n}\,. \end{array}$$

$$\mathbf{j}_0 = \text{charge operator} \qquad \text{charge}(\mathbf{b}) = -1 \qquad \text{charge}(\mathbf{c}) = +1$$

$$\mathbf{j}(z)\ \mathbf{j}(w) \sim \epsilon\,(z-w)^{-2} \qquad [\mathbf{j}_m, \mathbf{j}_n] = \epsilon\, m\, \delta_{m+n}\,. \qquad (5.3.1)$$

The algebra of the chiral current and the stress-energy tensor is anomalous:

$$\begin{aligned} T^{\mathrm{bc}}(z)\,\mathbf{j}(w) &\sim Q\,(z-w)^{-3} + (z-w)^{-2}\,\mathbf{j}(z) \\ [L^{\mathrm{bc}}_m, \mathbf{j}_n] &= -n\mathbf{j}_{m+n} + \tfrac{1}{2}Q\,m(m+1)\delta_{m+n} \end{aligned} \qquad (5.3.2)$$

so $\mathbf{j}(z)$ is scale and translation covariant ($m = 0, -1$) but *not* conformally covariant. The anomaly coefficient Q can be interpreted as a background charge on the sphere (see also equations 5.4.2 and5.6.3 below):

$$\mathbf{j}_0^\dagger = -[L^{\mathrm{bc}}_{-1}, \mathbf{j}_1]^\dagger = -[L^{\mathrm{bc}}_1, \mathbf{j}_{-1}] = -\mathbf{j}_0 - Q\,. \qquad (5.3.3)$$

There is no normal ordering ambiguity in $\mathbf{j}_m$ for $m \neq 0$, therefore

$$\mathbf{j}_m^\dagger = -\mathbf{j}_{-m} - Q\,\delta_{m,0}\,. \qquad (5.3.4)$$

5.4 The Fermi/Bose sea

A Fermi/Bose sea is a state $|q\rangle$ which splits the normal modes:

$$\begin{array}{ll} \mathbf{b}_n\,|q\rangle = 0 & n > \epsilon q - \lambda \\ \mathbf{c}_n\,|q\rangle = 0 & n \geq -\epsilon q + \lambda \end{array} \qquad (5.4.1)$$

where $q \in \mathbf{Z}$ for the NS sector and $q \in \frac{1}{2} + \mathbf{Z}$ for the R sector. By 5.1.2, the only nonzero inner products are

$$\langle q - Q\,|q\rangle = 1\,. \qquad (5.4.2)$$

The two point function in the sea $|q\rangle$ is

$$\langle \mathbf{c}(z)\,\mathbf{b}(w)\rangle_q = \sum_{m,n} z^{-m-(1-\lambda)} w^{-n-\lambda}\,\langle q - Q|\,\mathbf{c}_m\mathbf{b}_n\,|q\rangle = \left(\frac{z}{w}\right)^{\epsilon q} \frac{1}{z-w} \qquad (5.4.3)$$

from which,

$$\begin{aligned} \langle \mathbf{j}(z)\,\mathbf{j}(w)\rangle_q &= \epsilon(z-w)^{-2} \\ \left\langle T^{\mathrm{bc}}(z)\,\mathbf{j}(w)\right\rangle_q &= Q(z-w)^{-3} + (z-w)^{-2}\frac{q}{z} \\ \left\langle T^{\mathrm{bc}}(z)\,T^{\mathrm{bc}}(w)\right\rangle_q &= (z-w)^{-4}\tfrac{1}{2}c^{\mathrm{bc}} + (z-w)^{-2}\epsilon q(Q+q)\frac{1}{zw} \end{aligned} \qquad (5.4.4)$$

$$\begin{array}{llll} L^{\mathrm{bc}}_{+n}\,|q\rangle = 0 & \langle \mathbf{j}(z)\rangle_q = q z^{-1} & \left\langle T^{\mathrm{bc}}(z)\right\rangle_q = \frac{1}{2}\epsilon q(Q+q)z^{-2} \\ \mathbf{j}_{+n}\,|q\rangle = 0 & \mathbf{j}_0\,|q\rangle = q\,|q\rangle & L^{\mathrm{bc}}_0\,|q\rangle = \frac{1}{2}\epsilon q(Q+q)\,|q\rangle\,. \end{array} \qquad (5.4.5)$$

Thus the Bose/Fermi sea $|q\rangle$ has charge q, and it is apparent that 5.4.2 expresses the presence of a background charge Q. An SL_2 invariant state has $L_0 = 0$, so the only candidates are $|0\rangle$ and $|-Q\rangle$. Only a neutral state can be translation invariant, so $|0\rangle$ is the unique SL_2 invariant state.

Each Fermi sea can be obtained from any other by applying a monomial in the fields $\mathbf{b}$, $\mathbf{c}$. But this is not true for the Bose seas. The Bose seas $|q\rangle$ generate inequivalent representations of the $\mathbf{b}$, $\mathbf{c}$ algebra.

5.5 The U(1) stress-energy tensor

Define the $U(1)$ stress-energy tensor by

$$T^{\mathbf{j}}(z) = \tfrac{1}{2}\epsilon\,(\mathbf{j}(z)^2 - Q\partial\mathbf{j})\,, \tag{5.5.1}$$

subtracting the singularity in the $\mathbf{j}, \mathbf{j}$ operator product 5.3.1. The linear term in 5.5.1 is designed so that $T_B^{\mathbf{j}}$ and $T_B^{\mathbf{bc}}$ will have the same commutation relations with $\mathbf{j}$:

$$\begin{aligned} &T^{\mathbf{j}}(z)\,\mathbf{j}(w) \sim (z-w)^{-3}Q + (z-w)^{-2}\mathbf{j}(z) \\ &T^{\mathbf{j}}(z)\,T^{\mathbf{j}}(w) \sim (z-w)^{-4}\tfrac{1}{2}c^{\mathbf{j}} + (z-w)^{-2}2T^{\mathbf{j}}(w) + (z-w)^{-1}\partial_w T^{\mathbf{j}} \end{aligned} \tag{5.5.2}$$

$$c^{\mathbf{j}} = 1 - 3\epsilon Q^2 = \begin{cases} c^{\mathbf{bc}} & \text{Fermi statistics} \\ c^{\mathbf{bc}} + 2 & \text{Bose statistics.} \end{cases} \tag{5.5.3}$$

In the Fermi case the $U(1)$ current algebra gives the complete dynamics ($T_B^{\mathbf{j}} = T_B^{\mathbf{bc}}$), but in the Bose case it does not. For Bose systems, define[20]

$$T^{[-2]}(z) = T^{\mathbf{bc}}(z) - T^{\mathbf{j}}(z)\,. \tag{5.5.4}$$

By 5.3.2 and 5.5.2, $T^{[-2]}(z)$ commutes with $\mathbf{j}(w)$ and therefore with $T^{\mathbf{j}}(w)$, and $T^{[-2]}(z)$ generates a conformal algebra with central charge $c^{[-2]} = -2$.

5.6 Bosonization

Use the $U(1)$ current to define a chiral scalar field $\phi(z)$:

$$\mathbf{j}(z) = \epsilon\partial\phi(z) \qquad \phi(z) = \epsilon\int^{z} dw\,\mathbf{j}(w)$$

$$\mathbf{j}(z)\,\phi(w) \sim (z-w)^{-1} \qquad \phi(z)\,\phi(w) \sim \epsilon\,\ln(z-w)$$

$$\mathbf{j}(z)\,e^{q\phi(w)} \sim q(z-w)^{-1}\,e^{q\phi}(w) \qquad [\mathbf{j}_0,\,e^{q\phi(w)}] = q\,e^{q\phi(w)}$$

$$T^{\mathbf{j}}(z)\,e^{q\phi(w)} \sim \left[\tfrac{1}{2}\epsilon q(Q+q)\,(z-w)^{-2} + (z-w)^{-1}\,\partial_w\right] e^{q\phi(w)} \tag{5.6.1}$$

$$\text{charge}(e^{q\phi}) = q \qquad \text{weight}(e^{q\phi}) = \tfrac{1}{2}\epsilon q(Q+q)\,.$$

$$e^{q\phi(z)}\;\; e^{q'\phi(w)} \sim (z-w)^{\epsilon qq'}e^{q\phi(z)+q'\phi(w)}$$

$$e^{q\phi(0)}\,|0\rangle = |q\rangle \tag{5.6.2}$$

The soliton operator $e^{q\phi}$ shifts the Fermi/Bose sea level by q units of charge. Equation 5.4.2 gives

$$\langle 0|\,e^{-Q\phi(z)}\,|0\rangle = 1 \tag{5.6.3}$$

which again shows the need for charge $-Q$ to absorb the background charge Q on the sphere.

The $U(1)$ current can be fermionized in terms of fundamental solitons $e^{\pm\phi}$. In the Fermi case the fundamental solitons are exactly the original $\mathbf{b}$, $\mathbf{c}$ fields

$$\mathbf{b}(z) = e^{-\phi(z)} \qquad \mathbf{c}(z) = e^{\phi(z)} \qquad \text{(Fermi statistics)}\,. \tag{5.6.4}$$

The $e^{\pm\phi}\,e^{q\phi}$ operator products, given by equation 5.6.1, can be compared with 5.4.1 to confirm 5.6.2 for the Fermi systems. In the Bose case the $U(1)$ solitons cannot give the original fields, because of the missing central charge -2 and because the soliton fields $e^{\pm\phi(z)}$ are fermionic, while $\mathbf{b}$, $\mathbf{c}$ are bosonic.

5.7 The c = −2 system

Define

$$\eta = \partial c\,e^{-\phi} \qquad \partial\xi = \partial b\,e^{+\phi}\,. \tag{5.7.1}$$

$\eta(z)$ and $\xi(z)$ are conjugate free fermion fields of conformal weights 1 and 0 respectively:

$$\begin{aligned} &\eta(z)\,\xi(w) \sim (z-w)^{-1} \sim \xi(z)\,\eta(w) \\ &\eta(z) = \textstyle\sum_n z^{-n-1}\eta_n \qquad \xi(z) = \sum_n z^{-n-1}\xi_n \qquad [\eta_m,\,\xi_n]_+ = \delta_{m+n}\,. \end{aligned} \tag{5.7.2}$$

η and ξ commute with ϕ and have

$$\lambda^{\eta\xi} = 1 \qquad Q^{\eta\xi} = -1 \qquad c^{\eta\xi} = -2\,. \tag{5.7.3}$$

Thus every first order Bose system consists of its own $U(1)$ current algebra along with the $\lambda = 1$ first order Fermi system η, ξ. The Bose fields can be written

$$\mathbf{b} = e^{-\phi}\partial\xi \qquad \mathbf{c} = e^{\phi}\eta \qquad \text{(Bose statistics)}\,. \tag{5.7.4}$$

and again the operator products of the exponentials confirm 5.6.2.

The η, ξ system contains its own chiral $U(1)$ current, which gives a second chiral scalar $\chi(z)$:

$$\partial\chi = \eta\xi \qquad \chi(z)\,\chi(w) \sim \ln(z-w) \qquad \eta = e^{-\chi} \qquad \xi = e^{\chi}\,. \tag{5.7.5}$$

The zero-mode algebra $[\eta_0, \xi_0] = 1$ forces the ground state of the η, ξ system to be twofold degenerate. The two ground states are the SL_2 invariant $|0\rangle$ and its hermitian conjugate $|-Q\rangle_{\eta\xi}$. Since $\xi = e^{\chi}$,

$$\langle 0|\,\xi(z)\,|0\rangle_{\eta\xi} = \langle 0|\,\xi_0\,|0\rangle_{\eta\xi} = 1\,. \tag{5.7.6}$$

But in the construction 5.7.4 for the original fields $\mathbf{b}$, $\mathbf{c}$ only $\rho = \partial\xi$ appears. Therefore the $c = -2$ system is really η, ρ and not η, ξ. In the η, ρ system, it is consistent to fix $\eta_0 = 0$. Then the η, ρ system has a unique ground state:

$$\langle 0|\cdots|0\rangle_{\eta\rho} = \langle 0|\,\xi_0\cdots|0\rangle_{\eta\xi}\,. \tag{5.7.7}$$

5.8 The chiral scalar and Riemann-Roch

The anomalous operator product of the stress-energy tensor with the chiral $U(1)$ current, equation 5.3.2, is equivalent to the anomalous conservation law for the chiral current:

$$\bar\partial \mathbf{j}(z) = \epsilon\,\bar\partial\partial\phi(z) = \tfrac{1}{8}QR^{(2)}\sqrt{g} \tag{5.8.1}$$

where $R^{(2)}\sqrt{g}$ is the two dimensional scalar curvature density. The operator product 5.3.2 is derived from the conservation law 5.8.1 by differentiating with respect to the two-metric to get a Ward identity, and using the Ward identity to determine the singular part of the operator product expansion[7].

The anomalous conservation law is the equation of motion for ϕ, derived from the action

$$S(\phi) = \frac{1}{2\pi}\int d^2z\,\left(-\epsilon\bar\partial\phi\partial\phi - \tfrac{1}{4}Q\sqrt{g}R^{(2)}\phi\right)\,. \tag{5.8.2}$$

Note that the action is well-behaved for Fermi systems ($\epsilon = 1$) if $\phi \to i\phi$. The exponentials then take the familiar form $e^{iq\phi}$.

To find the background charge on an arbitrary Riemann surface, note that the action 5.8.2 gives expectation values $\left\langle e^{q\phi}\right\rangle = 0$ unless $q + q^{back} = 0$ with the background charge given by

$$q^{back} = Q\frac{1}{8\pi}\int d^2\,\sqrt{g}\,R^{(2)} = Q(1-g) \tag{5.8.3}$$

where g is the genus of the Riemann surface, and $2(1-g)$ is the Euler number given by the Gauss-Bonnet formula

$$\frac{1}{4\pi}\int d^2\,\sqrt{g}\,R^{(2)} = 2(1-g)\,. \tag{5.8.4}$$

The sphere has $g = 0$, which gives the background charge $q^{back} = Q$, as already seen in the operator representation. The background charge is related to the number of solutions of the equations of motion 5.1.1:

$$\#\text{ of }\mathbf{b}\text{ solutions} - \#\text{ of }\mathbf{c}\text{ solutions} = \epsilon q^{back} = (1-2\lambda)(g-1) \tag{5.8.5}$$

which is the Riemann-Roch formula.

6. THE SUPERCONFORMAL GHOSTS

6.1 Bosonisation

Specialising the constructions of the previous section to the superconformal ghosts,

$$\begin{aligned} \beta &= e^{-\phi}\partial\xi = e^{-\phi+\chi}\partial\chi & \gamma &= e^{\phi}\eta = e^{\phi-\chi} \\ \xi &= e^{\chi} & \eta &= e^{-\chi} \\ b &= e^{-\sigma} & c &= e^{\sigma}\,. \end{aligned} \tag{6.1.1}$$

The properties of the chiral scalars are

$$\begin{array}{llll} \phi(z)\,\phi(w) = -\ln(z-w) & Q^{\phi} = 2 & c^{\phi} = 13 & \mathrm{wt}(e^{q\phi}) = -\tfrac{1}{2}q(q+2) \\ \chi(z)\,\chi(w) = +\ln(z-w) & Q^{\chi} = -1 & c^{\phi} = -2 & \mathrm{wt}(e^{q\chi}) = +\tfrac{1}{2}q(q-1) \\ \sigma(z)\,\sigma(w) = +\ln(z-w) & Q^{\sigma} = -3 & c^{\sigma} = -26 & \mathrm{wt}(e^{q\sigma}) = +\tfrac{1}{2}q(q-3) \end{array} \tag{6.1.2}$$

The total ghost charge is ϕ-charge plus σ-charge; the ghost charge operator is $j_0^{\phi} + j_0^{\sigma}$. The inequivalent representations of the β, γ algebra are indexed by the $\phi+\chi$-charge, since β and γ both commute with $j_0^{\phi} + j_0^{\chi}$. One way of picturing this extra quantum number is to fermionise the current $\beta\gamma$, giving *two* charged solitons, $e^{\pm\phi}$ and $e^{\pm\chi}$. The $e^{\pm\chi}$ solitons are free, but the $e^{\pm\phi}$ solitons cannot be free, since their dimensions do not add up to 1.

6.2 Spin fields

The integer weight ghost fields b, c, η, $\partial\xi$ are not affected by the spin fields, so the ghost contribution to the spin fields comes only from ϕ. The exponentials $e^{q\phi}$ are components of superfields (NS operators) for $q \in \mathbf{Z}$, and spin fields (R operators) for $q \in \frac{1}{2} + \mathbf{Z}$. Note that these spin fields, like the S_α of the matter sector, have fractional weights and do not by themselves form a local algebra of fields. Only the combined ϕ, ψ^μ system will have a local algebra of spin fields.

The integer weight ghost fields all commute with the chirality operator Γ. The identity has even chirality, as does the corresponding state, the SL_2 invariant vacuum $|0\rangle_\phi$. The solitons $e^{\pm\phi}$ are odd, so

$$\Gamma\,|q\rangle_\phi = (-1)^q\,|q\rangle_\phi \qquad q \in \mathbf{Z}\,. \tag{6.2.1}$$

For the Ramond states, up to a conventional choice of overall sign,

$$\Gamma\,|q\rangle_\phi = (-1)^{q+1/2}\,|q\rangle_\phi \qquad q \in \tfrac{1}{2} + \mathbf{Z}\,. \tag{6.2.2}$$

In terms of the spin fields,

$$\Gamma e^{q\phi}\Gamma^{-1} = e^{q\phi}\begin{cases} (-1)^q & q \in \mathbf{Z} \\ (-1)^{q+1/2} & q \in \frac{1}{2} + \mathbf{Z} \end{cases} \tag{6.2.3}$$

which is obviously inconsistent with the operator products of the exponentials, in the same way that $\Gamma S_\alpha \Gamma^{-1} = (\gamma_{11}S)_\alpha$ is inconsistent with the operator products of S_α. Only in the combined ϕ, ψ^μ system can the chirality operator be extended to the spin fields, with the $\Gamma = 1$ projection giving a local field theory.

The spin field $e^{\phi/2}$, of weight $-5/8 = \hat{c}^{BC}/16$, corresponds to the unique state $|1/2\rangle_\phi \otimes |0\rangle_{\sigma\chi}$ of unbroken Ramond supersymmetry in the ghost system. The Witten index is thus $+1$ in the Ramond sector of the ghosts. This is responsible for spacetime chirality in the covariant formulation of the fermionic string theory. If the ghost states were all paired in chirality, then the $\Gamma = 1$ projection would produce only states of paired spacetime chirality.

6.3 The BRST current

The BRST supercurrent is

$$J_{BRST} = DC(C\,DB - \tfrac{3}{4}\,DC\,B) \tag{6.3.1}$$

where normal ordering is done with respect to the $\widehat{SL}_2$ invariant state, in which $\langle B(\mathbf{z}_1)\,C(\mathbf{z}_2)\rangle = \theta_{12}/z_{12}$. In Feynman diagrams for correlation functions on the sphere which involve J_{BRST}, no self-contractions are included. The BRST charge is

$$\epsilon\, Q_{BRST} = \frac{1}{2\pi i}\oint dz\, d\theta\, \epsilon\, J_{BRST}(\mathbf{z}) \qquad Q^{\dagger}_{BRST} = Q_{BRST}\,. \tag{6.3.2}$$

The BRST current is completely specified by three conditions:

1. the BRST transformation laws of the superconformal matter fields of weight h:

$$[\epsilon Q_{BRST},\, \Phi_{matter}] = [\,\epsilon C + \tfrac{1}{2}D(\epsilon C)D + h\partial(\epsilon C)]\,\Phi_{matter}\,, \tag{6.3.3}$$

2. the transformation laws of the ghost fields:

$$\begin{aligned}[\epsilon Q_{BRST},\, C] &= \epsilon\left(C\partial C - \tfrac{1}{4}\,DC\,DC\right)\\ [\epsilon Q_{BRST},\, B] &= -\epsilon T\end{aligned} \tag{6.3.4}$$

3. the requirement that J_{BRST} be an anomaly free supercurrent, i.e. a superconformal field of weight 1/2:

$$T(\mathbf{z}_1)\, J_{BRST}(\mathbf{z}_2) \sim \tfrac{1}{2}\theta_{12}z_{12}^{-2}\, J_{BRST}(\mathbf{z}_2) + \tfrac{1}{2}z_{12}^{-1}D_2 J_{BRST} + \theta_{12}z_{12}^{-1}\,\partial_2 J_{BRST}\,. \tag{6.3.5}$$

The last condition ensures that J_{BRST} is analytic (conserved) on any world-surface, and that its contour integrals are conformal invariants. From properties 1–3 it follows that

$$Q^2_{BRST} = 0\,, \tag{6.3.6}$$

because, by the BRST transformation laws, Q^2_{BRST} commutes with all the matter fields and C, and $[Q^2_{BRST}, B] = -[T, Q_{BRST}]_+ = 0$ because J_{BRST} is a conformal supercurrent. Therefore Q^2_{BRST} commutes with all the fields and must be a multiple of the identity. But it has total ghost charge +2 while the identity is neutral. Therefore $Q^2_{BRST} = 0$.

The procedure for finding (or verifying) equation 6.3.1 for J_{BRST} is to write the most general superfield of total ghost charge +1, and then to fix its coefficients by evaluating the operator products needed to verify equations 6.3.3-6.3.5, using the two-point function θ_{12}/z_{12}. Only if $d = 10$ is it possible to satisfy all of the defining properties simultaneously. Rewriting Q_{BRST} in the form

$$\epsilon Q_{BRST} = \frac{1}{2\pi i}\oint dz\, d\theta\, \epsilon\left(CT^{[X]} + \tfrac{1}{2}CT^{[BC]}\right) \tag{6.3.7}$$

makes it easy to derive 6.3.3. From 6.3.3 it is obvious that any $h = 1/2$ superconformal field in the matter sector is a BRST invariant vertex operator. These vertex operators are enough to give the complete S-matrix of the spacetime bosons (the NS sector). It is also useful to rewrite Q_{BRST} again:

$$\begin{aligned}Q_{BRST} &= Q^{(0)}_{BRST} + Q^{(1)}_{BRST} + Q^{(2)}_{BRST}\\ Q^{(0)}_{BRST} &= \frac{1}{2\pi i}\oint dz\,\left(cT_B^{[X\beta\gamma]} - c\partial c b\right)\\ Q^{(1)}_{BRST} &= \frac{1}{2\pi i}\oint dz\, \tfrac{1}{2}\gamma\psi_\mu\partial x^m u \quad = \quad \frac{1}{2\pi i}\oint dz\, \tfrac{1}{2}e^{\phi-\chi}\psi_\mu\partial x^\mu\\ Q^{(2)}_{BRST} &= \frac{1}{2\pi i}\oint dz\, \tfrac{1}{4}\gamma^2 b \quad = \quad \frac{1}{2\pi i}\oint dz\, \tfrac{1}{4}e^{2\phi-2\chi-\sigma}\end{aligned} \tag{6.3.8}$$

6.4 *BRST* invariant expectation values

The vacuum $|0\rangle$ is the SL_2 invariant state (for all of the fields). All charge operators annihilate the vacuum. The vacuum expectation values

$$\langle\cdots\rangle_Q = \langle 0|\cdots|0\rangle \tag{6.4.1}$$

are the correlation functions of the conformally invariant quantum field theory on the sphere (or plane or cylinder). These expectation values vanish unless they contain ghost operators which exactly soak up the ghost background charges. In particular, $\langle 1\rangle_Q = 0$. The disadvantage of these expectation values is the background b, c charge. It requires that three of the vertex operators contain a factor of c. In tree amplitudes this is not a problem of principle, since the BRST quantization treats all operators of the matter – ghost system equally. But it is an inconvenience, because the asymmetry of the distribution of the b, c charges among the vertices obscures the duality of the scattering amplitudes, and because it would be attractive to have the full S-matrix entirely in terms of the x, ψ, ϕ system. In loops the problem is serious, since there are no vertex operators with the negative b, c charge needed to neutralise the background.

Let

$$\langle -Q| = \langle 0|\, e^{3\sigma-2\phi}(\infty) \qquad \langle -Q \mid 0\rangle = 1 \tag{6.4.2}$$

be the state conjugate to the vacuum, in which the ghost background charges have been neutralised. Note that $e^{3\sigma-2\phi}$ is a conformal field of weight 0. The vacuum and its hermitian conjugate are both BRST invariant

$$Q_{BRST}\,|0\rangle = 0 \qquad \langle -Q|\, Q_{BRST} = 0\,. \tag{6.4.3}$$

The vacuum is invariant because the conformal generators commute with Q_{BRST} and the SL_2 invariant state is unique. To show that its conjugate is invariant, first show directly from

$$\begin{array}{ll} b_n|0\rangle = 0 \;\; n \geq -1 & c_n|0\rangle = 0 \;\; n \geq 2 \\ \beta_n|0\rangle = 0 \;\; n \geq -1/2 & \gamma_n|0\rangle = 0 \;\; n \geq 3/2 \end{array} \tag{6.4.4}$$

that

$$Q^0_{BRST}\; e^{3\sigma(0)}\,|0\rangle = 0\,. \tag{6.4.5}$$

Then use version 6.3.8 for Q_{BRST}, and the standard operator products of exponentials.

Both the vacuum expectation value $\langle\cdots\rangle_Q$ and the expectation value

$$\langle\cdots\rangle_0 = \langle -Q|\cdots|0\rangle = \langle e^{3\sigma-2\phi}(\infty)\cdots\rangle_Q \tag{6.4.6}$$

are BRST invariant. The advantage of $\langle\cdots\rangle_0$ is charge neutrality; $\langle 1\rangle_0 = 1$. On higher genus Riemann surfaces, there is a manifestly BRST invariant expectation

value $\langle\cdots\rangle_{Q(1-g)}$ with background charge $Q(1-g)$. For calculating loop amplitudes, the problem is to screen the background charge to get a BRST invariant, neutral expectation value on an arbitrary Riemann surface[12].

The correlation functions on the sphere of exponentials of the chiral scalars ϕ, χ, σ are calculated using two-point functions in the simplest possible form, $\pm\ln(z-w)$, ommitting self-contractions exactly as if there were no background charge. The only effect of the background charge is to determine which expectation values of exponentials are nonzero, namely those which neutralize the background charges.

Note that these $BRST$-invariant expectation values are for the small algebra of η, $\rho = \partial\xi$, *not* for the large algebra of η, ξ which includes the ξ_0 zero mode. In the large algebra, the neutraliser of the background charge is $e^{3\sigma-2\phi+\chi}$, but

$$[Q_{BRST},\, e^{3\sigma-2\phi+\chi}] \neq 0\,. \tag{6.4.7}$$

This is a key point in the construction of the fermion vertex.

7. The Fermion Vertex and Spacetime Supersymmetry

The object is to construct BRS invariant vertex operators for spacetime fermions, and to construct a two dimensional chiral current for spacetime supersymmetry. The vertex operators must be spin fields in the matter ghost system so they should combine the spin fields S_α of the ψ^μ system and the spin fields $e^{\pm\phi/2}$ of the β, γ system. The fermion vertex operator should be a fermion field on the world surface in order that the fermion amplitudes have the antisymmetry properties appropriate to spacetime Fermi statistics.

It is enough to construct the vertex for massless fermions, since the scattering amplitudes of all the other states appear as residues of the massless fermion amplitudes at poles in the intermediate momenta. In the language of two dimensional field theory, the vertex operators for the massless fermions generate through their operator products the algebra of vertex operators for all physical states.

7.1 $\mathbf{V_{-1/2}}$

It would be simplest if the fermion vertex did not couple to the ordinary conformal ghosts b, c. The fermion vertex operator must then anticommute (up to a total derivative) with each of $Q_{BRST}^{(0,1,2)}$ because each has a different b, c charge. The vertex operator should be an ordinary conformal field of weight 1 in order to anticommute with $Q_{BRST}^{(0)}$.

The simplest candidate for the matter part of the massless fermion vertex is $u^\alpha S_\alpha e^{ik\cdot x}$, which has dimension $5/8 + k^2/2 = 5/8$. The ghost sector must supply the missing 3/8 weight:

$$V_{-1/2} = u^\alpha e^{-\phi/2} S_\alpha e^{ik\cdot x} . \tag{7.1.1}$$

$V_{-1/2}$ is an ordinary conformal field of weight 1 if $k^2 = 0$. For invariance under $Q_{BRST}^{(1)}$ u will have to satisfy the massless Dirac equation $\not k u = 0$.

The $\Gamma = 1$ projection requires that u^α be left handed. Henceforth the convention will be that S_α is left handed, i.e., $\Gamma S_\alpha \Gamma^{-1} = S_\alpha$, and S^α is right handed. After the chiral projection, $V_{-1/2}(z)$ becomes a local fermionic field. It is fermionic because

$$V_{-1/2}(z)\, V_{-1/2}(w) \sim (z-w)^{-1} u^\alpha u^\beta \gamma^\mu_{\alpha\beta} e^{-\phi} \psi_\mu e^{2ik\cdot x}(w) , \tag{7.1.2}$$

which is odd under $z \leftrightarrow w$.

Because $V_{-1/2}$ is a conformal field of weight 1,

$$[Q_{BRST}^{(0)}, V_{-1/2}]_+ = \partial(c V_{-1/2}) . \tag{7.1.3}$$

The spacetime Dirac equation $\not k u = 0$ implies $[Q_{BRST}^{(1)}, V_{-1/2}(z)]_+ = 0$ because

$$e^{\phi-\chi} \psi_\mu \partial_z x^\mu\, V_{-1/2}(w) \sim (z-w)^{-1} (-i \not k u)_\alpha e^{\frac{1}{2}\phi-\chi} S^\alpha e^{ik\cdot x} \sim (z-w)^0 . \tag{7.1.4}$$

The last piece of the BRST invariance, $[Q_{BRST}^{(2)}, V_{-1/2}(z)]_+ = 0$, follows from the nonsingularity of the operator product expansion

$$e^{2\phi-2\chi} b(z)\, V_{-1/2}(w) \sim (z-w)^{+1} e^{\frac{3}{2}\phi-2\chi} b u^\alpha S_\alpha e^{ik\cdot x}(w) . \tag{7.1.5}$$

Combining the three pieces gives BRST invariance of $V_{-1/2}$:

$$[Q_{BRST}, V_{-1/2}]_+ = \partial(c V_{-1/2}) . \tag{7.1.6}$$

7.2 $\mathbf{V_{1/2}}$

$V_{-1/2}$ cannot be the entire fermion vertex operator because it has nonzero ϕ charge. The correlation functions $\langle V_{-1/2}(z) \cdots \rangle_0$ on the sphere all vanish by charge conservation. The correlation functions $\langle V_{-1/2}(z) \cdots \rangle_Q$ vanish except for the four point function. This difficulty can be avoided if there is a second fermion vertex, $V_{1/2}(z)$, having the opposite ϕ charge. $e^{\phi/2}$ has dimension $-5/8$ and odd chirality, so the spin field with even chirality, $e^{\phi/2} S^\alpha$ has dimension 0 and is righthanded in spacetime. To get even chirality, weight 1 and lefthandedness in spacetime, write a vertex of the form $e^{\frac{1}{2}\phi} u^\alpha \gamma^\mu_{\alpha\beta} S^\beta \partial x_\mu e^{ik\cdot x}$. The question now becomes BRST invariance.

A BRST invariant vertex operator $V_{1/2}$ can be constructed from $V_{-1/2}$ using the extension of the matter-ghost system which contains the field $\xi(z)$. Recall that the ghost system contains $\partial\xi$, but not ξ itself. Thus $\xi V_{-1/2}$ is not in the matter-ghost system, but $[Q_{BRST}, \xi V_{-1/2}]$ *is*, because the commutation with Q_{BRST} can absorb the zero mode of ξ. Since $Q_{BRST}^2 = 0$, the commutator is automatically BRST invariant. Normally, commutation with Q_{BRST} gives vertex operators for BRST-exact states, which are null and decouple from physical states. But here the algebra of fields has been expanded so that all BRST invariant states are BRST-exact in the large algebra. Thus any BRST-closed state in the small algebra can be represented as a commutator with Q_{BRST} in the large algebra.

To be precise, define $V_{1/2}$ by

$$
\begin{aligned}
V_{1/2} &= 2[Q_{BRST}, \xi V_{-1/2}] - \partial(2c\xi V_{-1/2}) \\
&= 2[Q^{(1)}_{BRST}, V_{-1/2}] + \tfrac{1}{2} b e^{3\phi/2-\chi} u^\alpha S_\alpha e^{ik\cdot x} .
\end{aligned}
\tag{7.2.1}
$$

The total derivative is subtracted because it contains ξ, whereas $V_{1/2}$ should be in the small algebra. This modification does not affect BRST invariance because the BRST commutator is still a total derivative:

$$
[Q_{BRST}, V_{1/2}]_+ = -\partial[Q_{BRST}, 2c\xi V_{-1/2}]_+ . \tag{7.2.2}
$$

The term in 7.2.1 containing $b(z)$ will never contribute to correlation functions because neither $V_{-1/2}$ nor $V_{1/2}$ contains $c(z)$. So $V_{1/2}$ might as well be defined as $2[Q^{(1)}_{BRST}, V_{-1/2}]$, which can be calculated using the operator product 4.8.11:

$$
V_{1/2} = e^{\phi/2} u^\alpha \gamma^\mu_{\alpha\beta} (\partial x_\mu + \tfrac{i}{4} k\cdot\psi\psi_\mu) S^\beta e^{ik\cdot x} . \tag{7.2.3}
$$

7.3 Scattering amplitudes

The two fermion vertex operators, $V_{-1/2}$ and $V_{1/2}$, give tree level fermion scattering amplitudes by formulas of the form

$$
A(1,\ldots,N) = \int dz_1 \cdots dz_N \, \langle V_{q_1} \cdots V_{q_N} \rangle_0 , \tag{7.3.1}
$$

but it must be shown that these formulas do not depend on the choice of ϕ charges $q_i = \pm\frac{1}{2}$, $\sum q_i = 0$. This will be done in section 7.4, by showing that the expectation value in equation 7.3.1 is invariant under rearrangements of the q_i.

First note that, given this rearrangement lemma, the fermion amplitudes factorize, as they should, on the Neveu-Schwarz amplitudes for spacetime bosons:

$$
\begin{aligned}
V_{-1/2}(z_1)\, V_{1/2}(z_2) &\sim -2[Q^{(1)}_{BRST}, V_{-1/2}(z_1)\, \xi V_{1/2}(z_2)] \\
&\sim 2[Q^{(1)}_{BRST}, (z_1 - z_2)^{-1+k_1\cdot k_2} u_1^\alpha u_2^\beta \gamma^\mu_{\alpha\beta} \xi e^{-\phi} \psi_\mu e^{i(k_1+k_2)\cdot x}(z_2)] \\
&\sim (z_1 - z_2)^{-1+k_1\cdot k_2}\, u_1^\alpha u_2^\beta \gamma^\mu_{\alpha\beta} \\
&\qquad \times (\partial x_\mu + i(k_1 + k_2)\cdot\psi\psi_\mu)\, e^{i(k_1+k_2)\cdot x}(z_2) ,
\end{aligned}
\tag{7.3.2}
$$

so the integral over z_1 near z_2 gives a pole at $(k_1 + k_2)^2 = 0$ whose residue is a NS massless vector boson vertex operator of the form $V_0(z_2) = \int d\theta \cdot DX e^{ik\cdot X}$.

On the other hand, the operator products $V_{1/2}\, V_{1/2}$ and $V_{-1/2}\, V_{-1/2}$ factorize the amplitudes on vertex operators $V_{\pm 1}$ with ϕ charge ± 1. For example, $V_{-1/2}\, V_{-1/2} \sim V_{-1} = e^{-\phi}\psi e^{ik\cdot x}$. But any factorization on $V_{\pm 1}$ is exactly equivalent to a factorization on V_0, since each pair of fermion vertices which are brought close together can be taken to be of opposite charge by the rearrangement lemma. Therefore $V_{\pm 1}$ are alternative forms of the massless boson vertex. Equation 7.3.2 shows explicitly that V_0 is derived from V_{-1} exactly as $V_{1/2}$ is derived from $V_{-1/2}$. A more complicated calculation shows that V_{+1} is derived in the same way from V_0. In the classical formulation of the fermionic string, the physical states corresponding to the vertex operators V_{-1} and V_0 were discussed as two equivalent "pictures" for the string states, although, because the ghosts were missing, there was no weight 1 vertex operator V_{-1}.

It is clear now that there are infinitely many pictures, corresponding to the infinitely many Bose seas which give the inequivalent representations of the superconformal ghost fields β, γ. The infinite number of equivalent vertex operators for each physical state are derived by the picture changing operation $V_{q+1} = [Q_{BRST}, \xi V_q]$, and by its inverse. The rearrangement lemma given below for $V_{\pm 1/2}$ can be generalized to show the equivalence of all the pictures for both fermions and bosons.

7.4 The rearrangement lemma

The object is to show that

$$\left\langle \cdots V_{1/2}(z) \cdots V_{-1/2}(w) \cdots \right\rangle_0 = \left\langle \cdots V_{-1/2}(z) \cdots V_{1/2}(w) \cdots \right\rangle_0 . \tag{7.4.1}$$

The idea is to use the equivalence of expectation values in the small and large algebras and the contour integral form of the BRST transformation:

$$\begin{aligned} \langle \cdots A(z) \cdots [\epsilon Q_{BRST}, \xi A(w)] \cdots \rangle_{\partial\xi} &= \frac{1}{2\pi i} \oint_{C_w} dz' \, \langle \xi(\infty) \cdots A(z) \cdots j_{BRST}(z') \, \xi(w) A(w) \rangle \\ &= \frac{1}{2\pi i} \oint_{C_z} dz' \, \langle \cdots \xi(z) A(z) \cdots j_{BRST}(z') \, \xi(w) A(w) \rangle \\ &= \langle \cdots [\epsilon Q_{BRST}, \xi A(w)] \cdots A(z) \cdots \rangle_{\partial\xi} . \end{aligned} \tag{7.4.2}$$

The contour can be deformed by BRST invariance of the expectation value and of the operators represented by ellipses. The ξ field can be moved because its dimension is zero and only its zero mode participates in the expectation value.

7.5 Spacetime supersymmetry

The spacetime supersymmetry current $q_\alpha(z)$ is simply the fermion vertex at zero momentum. It takes the forms

$$\cdots \quad q_{\alpha,-1/2}(z) = e^{-\phi/2} S_\alpha \qquad q_{\alpha,1/2} = e^{\phi/2} \gamma^\mu_{\alpha\beta} S^\beta \partial x_\mu \quad \cdots . \tag{7.5.1}$$

in the various pictures. $q_\alpha(z)$ is a BRST invariant dimension 1 conformal field, so its contour integral

$$Q_{\alpha,q} = \frac{1}{2\pi i} \oint dz \, q_{\alpha,q}(z) \tag{7.5.2}$$

is invariantly defined. The operator product 7.3.2 gives

$$[Q_{\alpha,-1/2}, Q_{\beta,1/2}]_+ = \gamma^\mu_{\alpha\beta} p_\mu \tag{7.5.3}$$

where p_μ is the spacetime momentum operator, and $[Q_{\alpha,1/2}, V_{-1/2}] = V_0$, i.e.,

$$[Q_{\alpha,1/2}, e^{-\phi/2} S_\beta e^{ik\cdot x}]_+ = \gamma^\mu_{\alpha\beta} \psi_\mu e^{ik\cdot x} , \tag{7.5.4}$$

showing that Q_α is the spacetime supersymmetry generator. Since Q_α commutes with the screening operator $e^{3\sigma - 2\phi}$ for the background charge, the contour argument shows that the expectation values $\langle \cdots \rangle_0$ on the sphere are invariant under supersymmetry. Thus the tree amplitudes are supersymmetric. All that is needed to show supersymmetry of the loop expansion, since $q_\alpha(z)$ is a conformal current, is to show that the screening charges on arbitrary Riemann surfaces preserve spacetime supersymmetry.

REFERENCES

1. S.H. Shenker, this volume.
2. D. Friedan, Z. Qiu, and S. Shenker, *Proc. of the Santa Fe Meeting of the APS Div. of Particles and Fields, October 31 - November 3, 1984*, T. Goldman and M. Nieto (eds.), World Scientific (1985).
3. D. Friedan, Z. Qiu, and S. Shenker, *Phys. Lett.* 151B (1985) 37.
4. D. Friedan, E. Martinec, and S. Shenker, *Phys. Lett.* 160B (1985) 55.
5. D. Friedan, E. Martinec, and S. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Fermi Institute preprint EFI 85-89 and Princeton preprint, submitted to Nucl. Phys. B.
6. J. Cohn, D. Friedan, Z. Qiu, and S. Shenker, EFI preprint 85-90 (presented in reference 2).
7. D. Friedan, in 1982 Les Houches summer school *Recent Advances in Field Theory and Statistical Mechanics*, J-B. Zuber and R. Stora (eds.), North-Holland (1984).
8. A.M. Polyakov, *Phys. Lett.* 103B (1981) 207.
9. K. Fujikawa, *Phys. Rev.* D25 (1982) 2584; M. Kato and K. Ogawa, *Nuc. Phys.* B212 (1983) 443; and S. Hwang, *Phys. Rev.* D28 (1983) 2614.
10. See Mandelstam's lectures in this volume.
11. A.M. Polyakov, *Phys. Lett.* 103B (1981) 211; elaborated in E. Martinec, *Phys. Rev.* D28 (1983) 2604.
12. E. Martinec, in preparation.

13. D. Friedan and P. Windey, *Nucl. Phys.* B235[FS11] (1984) 395.
14. Cf. R.C. Gunning, *Lectures on Riemann Surfaces*, Princeton Univ. Press (1966).
15. A.A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Nuc. Phys.* B241 (1984) 333.
16. D. Friedan, Z. Qiu, and S. Shenker in *Vertex Operators in Mathematics and Physics*, J. Lepowsky et.al. (eds.), Springer-Verlag (1984); *Phys. Rev. Lett.* 52 (1984) 1575.
17. G. Segal, *Comm. Math. Phys.* 80 (1981) 301;
18. V. Knizhnik and A. B. Zamolodchikov, *Nuc. Phys.* B247 (1984) 83.
19. A. Chodos and C. Thorn, *Nuc. Phys.* B72 (1974) 509; B. L. Feigin and D. B. Fuchs, Moscow preprint (1983) and *Funct. Anal. Appl* 16 (1982) 114; V. L. Dotsenko and V. Fateev, *Nuc. Phys.* B240 (1984) 312; C. Thorn, *Nucl. Phys.* B248 (1985) 551.
20. Cf. P. Goddard and D. Olive, *Nucl. Phys.* B257[FS14] (1985) 83.