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The Space of Conformal Field Theories and the Space of Classical String Ground States

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ABSTRACT

A formal (and speculative) construction is given of the space of all conformal field theories of given conformal central charge – and also of the space of all classical ground states of string – using only intrinsic structure in the space of all closed Riemann surfaces.

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FOREWORD

Daniel Friedan

I met Dima Knizhnik only three times – first in Moscow in the spring of 1983, again in Moscow at the Landau-Nordita meeting in June, 1984 and, finally, at the Yukawa Symposium in Kyoto in October, 1987. Because of the political situation in those years, direct contacts were infrequent between physicists working in the Soviet Union and physicists working in the U.S. or even Europe.

On several occasions Dima and I were thinking independently about the same subjects at the same times, usually from somewhat different points of view. It might have been interesting if we had had more freedom to interact. Beginning in 1984, Steve Shenker and I made Dima a standing offer of a position, but he was not able even to consider visiting until just before his death.

Consequently, I was forced to know Dima mostly from his published work. It is clear that he was in a state of explosive intellectual growth throughout his short career. It is impossible to imagine where time might have led him.

Theoretical physics is, by and large, an improvisational ensemble work, although the psychology of physicists seems to require that the official history be written otherwise. When such a creative voice as Dima Knizhnik is lost, we are all irreversibly diminished.

1. Introduction

A succinct abstract characterization of the space of conformal field theories – and of its close relative, the space of classical ground states of string – might help with the classification of these objects and might also provide a starting point for a characterization of the quantum ground state of string.

This note is a formal and speculative attempt at such a characterization. The space CFT_c of unitary conformal field theories with conformal central charge c is conjectured to be exactly the spectrum $Spec(A_c)$ of a certain commutative $*$-algebra A_c constructed (formally) using only intrinsic structure in a line bundle L_c over the space of all closed Riemann surfaces. Recall that the spectrum of a commutative $*$-algebra is the space of all characters or, equivalently, irreducible representations of the algebra. The algebra can be interpreted as the algebra of functions on its spectrum. In other words, CFT_c is to be constructed by constructing its function algebra A_c from a certain line bundle over the space of all closed Riemann surfaces.

Similarly, the space CGS of all classical ground states of string is constructed (formally and speculatively) by constructing its function algebra A_+ from the space of all closed Riemann surfaces. This construction is purely intrinsic; it does not even refer to the fact that the points of the space stand for Riemann surfaces.

Both constructions are formal and both beg a crucial analytic question. The present goal is only to make a simple formulation; all the hard work is left for later.

The basic underlying suppositions are (1) that the partition function of a conformal field theory for all closed Riemann surfaces determines the conformal field theory uniquely, i.e., that no two conformal field theories have the same partition function, and (2) that any section of a certain line bundle over the space of closed Riemann surfaces satisfying a small number of intrinsic conditions is the partition function of some conformal field theory.

The reason for basing the construction on the *closed* Riemann surfaces is that it avoids introducing a Hilbert space of states as a fundamental object. In string theory, the Hilbert space varies with the classical ground state and might not even make sense in the full quantum theory. It seems attractive to have the Hilbert space be a derivative object in the classical theory.

It should be mentioned that the approach described in the present note has not yet led to any concrete progress on the problem of classifying conformal field theories in general nor on the problem of describing the quantum ground state of string.

The abstract characterization of conformal field theory and classical string theory sketched in this note is essentially a refined version of that given in references 1-3 several years ago. More details on some points can be found there. The idea of seeing the string partition function as an object on the moduli space of Riemann surfaces was independently arrived at by Belavin and Knizhnik[4].

2. The universal moduli space

We start by defining the *universal moduli space* of Riemann surfaces, M. It is the space of all smooth, compact, not necessarily connected Riemann surfaces without boundary, none of whose connected components are 2-spheres, plus one extra point, the 2-sphere itself. We will see that M has the structure of a connected, analytic, commutative $*$-semigroup with the 2-sphere $\mathbf{P}$ as its identity element.

It is convenient to define M by way of two larger moduli spaces. Let M_{sm} be the space of all smooth, compact, not necessarily connected Riemann surfaces without boundary. Let $\widetilde{M}$ be the space of all compact, not necessarily connected Riemann surfaces without boundary, which are smooth except for at most a finite number of nodes. The space of surfaces with at least one node is $D = \widetilde{M} - M_{sm}$.

A node in a Riemann surface can be thought of as a circle on the surface which has been pinched down to a single point. Equivalently, a node can be pictured as an infinitely long tube. A more detailed description is given below.

The partition function Z of a conformal field theory is a section of the line bundle $L_c = (EE^*)^{c/24}$ over M_{sm}. Here $E = (\lambda_H)^{12}$ is the holomorphic line bundle over M formed by taking the 12^{th} power of the determinant of the holomorphic differentials on the Riemann surfaces comprising M; E^* is the complex conjugate of E and c is the conformal central charge of the theory. In reference 1 the holomorphic line bundle E is defined in a way more natural to conformal field theory, in terms of families of projective connections on Riemann surfaces.

The partition function satisfies

(I) Z is nonsingular on the surfaces with nodes and thus extends to $\widetilde{M}$,

(II) $Z(\Sigma) = Z(\nu\Sigma)$,

(III) $Z(\mathbf{P}) = 1$,

(IV) $Z(\Sigma^*) = Z(\Sigma)^*$,

(V) $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1)\, Z(\Sigma_2)$,

where Σ, Σ_1 and Σ_2 are arbitrary Riemann surfaces (in $\widetilde{M}$), Σ^* is the complex conjugate of Σ and $\nu\Sigma$ is the smooth Riemann surface obtained by removing all nodes in Σ (the *normalization* of Σ). These conditions make sense because $(L_c)_{\mathbf{P}} = \mathbf{C}$, the complex numbers; $(L_c)_{\Sigma^*} = (L_c^*)_\Sigma$; L_c extends to $\widetilde{M}$ and $(L_c)_{\Sigma_1 \cup \Sigma_2} = (L_c)_{\Sigma_1} \otimes (L_c)_{\Sigma_2}$.

Condition (I) follows from locality, homogeneity and the fact that the conformal weights of the field theory are nonnegative. This is discussed in more detail below.

Condition (II) follows from locality, homogeneity and the uniqueness of the ground state. This is also discussed below.

Condition (III) follows from the fact that the vacuum state of the conformal field theory has nonzero norm. The partition function can then be normalized to take the value 1 on the 2-sphere.

Condition (IV) follows from CPT invariance. In a functional integral formulation of the field theory, CPT invariance means that the action is invariant under complex conjugation combined with orientation reversal of the surface.

Condition (V) is a kind of cluster decomposition property. It states the obvious fact that disconnected components of a Riemann surface are decoupled in a conformal field theory.

To explain conditions (I) and (II) we need a detailed description of a node in a Riemann surface. The neighborhood of a node can be parametrized by two local coordinates z_1 and z_2 satisfying a patching equation $(z_1 - x_1)(z_2 - x_2) = 0$. The neigborhood of the node thus consists of the two smooth neighborhoods parametrized by z_1 and z_2 with the point $z_1 = x_1$ in the first neighborhood glued to the point $z_2 = x_2$ in the second neighborhood.

Now restrict the coordinates z_1 and z_2 to annuli $|q| < |z_i - x_i| < 1$ and replace the above patching equation by $(z_1 - x_1)(z_2 - x_2) = q$. For $q \neq 0$ the patching equation is nowhere singular. Then the two coordinates z_1 and z_2 can be replaced by a single coordinate $z = q^{-1/2}(z_1 - x_1) = q^{1/2}(z_2 - x_2)^{-1}$ ranging over the annulus $|q|^{1/2} < |z| < |q|^{-1/2}$. The node has become a tube of length $-\ln|q|$. The limit $q \rightarrow 0$ exhibits the node as an infinitely long tube.

The space D of Riemann surfaces with nodes is generically of complex codimension 1 in $\widetilde{M}$, because the surfaces with node are specified by the one equation $q = 0$. The surfaces with n nodes are specified by n equations of the form $q_i = 0$, with $q_i \rightarrow 0$ describing the closing of the i^{th} node. It is crucial that the closing of multiple nodes is described by the vanishing of *independent* parameters q_i.

Because of the locality and homogeneity of the conformal field theory, the annulus or tube parametrized by q can be represented in the operator formulation of the theory as $q^{L_0}(\bar{q})^{\overline{L}_0}$ acting on the states flowing through the tube (L_0 and $\overline{L}_0$ being the usual Virasoro operators). At $q = 0$ this operator becomes the projection on the ground state(s) of the theory, assuming that L_0 and $\overline{L}_0$ are both nonnegative (which follows from unitarity). The partition function is thus nonsingular in the limit $q \to 0$. This gives condition (I).

Assuming that the ground state is unique, the node is represented by the projection $|0\rangle\langle 0|$ on the $SL_2 \times SL_2$-invariant ground state $|0\rangle$. The ground state is thus the only state which flows through a node. Picturing the node as an infinitely long tube makes this obvious.

By the operator analysis, the partition function of the Riemann surface with node, i.e., at $q = 0$, is the same as the partition function of the surface obtained by forgetting the identification of the two points $z_1 = x_1$, $z_2 = x_2$, and using the ground state to provide boundary conditions at the two punctures $z_1 = x_1$ and $z_2 = x_2$. The ground state boundary condition at a puncture is equivalent to insertion of the identity quantum field at the puncture, which in turn is equivalent to forgetting the puncture entirely.

The partition function of a Riemann surface Σ with nodes is therefore the same as the partition function of the smooth surface $\nu\Sigma$ obtained by forgetting the patching equations for the nodes, i.e., by removing the nodes and filling in the resulting punctures. This is condition (II).

Now define an equivalence relation on $\widetilde{M}$ by requiring that, for all Riemann surfaces Σ in $\widetilde{M}$,

$$\Sigma \sim \nu\Sigma \sim \Sigma \cup \mathbf{P}\,. \tag{1}$$

The *universal moduli space* is defined to be the quotient $M = \widetilde{M}/\!\sim$. The line bundle L_c respects the equivalence relation and can be regarded as a line bundle over M.

Under the equivalence relation, every Riemann surface is equivalent to a smooth surface, and every surface except $\mathbf{P}$ is equivalent to a smooth surface none of whose connected components are 2-spheres. Thus, as a point set, M is the space of all smooth, compact, not necessarily connected Riemann surfaces without boundary, none of whose connected components are 2-spheres, plus one extra point, the 2-sphere itself.

It follows from conditions (I)-(III) and (V) that the partition function of a conformal field theory is a section of the line bundle $L_c \to M$.

Henceforth, when we write Σ we mean the equivalence class in M. The equivalence relation respects complex conjugation and disjoint union, so Σ^* and $\Sigma_1 \cup \Sigma_2$ still make sense. In fact, the operation of disjoint union makes M into a commutative $*$-semigroup with product

$$\Sigma_1\Sigma_2 = \Sigma_1 \cup \Sigma_2, \tag{2}$$

identity element

$$1 = \mathbf{P} \tag{3}$$

and conjugation

$$\Sigma \mapsto \Sigma^* . \tag{4}$$

In addition to this algebraic structure, the universal moduli space M is a connected analytic space. This was shown in reference 1 by constructing M from the stable compactifications of the moduli spaces M_g of smooth, connected compact Riemann surfaces of genus g. A more efficient way is to observe* that M is the direct limit as $g \to \infty$ of the Satake compactification M_g^{sat} of the moduli space M_g. The Satake compactification[5] is constructed by embedding M_g in the moduli space of abelian varieties and compactifying that space by algebraic means. The result is that M_g^{sat} is a connected projective algebraic variety for each genus g.

*as was pointed out by S. Bloch and by P. Deligne.

What is added to M_g to obtain M_g^{sat} is precisely the set of Riemann surfaces obtainable from those in M_g by forming and removing nodes and then discarding the genus 0 components (2-spheres) of the resulting surface. In particular, there is a consistent system of natural embeddings $M_{g-1}^{sat} \to M_g^{sat}$. The universal moduli space is the direct limit

$$M = \lim_{g \to \infty} M_g^{sat} . \tag{5}$$

M is thus a connected analytic space and, in fact, the direct limit of projective varieties.

The line bundle L_c is real-analytic on M since $E = (\lambda_H)^{12}$ is a well-defined holomorphic line bundle on M. L_c is itself a commutative $*$-semigroup with identity, since $(L_c)_{\Sigma_1\Sigma_2} = (L_c)_{\Sigma_1} \otimes (L_c)_{\Sigma_2}$, $(L_c)_{\mathbf{P}} = \mathrm{C}$ and $(L_c)_{\Sigma^*} = (L_c^*)_{\Sigma}$. The line bundle map $L_c \to M$ respects the semigroup structures, with the multiplication and conjugation laws in L_c being linear in the fibers.

Given that M is a connected topological semigroup with identity, elementary algebraic topology tells us that $\pi_1(M)$ is abelian and thus equal to the first homology group $H_1(M)$. It should be possible to show that $H_1(M) = 0$ and thus that M is simply-connected. This seems a fundamental point to establish, since a non-contractible loop in M would allow global anomalies which could not be eliminated by conditions local on M. Establishing $H_1(M) = 0$ would have the added benefit of leaving the holomorphic line bundles on M classified by $H^2(M)$. It seems possible to show that $H^2(M) = \mathbf{Z}$ (see reference 1), which would then imply that the powers of $E = (\lambda_H)^{12}$ are the only holomorphic line bundles nonsingular on M. The line bundle L_c could then be said to be intrinsic to M.

3. The space of conformal field theories

We have seen that the universal moduli space of Riemann surfaces, M, is a connected, analytic, commutative $*$-semigroup with identity and that the line bundle $L_c \to M$ is a morphism of commutative $*$-semigroups with identity such that the multiplication and conjugation laws in L_c are linear in the fibers.

Conditions (I)-(V) on the partition function Z of a conformal field theory can now be condensed into the condition that $Z : M \to L_c$ should be a section of $\pi : L_c \to M$ which is at the same time a morphism of $*$-semigroups, i.e.,

(1) $\quad \pi \circ Z = id,$

(2) $\quad Z(1) = 1,$

(3) $\quad Z(\Sigma^*) = Z(\Sigma)^*,$

(4) $\quad Z(\Sigma_1 \Sigma_2) = Z(\Sigma_1)\, Z(\Sigma_2).$

Conditions (1)-(4) become a formal characterization of the space of conformal field theories, given two suppositions which were made in reference 1:

(S1) A conformal field theory is completely determined by its partition function $Z : M \to L_c$. No two conformal field theories have the same partition function.

(S2) There exist some intrinsic criteria which specify a linear subspace of the local analytic sections of $L_c \to M$ such that *every* global analytic section satisfying Z those local criteria and also satisfying conditions (2)-(4) above is the partition function of some unitary conformal field theory. That is, conditions (1)-(4) plus some appropriate local analyticity condition suffice to permit reconstructing a unique conformal field theory from Z. (The idea that unitarity should follow automatically was not suggested in reference 1. Some support for the idea is given below.)

Given (S1) and (S2), the space CFT_c is exactly the space of analytic (in an as yet unspecified sense) sections of $L_c \to M$ which are morphisms of $*$-semigroups. Not least of the remarkable consequences would be that any such section, restricted to genus 1 surfaces, would have an expansion in powers of q and $\bar{q}$ with nonnegative integer coefficients, since the coefficients would be the multiplicities of representations of the Virasoro algebras.

Neither of these suppositions has been proved (but see reference 1 and below for some arguments in their favor). Supposition (S1) is a reasonable conjecture – it is precisely stated and there are no known counterexamples. (S2) is not a precise conjecture and the essential missing ingredient which must be supplied before it becomes one is the appropriate local analyticity criterion.

It is now a simple formal exercise to translate conditions (1)-(4) into a construction of a commutative $*$-algebra A_c whose spectrum is CFT_c. Let $\Gamma(M, L_c)$ be the linear space of global analytic sections of $L_c \to M$ satisfying the as yet unspecified local analyticity condition. Define A_c to be the dual linear space $\Gamma(M, L_c)^*$ so that $\Gamma(M, L_c) = A_c^*$.

To see that A_c is a commutative $*$-algebra with identity, first construct a dual multiplication on $\Gamma(M, L_c)$. For $s \in \Gamma(M, L_c)$ define $m^*s \in \Gamma(M, L_c) \otimes \Gamma(M, L_c)$ by setting $m^*s(\Sigma_1, \Sigma_2) = s(\Sigma_1\Sigma_2)$. This makes sense because $(L_c)_{\Sigma_1} \otimes (L_c)_{\Sigma_2} = (L_c)_{\Sigma_1\Sigma_2}$. Now define the product of $\alpha_1, \alpha_2 \in A_c$ by $(\alpha_1\alpha_2, s) = (\alpha_1 \otimes \alpha_2, m^*s)$ for all $s \in \Gamma(M, L_c)$. The identity $1 \in A_c$ is given by $(1, s) = s(\mathbf{P})$ for all $s \in \Gamma(M, L_c)$. The conjugation in A_c is given by $(\alpha^*, s) = (\alpha, s^*)^*$.

Since $\Gamma(M, L_c) = A_c^*$, conditions (1)-(4) on the partition function are equivalent to stating that the partition function of a conformal field theory is an element $Z \in A_c^*$ satisfying

$$\text{(i)} \quad Z(1) = 1,$$

$$\text{(ii)} \quad Z(\alpha^*) = Z(\alpha)^*,$$

$$\text{(iii)} \quad Z(\alpha_1 \alpha_2) = Z(\alpha_1) Z(\alpha_2) \ .$$

The elements of A_c^* satisfying (i)-(iii) are the characters of A_c or, equivalently, the irreducible representations or maximal ideals. The space of all irreducible representations is the spectrum $Spec(A_c)$. A_c can be interpreted as the commutative $*$-algebra of functions on its spectrum.

The suppositions (S1) and (S2) now amount to the suggestion that $CFT_c = Spec(A_c)$. This characterization of CFT_c is extremely formal and depends on the imprecise and unproved suppostitions (S1) and (S2). If $\Gamma(M, L_c)$ can be defined precisely, it should then be possible to put a norm or norms on it such that $Spec(A_c)$ becomes a topological space or even a real analytic space. This would be a precise expression of the idea that conformal field theories are close together in CFT_c if their partition functions are close on M.

From what is known of examples, it seems possible that CFT_c is in fact an algebraic variety. It might well be that the solvable conformal field theories are too special. But, for example, the gaussian models, a subset of CFT_c for $c = n$ an integer, form the algebraic variety $O(n, n, \mathbf{Z}) \backslash O(n, n, \mathbf{R}) / O(n, \mathbf{R}) \times O(n, \mathbf{R})$. It is difficult to see how the present approach would provide such an algebraic structure for $Spec(A_c)$.

This formalism generalizes readily to give the space $SCFT_c$ of superconformal field theories entirely in terms of intrinsic structure in a line bundle over the universal moduli space of super Riemann surfaces[7]. A concrete result in this direction could be of some conventional mathematical interest, since the boundary of $SCFT_c$ consists of the Calabi-Yau spaces of dimension c in the limit of large volume. Reconstructing the superconformal field theories would lead to a construction of the Calabi-Yau metrics. This would be done by considering the conformal fields whose weights approach zero as the field theory approaches the boundary of $SCFT_c$. In the limit, these fields form a commutative associative operator product algebra whose spectrum is the Calabi-Yau space. The conformal weights of these fields approach the eigenvalues of the laplacian on the Calabi-Yau space, from which the metric could be reconstructed, in principle. Reference 6 provides the dictionary, in the large volume limit of a manifold, between the eigenfunctions of the laplacian on the manifold and the fields whose dimensions are the eigenvalues of the laplacian, and between the multiplication of functions on the manifold and the operator products of the corresponding fields.

Comments on unitarity

It might seem surprising to suggest that no additional positivity conditions are needed to ensure unitarity of the reconstructed conformal field theory. There are some obvious positivity conditions which are consequences of unitarity. Suppose Σ to be a doubled surface – a closed surface made by gluing a surface with boundary to its complex conjugate surface. The partition function of a unitary conformal field theory is positive at such a surface Σ. However, it seems impossible to describe the space of doubled surfaces in terms intrinsic to M. Even if this were possible, it is hard to see how such a positivity condition would imply unitarity, at least in as straightforward a way as the reflection positivity condition implies unitarity in euclidean quantum field theory.

Physical considerations suggest that *no* positivity conditions are needed to ensure unitarity. The normalization condition $Z(1) = 1$ and the nonsingularity of Z together imply that a Landau-Ginsburg model exists. That is, the logarithm of the partition function is finite and thus can be expanded in derivatives of order parameters. The reality condition $Z(\Sigma^*) = Z(\Sigma)^*$ implies that the Landau-Ginsburg model will be CPT invariant. By power counting, any such Landau-Ginsburg effective action contains no more than two derivatives of the order parameters and is manifestly unitary.

The available evidence supports this argument. The normalization condition $Z(1) = 1$ and the nonsingularity of Z on M ensure that any reconstructed conformal field theory would have a unique ground state and nonnegative conformal weights. Every known conformal field theory with these two properties is in fact unitary.

Comments on the suppositions

A sketch of a method for substantiating (S1) was given in reference 1. The partition function of a conformal field theory can be expanded in powers of the coordinates q_i and $\bar{q}_i$ which parametrize the opening of nodes. The coefficients in these q-expansions are sums of products of correlation functions of local fields. It should be possible to reconstruct the correlation functions from these coefficients.

The situation is especially simple when the representations of the conformal algebra occur with multiplicity at most 1. The partition function in genus 1 gives the conformal weights. The partition function in genus 2 gives the squares of the 3-point correlation functions and thus determines the operator product coefficients up to signs. It seems plausible that enough information is available at higher genus to fix the signs (up to Z_2 symmetries of the theory). This would determine the theory completely.

A sketch of an argument in favor of supposition (S2) was also given in reference 1. Any section $Z \in \Gamma(M, L_c)$ satisfying conditions (2)-(4) will have q-expansions from whose coefficients correlation functions can be extracted. These will satisfy – by virtue of (2)-(4) and analyticity – the axioms of conformal field theory. The correlation functions of the analytic stress tensor will then be derived by taking derivatives on M.

Comments on $\Gamma(M, L_c)$

If we admit as the sections of $L_c \rightarrow M$ all the real analytic sections which can be expressed locally as finite sums of analytic times analytic functions on M, then $Spec(A_c)$ will be the space of so-called *rational* unitary conformal field theories – modulo (S1) and (S2). In the example of the gaussian models, the rational conformal field theories are the rational points in the space of conformal field theories (this was one motivation for the nomenclature). In these special theories it is possible to describe the partition function as a sesquilinear pairing of holomorphic sections of finite rank projectively flat vector bundles over M, as in reference 1. Reference 8 reviews the considerable progress which has been made towards classifying the rational theories.

The rank of the projectively flat vector bundle jumps wildly even when the conformal field theory – the partition function – changes very little. Since our object is to characterize the whole space of conformal field theories, the vector bundle language seems inappropriate. There might be, however, a way to specify $\Gamma(M, L_c)$ based on some analytically precise notion of sesquilinear pairings of sections of infinite rank projectively flat vector bundles. The rational theories would merely be special points at which the infinite rank bundle degenerates to finite rank.

It might seem natural to take as $\Gamma(M, L_c)$ all nonsingular real analytic sections of L_c which have expansions in powers of q and $\bar{q}$. There seems, however, to be a difficulty here. Suppose that Z is the partition function of a conformal field theory with central charge c, such that Z is nowhere zero on M – for example, a partition function defined via a functional integral with positive measure. Any real power Z^{γ} would be nonsingular and real analytic, would have a q expansion and would satisfy conditions (1)-(4) with central charge γc. But we know that unitarity permits only a discrete set of values of the central charge less than 1. Thus either a simple real analyticity condition for $\Gamma(M, L_c)$ is too weak, or unitarity requires additional conditions beyond (1)-(4).

4. The space of classical string ground states

The classical ground states of string theory differ from conformal field theories in a few respects.

First, the partition function of string theory cannot be normalized to 1 on the 2-sphere. Its value on the 2-sphere is $Z(\mathbf{P}) = \lambda^{-2}$ where λ is the string coupling constant, which is a parameter of the classical ground state. To take the string coupling constant into account it is necessary to define an augmented universal moduli space M_+.

Let $\widetilde{M}_+$ be the compact, not necessarily connected Riemann surfaces without boundary which are smooth except for at most a finite number of nodes and which have no components of genus 0 (no components which are 2-spheres). The empty surface is included in $\widetilde{M}_+$.

For $\Sigma \in \widetilde{M}_+$, let $\nu_+\Sigma$ be the smooth surface obtained by removing the nodes in Σ and discarding all the resulting components of genus 0. Write $\chi(\Sigma)$ for the Euler number of Σ. The difference $[\chi(\nu_+\Sigma) - \chi(\Sigma)]/2$ is equal to the number of nodes removed from Σ minus the number of genus 0 components discarded. It is always nonnegative.

Introduce a new, abstract element $x = x^*$ with Euler number $\chi(x) = -2$. This new element will play the formal role of $\mathbf{P}^{-1}$ in the semigroup. Let $F(x) = \{1, x, x^2, \ldots\}$ be the free commutative $*$-semigroup on x. Define an equivalence relation on $F(x) \times \widetilde{M}_+$ by requiring that

$$x^n \Sigma \sim x^{n+[\chi(\nu_+\Sigma)-\chi(\Sigma)]/2} (\nu_+\Sigma) . \tag{6}$$

The augmented universal moduli space M_+ is defined to be the quotient $F(x) \times \widetilde{M}_+/\sim$.

As before, M_+ is a $*$-semigroup with identity. The product operation is again the disjoint union of Riemann surfaces. But now the identity element is the empty surface, which is equivalent to the singular torus, since removing the node from the singular torus leaves $\mathbf{P}$, which is discarded to give the empty surface. $F(x)$ is a sub-semigroup of M_+. The previously defined universal moduli space M is the quotient $M_+/F(x)$.

M_+ is analytic but it is not connected – in fact it is the infinite symmetric product of the union of all the Satake compactifications M_g^{sat}. The Euler number is a well-defined continuous morphism from $\widetilde{M}_+$ to the additive semigroup of nonpositive even numbers. The connected components of $\widetilde{M}_+$ are the sets of fixed Euler number.

The partition function Z_{str} of a classical string ground state is a section of the line bundle $L_+ \to M_+$, where L_+ is the line bundle of densities or volume elements on M_+. In terms of holomorphic objects, $L = K_{M_+} K^*_{M_+}$, where K_{M_+} is the canonical line bundle of M_+ – the determinant of the holomorphic cotangent bundle. Suppose (q, x_1, x_2) parametrizes a node in a Riemann surface Σ. A local holomorphic section ω of K_{M_+} has a double pole at $q = 0$ of the form

$$\omega(q, x_1, x_2, \Sigma) = q^{-2} dq dx_1 dx_2 \, \omega(\Sigma) + O(q^{-1}) . \tag{7}$$

This is the only definition of the canonical bundle which respects the equivalence relation used to define M_+.

The line bundle $L_+ \to M_+$ is again a morphism of $*$-semigroups. The space CGS of string classical ground states is – modulo the familiar suppositions – the space of all sections $Z_{str} : M_+ \to L_+$ which are $*$-semigroup morphisms.

The coupling constant is given by $Z_{str}(x) = \lambda^2$. The coupling constant is a free parameter of the classical ground state because the Euler number is continuous on M_+. This allows the string partition function to be multiplied by $(\lambda'/\lambda)^{\chi(\Sigma)}$, changing the coupling constant from λ to λ'.

Just as before, the linear space $A_+ = \Gamma(M_+, L_+)^*$ is a $*$-algebra with identity and (conjecturally) $CGS = Spec(A_+)$.

The algebra A_+ is related to the algebra A_{26} by the exact sequence

$$0 \to \mathbf{C}(x) \to A_+ \to A_{26} \to 0\,. \tag{8}$$

This follows from the fact that the central charge of the conformal ghost system is -26.

In order to obtain the analogous abstract characterization of the classical ground states of fermionic string theory, the augmented universal moduli space of ordinary Riemann surfaces should be replaced by the analogous construction for super Riemann surfaces[7].

The string construction is completely intrinsic to the space M_+. Once M_+ is obtained, the fact that the points of M_+ represent Riemann surfaces can be forgotten. Abstracting the string ground state away from the notion of Riemann surface might be desirable, since the interpretation as a theory of strings might make sense only at weak coupling.

In reference 2 it was suggested that the quantum ground state of string might be described in the same language as the classical ground states, after "completing" the universal moduli space, now M_+, to include "infinite genus" surfaces. There is still nothing particularly useful to say about this suggestion, except possibly that the completion of M_+ ought to be connected. Then the Euler number would no longer be continuous and the coupling constant no longer arbitrary.

A more accessible problem might be to construct the perturbative string S-matrix in this abstract approach. There would have to be some way to understand

Wick rotation abstractly (perhaps as analytic continuation in CGS). It might also be interesting to try to extend the abstract characterization of the classical ground states described in this note to an analogous characterization of at least the perturbative quantum ground states, to see if the formal simplicity can be maintained.

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