# Quantum field theories of extended objects 

Daniel Friedan<br>New High Energy Theory Center<br>and Department of Physics and Astronomy<br>Rutgers, The State University of New Jersey<br>Piscataway, New Jersey 08854-8019, USA.<br>The Science Institute<br>The University of Iceland<br>Reykjavik, Iceland


#### Abstract

First steps are taken in a project to construct a general class of conformal and perhaps, eventually, non-conformal quantum field theories of ( $n-1$ )-dimensional extended objects in a $d=2 n$ dimensional conformal space-time manifold $M$. The fields live on the spaces $\mathcal{E}_{\partial \xi}$ of relative integral $(n-1)$-cycles in $M$. These are the integral $(n-1)$-currents of given boundary $\partial \xi$. Each $\mathcal{E}_{\partial \xi}$ is a complete metric space geometrically analogous to a Riemann surface $\Sigma$. For example, if $M=S^{d}, \Sigma=S^{2}$. The quantum fields on $\mathcal{E}_{\partial \xi}$ are to be mapped to observables in a 2 d CFT on $\Sigma$. The correlation functions on $\mathcal{E}_{\partial \xi}$ are to be given by the 2 d correlation functions on $\Sigma$. The goal is to construct a CFT of extended objects in $d=2 n$ dimensions for every 2d CFT, and eventually a non-conformal QFT of extended objects for every non-conformal 2d QFT, so that all the technology of 2d QFT can be applied to the construction and analysis of quantum field theories of extended objects. The project depends crucially on settling some mathematical questions about analysis in the spaces $\mathcal{E}_{\partial \xi}$. The project also depends on extending the observables of 2d CFT from the finite sets of points in a Riemann surface to the integral 0 -currents.


## Contents

1 Introduction ..... 5
1.1 The free $n$-form on a conformal space-time manifold $M$ of dimension $d=2 n$1.2 Currents in $M$ and the boundary operator $\partial$ on currents6
1.3 Reformulating the free $n$-form field theory ..... 7
1.4 The path of generalization ..... 7
1.5 Geometric Measure Theory - flat currents and integral currents ..... 8
1.6 The bundle $\mathcal{E} \xrightarrow{\partial} \mathcal{B}$ of extended objects: $\mathcal{E}=\mathcal{D}_{n-1}^{\text {int }}(M), \mathcal{B}=\partial \mathcal{D}_{n-1}^{\text {int }}(M)$ ..... 10
1.7 Disclaiming rigor ..... 10
1.8 Constructing a QFT of extended objects from a 2 d QFT ..... 11
2 Geometry of the space $\mathcal{D}_{k}^{\text {int }}(M)$ of integral $k$-currents in $M$ ..... 11
2.1 The principal fiber bundle $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M) \subset \mathcal{D}_{k-1}^{\text {int }}(M)$ ..... 11
2.2 The local metric geometry of $\mathcal{D}_{k}^{\mathrm{int}}(M)$ is the same at every point ..... 11
2.3 The tangent spaces of the fibers $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ all equal $\mathcal{V}_{k+1} \subset \mathcal{D}_{k+1}^{\text {flat }}(M)$ ..... 12
3 The Hodge *-operator on the tangent spaces $T_{\xi}\left(\mathcal{E}_{\partial \xi}\right)=\mathcal{V}_{n} \subset \mathcal{D}_{n}^{\text {flat }}(M)$ ..... 13
4 Currents in $\mathcal{D}_{k}^{\text {int }}(M)$ ..... 14
$4.1 \quad \partial_{*}: \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)\right) \rightarrow \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k-1}^{\text {int }}(M)\right)$ ..... 14
$4.2 \quad \Pi_{*}: \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)\right) \rightarrow \mathcal{D}_{j+k}^{\text {int }}(M)$ ..... 15
$4.3 \quad \Pi_{*}: \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}\right) \rightarrow \mathcal{D}_{j+k}^{\text {int }}(M)$ commutes with $\partial$ ..... 15
$4.4 \Pi_{*}$ is an isomorphism on the homology groups ..... 15
4.5 Tangent vectors as infinitesimal 1-currents ..... 16
4.6 The Hodge *-operator on 1-currents in $\mathcal{E}_{\partial \xi}$ ..... 16
5 An analog of a 2 d conformal field theory on each $\mathcal{E}_{\partial \xi}$ ..... 16
$5.1 n$-forms on space-time as 1 -forms on $\mathcal{E}_{\partial \xi}$ ..... 16
5.2 Scalar fields and vertex operators on $\mathcal{E}_{\partial \xi}$ ..... 17
5.3 Global symmetry on $\mathcal{E}_{\partial \xi}$ ..... 17
5.4 Space-time gauge symmetries are special collections of global symmetries ..... 18
5.5 An analog of the $U(1) \times U(1) 2$ d gaussian model on each $\mathcal{E}_{\partial \xi}$ ..... 18
6 Synopsis ..... 18
7 More on currents in $M$ ..... 21
7.1 Intersection of currents ..... 21
7.2 The operator $J=\epsilon_{n} *$ on $n$-currents ..... 21
7.3 The chiral projection operators $P_{ \pm}$ ..... 22
7.4 The skew-hermitian intersection form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ on currents ..... 22
8 Quantum field theory of the free $n$-form on $M$ ..... 23
8.1 The chiral fields ..... 23
8.2 The Schwinger-Dyson equations in terms of $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ ..... 23
$9 \mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as an almost-complex space ..... 24
10 The Schwinger-Dyson equations on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ ..... 25
11 Quasi two-dimensionality of $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ ..... 25
12 Quasi Riemann surfaces ..... 26
12.1 Definition ..... 26
$12.2 \mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as a quasi Riemann surface ..... 28
12.3 The bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ of quasi Riemann surfaces ..... 30
12.4 A Riemann surface $\Sigma$ as a quasi Riemann surface ..... 30
12.5 Definition (more) ..... 31
12.6 Morphisms ..... 32
12.7 Isomorphisms and automorphisms ..... 33
12.8 Morphisms and homology ..... 33
12.9 Quasi-holomorphic curves ..... 34
13 2d CFT on a quasi-holomorphic curve ..... 35
14 A wishful conjecture ..... 36
15 Correlation functions from extended 2d CFT ..... 37
16 Perturbation theory ..... 38
16.1 Varying the parameter $R$ of the gaussian model ..... 38
16.2 Perturbing by a general (1,1)-form ..... 39
17 Gauge symmetry of the classical free $n$-form ..... 40
17.1 The local gauge transformations over $\mathcal{B}$ ..... 41
17.2 The reconstruction of the space-time gauge potentials ..... 42
18 Connecting the $\mathcal{E}_{\partial \xi}$ ..... 43
18.1 The natural connection in $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M)$ ..... 43
18.2 The connection in the bundle of theories over $\mathcal{B}$ ..... 44
18.3 Transport of observables between fibers $\mathcal{E}_{\partial \xi}$ ..... 45
19 Explorations ..... 46
19.1 The $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ ..... 47
19.2 Reconstruct $\mathcal{Q}$ from $\mathcal{Q}_{1}$ ..... 48
19.3 Augment $\Sigma$ ..... 49
19.4 Isomorphisms of the $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ to $\mathcal{Q}\left(\Sigma_{+}\right)$ ..... 50
19.5 The universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ of quasi Riemann surfaces ..... 50
19.6 Embed $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ in the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ ..... 51
19.7 Homogeneity of $\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0)$ ..... 51
20 Mathematical questions ..... 52
20.1 Does the Hodge $*$-operator act on $T_{0} \mathcal{D}_{n-1}^{\text {int }}(M)_{0}$ ? ..... 52
20.2 Are quasi Riemann surfaces classified by their homology data? ..... 53
20.3 Can a Riemann surface be augmented? ..... 53
20.4 What can be said about the universal bundle of quasi Riemann surfaces? ..... 54
20.5 How much function theory can be done on a quasi Riemann surface? ..... 54
20.6 Are there mathematical applications? ..... 55
21 Further steps ..... 55
21.1 Extended CFT on quasi Riemann surfaces ..... 55
21.2 Gauge invariance ..... 56
21.3 Connecting the $\mathcal{Q}(M)_{\partial \xi}$ ..... 57
21.4 The local fields in space-time ..... 58
21.5 Partition functions and variation of conformal structure ..... 58
21.6 Non-conformal extended quantum field theory ..... 59
22 Questions about history and references ..... 59
Acknowledgments ..... 61
Appendices ..... 62
A Construction of a path of integral currents a la Game of Thrones ..... 62
B The free complex $n$-form on euclidean $\mathbb{R}^{d}$ ..... 65
C Vertex operators and the Dirac quantization condition ..... 70
D Complex conjugation and reality conditions ..... 73
References ..... 75

## 1 Introduction

This paper reports the first steps of a project to construct and analyze a general class of quantum field theories of ( $n-1$ )-dimensional extended objects in a space-time manifold of dimension $d=2 n$. Much still remains to be done.

My referencing is surely inadequate. I hope to do better in future revisions. Section 22 asks for advice on history and references.

### 1.1 The free $n$-form on a conformal space-time manifold $M$ of dimension $d=2 n$

The first step is to reformulate the conformally invariant quantum field theory of a free $n$-form $F(x)$ on a space-time manifold $M$ of dimension $d=2 n$ as the quantum field theory of a free 1 -form on a space of $(n-1)$-dimensional extended objects. The general project extrapolates from this example.

Space-time is taken to be a compact real manifold $M$ of dimension $d=2 n$ with an orientation and with a conformal class of riemannian metrics. The main example is euclidean $\mathbb{R}^{d}$ or, rather, its conformal compactification, the $d$-sphere $S^{d}=\mathbb{R}^{d} \cup\{\infty\}$. Actually, all that is used of the conformal structure on $M$ is the Hodge *-operator acting in the middle dimension, on $n$-forms. This might be less structure than a conformal class of riemannian metrics.

The free $n$-form theory is the generalization to $d=2 n$ dimensions of free quantum electrodynamics in 4-d [1-3]. The field equations of the free $n$-form $F(x)$ are

$$
\begin{equation*}
d F=0, \quad d F^{*}=0, \quad F^{*}=i^{-1} * F \tag{1.1}
\end{equation*}
$$

where $*$ is the conformally invariant Hodge $*$-operator acting on $n$-forms,

$$
\begin{equation*}
* \omega_{\mu_{1} \ldots \mu_{n}}(x)=\epsilon_{\mu_{1} \ldots \mu_{n}}{ }_{\nu_{1} \ldots \nu_{n}}(x) \omega_{\nu_{1} \ldots \nu_{n}}(x), \quad *^{2}=(-1)^{n} . \tag{1.2}
\end{equation*}
$$

The integral of the $n$-form $F$ over an $n$-surface is the magnetic charge. The integral of $F^{*}$ is the electric charge. The gauge potential $A(x)$ and the dual gauge potential $A^{*}(x)$ are ( $n-1$ )-forms constructed by integrating

$$
\begin{equation*}
d A=F, \quad d A^{*}=F^{*} \tag{1.3}
\end{equation*}
$$

The gauge potentials are determined up to gauge transformations

$$
\begin{equation*}
A \rightarrow A+d f, \quad A^{*} \rightarrow A^{*}+d f^{*} \tag{1.4}
\end{equation*}
$$

given by ( $n-2$ )-forms $f, f^{*}$.
The extended objects of the free $n$-form theory are described by fields

$$
\begin{equation*}
V_{p, p^{*}}(\xi)=e^{i p \int_{\xi} A+i p^{*} \int_{\xi} A^{*}} \tag{1.5}
\end{equation*}
$$

which live on $(n-1)$-currents $\xi$ in space-time. They carry electric charge $p$ and magnetic charge $p^{*}$. Taking the gauge group $G$ to be compact, $G=U(1) \times U(1)$, the charges lie in integer lattices,

$$
\begin{equation*}
p=\frac{m}{R}, \quad p^{*}=\frac{m^{*}}{R^{*}}, \quad m, m^{*} \in \mathbb{Z} \tag{1.6}
\end{equation*}
$$

The Dirac quantization condition

$$
\begin{equation*}
R R^{*}=1 \tag{1.7}
\end{equation*}
$$

follows from the requirement that the correlation functions of the fields $V_{p, p^{*}}(\xi)$ should be single-valued.

### 1.2 Currents in $M$ and the boundary operator $\partial$ on currents

Currents will be the basic mathematical objects of this enterprise. A $k$-current $\xi$ in the space-time manifold $M$ is a linear function - a distribution - on $k$-forms $\omega$,

$$
\begin{equation*}
\int_{\xi} \omega=\int_{M} d^{d} x \xi^{\mu_{1} \ldots \mu_{k}}(x) \omega_{\mu_{1} \ldots \mu_{k}}(x) \tag{1.8}
\end{equation*}
$$

$\mathcal{D}_{k}(M)$ is the linear space of $k$-currents in $M$. A $k$-current $\xi$ is called smooth when $\xi(x)^{\mu_{1} \ldots \mu_{k}}$ is smooth. Equivalently, $\xi$ is smooth when it is represented by a smooth $(d-k)$ form $\omega_{\xi}$,

$$
\begin{equation*}
\int_{\xi} \omega=\int_{M} \omega_{\xi} \wedge \omega \tag{1.9}
\end{equation*}
$$

Since we will be considering fields that live on spaces of currents, it will be more congenial to write differential forms as linear functions of currents,

$$
\begin{equation*}
\omega(\xi)=\int_{\xi} \omega \tag{1.10}
\end{equation*}
$$

When the $n$-form $F(x)$ is considered as a quantum field, it is a distributional $n$-form acting as a linear function $\xi \mapsto F(\xi)$ on smooth $n$-currents in $M$.

The boundary operator on currents is dual to the exterior derivative on forms,

$$
\begin{gather*}
\partial: \mathcal{D}_{k}(M) \rightarrow \mathcal{D}_{k-1}(M)  \tag{1.11}\\
\int_{\partial \xi} \omega=\int_{\xi} d \omega, \quad(\partial \xi)^{\mu_{2} \ldots \mu_{k}}(x)=-\partial_{\mu_{1}} \xi^{\mu_{1} \ldots \mu_{k}}(x) . \tag{1.12}
\end{gather*}
$$

Examples of $k$-currents are given by $k$-dimensional submanifolds in $M$. The corresponding linear function on a $k$-form $\omega$ is the integral of $\omega$ over the submanifold. The boundary of the $k$-current corresponds to the boundary of the submanifold.

### 1.3 Reformulating the free $n$-form field theory

The reformulation of the free $n$-form theory stems from the realization that the space of ( $n-1$ )-currents in $M$ on which the $n$-forms live can be decomposed into a union of spaces $\mathcal{E}_{\partial \xi}$ each of which has the property that its tangent space at each point is a linear space of $n$-currents in space-time which is closed under the action of the Hodge *-operator. The spaces $\mathcal{E}_{\partial \xi}$ will be described in more detail shortly.

Since tangent vectors in $\mathcal{E}_{\partial \xi}$ are $n$-currents in space-time, the $n$-form fields $F$ and $F^{*}=i^{-1} * F$ on the space-time manifold $M$ become 1 -forms $j$ and $j^{*}=i^{-1} * j$ on $\mathcal{E}_{\partial \xi}$, where the Hodge $*$-operator on $n$-forms has become a linear operator acting on the 1 forms on $\mathcal{E}_{\partial \xi}$. The field equations become

$$
\begin{equation*}
d j=0, \quad d j^{*}=0 \tag{1.13}
\end{equation*}
$$

Scalar fields, i.e., 0 -forms, $\phi$ and $\phi^{*}$ on $\mathcal{E}_{\partial \xi}$ are constructed by integrating

$$
\begin{equation*}
d \phi=j, \quad d \phi^{*}=j^{*} . \tag{1.14}
\end{equation*}
$$

The scalar fields on $\mathcal{E}_{\partial \xi}$ express the gauge potentials on space-time,

$$
\begin{equation*}
\phi(\xi)=\int_{\xi} A, \quad \phi^{*}(\xi)=\int_{\xi} A^{*} \tag{1.15}
\end{equation*}
$$

The extended objects are described by the "vertex operator" fields on $\mathcal{E}_{\partial \xi}$,

$$
\begin{equation*}
V_{p, p^{*}}(\xi)=e^{i p \phi(\xi)+i p^{*} \phi^{*}(\xi)} . \tag{1.16}
\end{equation*}
$$

The free $n$-form theory on $M$ is thus reformulated as a free 1-form theory on $\mathcal{E}_{\partial \xi}$. The free $n$-form theory begins to look formally analogous to the conformal field theory of a free 1 -form on a two-dimensional manifold.

### 1.4 The path of generalization

The path of generalization will retrace the historical development of the general class of two-dimensional conformal and non-conformal quantum field theories starting from the theory of the free 1 -form. The 2 d conformal field theories that were constructed in that development include the theories of several 1 -form fields, the orbifolds of the 1-form theories, and the theories constructed from all these by conformal perturbation theory. Notable special cases are the 2d conformal field theories containing non-abelian current algebras. Along a sideline are the holomorphic conformal field theories made from 1-form theories, including the Monster 2d CFT. Finally, there are the non-conformal 2d quantum field theories constructed by perturbation theory governed by the 2 d renormalization group. If these constructions on the free 1-form in two dimensions can indeed be carried out on the free 1 -form on the spaces $\mathcal{E}_{\partial \xi}$ of $(n-1)$-currents in space-time, then such a menagerie of examples will strongly suggest that it should be possible to formulate axiomatically a general class of quantum field theories of extended objects in one-to-one correspondence with the 2 d quantum field theories. I will try here to describe the setting in which such quantum field theories of extended objects might be formulated, and develop mathematical conjectures that would provide a construction of such theories.

### 1.5 Geometric Measure Theory - flat currents and integral currents

The quantum field theories of extended objects envisioned here are all to be theories of fields on certain spaces $\mathcal{E}_{\partial \xi}$ of currents in space-time. The tools for defining $\mathcal{E}_{\partial \xi}$ and for doing analysis in $\mathcal{E}_{\partial \xi}$ come from a branch of mathematics called Geometric Measure Theory (GMT). Given a manifold $M$, Geometric Measure Theory defines the complete metric space of integral $k$-currents in $M$ 4. One of several equivalent definitions starts from maps $\sigma$ from the oriented $k$-simplex $\Delta^{k}$ to $M$. The $k$-current $[\sigma]$ defined by

$$
\begin{equation*}
\int_{[\sigma]} \omega=\int_{\Delta^{k}} \sigma^{*} \omega \tag{1.17}
\end{equation*}
$$

is a delta-function $k$-current concentrated on the image $\sigma\left(\Delta^{k}\right)$ in $M$. A linear combination $\sum_{\alpha} n_{\alpha}\left[\sigma_{\alpha}\right]$ of such $k$-currents with integer coefficients $n_{\alpha}$ is the $k$-current representing the singular $k$-chain $\sum_{\alpha} n_{\alpha} \sigma_{\alpha}$ in $M$. The singular $k$-currents realize a naive idea of the $k$ dimensional objects in $M$.

Next, a certain norm, called the flat norm, is put on $k$-currents. At each point $x \in M$, the riemannian metric at $x$ is used to define a euclidean length function $|\omega(x)|$ on the vector space of $k$-forms $\omega(x)$ at $x$. The co-mass $M_{k}^{*}(\omega)$ of a $k$-form is defined as

$$
\begin{equation*}
M_{k}^{*}(\omega)=\sup _{x \in M}|\omega(x)| . \tag{1.18}
\end{equation*}
$$

The mass $M_{k}(\xi)$ of a $k$-current is then defined as

$$
\begin{equation*}
M_{k}(\xi)=\sup _{M_{k}^{*}(\omega)=1}|\omega(\xi)| . \tag{1.19}
\end{equation*}
$$

When $\xi$ is the characteristic current of a submanifold of $M$, the mass $M_{k}(\xi)$ is the $k$ volume of the submanifold. The flat norm of a $k$-current is defined as

$$
\begin{equation*}
\|\xi\|_{f a t}=\inf _{\xi^{\prime}}\left[M_{k}\left(\xi-\partial \xi^{\prime}\right)+M_{k+1}\left(\xi^{\prime}\right)\right] \tag{1.20}
\end{equation*}
$$

where $\xi^{\prime}$ ranges over all $(k+1)$-currents. The flat norm gives the flat metric on the space of $k$-currents,

$$
\begin{equation*}
d_{f l a t}\left(\xi_{1}, \xi_{2}\right)=\left\|\xi_{1}-\xi_{2}\right\|_{f l a t} \tag{1.21}
\end{equation*}
$$

The completion in the flat metric of the vector space of finite norm $k$-currents is the vector space of flat $k$-currents. Roughly, a flat current is a distribution on $k$-forms that takes no derivatives.

The flat norm is a physically reasonable measure of the size of a $k$-dimensional object. A $k$-current $\xi$ of small flat norm is physically small in the sense that it can be shrunk away to nothing with little effort. If a singular $k$-current $\xi$ is small in the flat norm, then there is a $(k+1)$-current $\xi^{\prime}$ such that both $M_{k+1}\left(\xi^{\prime}\right)$ and $M_{k}\left(\xi-\partial \xi^{\prime}\right)$ are small. So part of $\xi$ can easily be shrunk away through $\xi^{\prime}$, which has small $(k+1)$-volume, and what remains of $\xi$ has small $k$-volume and can easily be shrunk away through itself. So the
flat metric is a physically reasonable measure of the difference between two $k$-dimensional objects. The definition of the flat metric requires a notion of distance in $M$, i.e., a choice of riemannian metric on $M$, but the flat metric topology on currents is the same for any choice of riemannian metric on $M$.

Completing the space of singular $k$-currents in the flat metric gives the space of integer rectifiable $k$-currents. Finally, an integral $k$-current is defined to be an integer rectifiable $k$-current $\xi$ whose boundary $\partial \xi$ has finite mass. The integral currents that are not singular currents are fractal objects.

We will write $\mathcal{D}_{k}^{\text {int }}(M)$ for the space of integral $k$-currents in $M$. A basic theorem of [4] states that $\mathcal{D}_{k}^{\text {int }}(M)$ is a complete metric space and that the boundary operator takes integral $k$-currents to integral ( $k-1$ )-currents,

$$
\begin{equation*}
\partial: \mathcal{D}_{k}^{\mathrm{int}}(M) \rightarrow \mathcal{D}_{k-1}^{\mathrm{int}}(M), \tag{1.22}
\end{equation*}
$$

acting continuously in the metric topology. Moreover, $\mathcal{D}_{k}^{\text {int }}(M)$ is a normed abelian group under addition of currents, i.e., the addition law is continuous in the flat metric topology.

We will be using only the $\mathcal{D}_{k}^{\text {int }}(M)$ with $k=n-2, n-1, n, n+1$. In order to handle the case $d=2, n=1$, we need to define the ( -1 )-currents,

$$
\begin{gather*}
\mathcal{D}_{-1}^{\mathrm{int}}(M)=\mathbb{Z}, \quad \mathcal{D}_{-1}(M)=\mathbb{C}  \tag{1.23}\\
\partial: \mathcal{D}_{0}^{\mathrm{int}}(M) \rightarrow \mathcal{D}_{-1}^{\mathrm{int}}(M), \quad \partial \eta=\int_{\eta} 1=1(\eta) . \tag{1.24}
\end{gather*}
$$

Also, there is a distinguished $d$-current which is the oriented manifold $M$ itself, which acts on $d$-forms $\omega$ by

$$
\begin{equation*}
\omega(M)=\int_{M} \omega \tag{1.25}
\end{equation*}
$$

We will use this $d$-current for the case $d=2$, when it lies in $\mathcal{D}_{n+1}^{\mathrm{int}}(M)$. So we define

$$
\begin{gather*}
\mathcal{D}_{d+1}^{\mathrm{int}}(M)=\mathbb{Z}, \quad \mathcal{D}_{d+1}(M)=\mathbb{C}  \tag{1.26}\\
\partial: \mathcal{D}_{d+1}^{\mathrm{int}}(M) \rightarrow \mathcal{D}_{d}^{\mathrm{int}}(M), \quad \partial 1=M \tag{1.27}
\end{gather*}
$$

Now we have the augmented de Rham complex of currents


The $\mathcal{D}_{k}(M)$ will be the complex currents, but they can be taken to be real for $n$ odd. The precise characterization of the linear space of complex currents - smooth, flat, distributional, etc. - will mostly be left unspecified, to be determined by the context.
1.6 The bundle $\mathcal{E} \xrightarrow{\partial} \mathcal{B}$ of extended objects: $\mathcal{E}=\mathcal{D}_{n-1}^{\text {int }}(M), \mathcal{B}=\partial \mathcal{D}_{n-1}^{\text {int }}(M)$

I am proposing to take as the space of extended objects the space

$$
\begin{equation*}
\mathcal{E}=\mathcal{D}_{n-1}^{\mathrm{int}}(M) \tag{1.29}
\end{equation*}
$$

of integral ( $n-1$ )-currrents in $M$. The space $\mathcal{E}$ forms a bundle

$$
\begin{align*}
& \mathcal{E}  \tag{1.30}\\
& \downarrow_{\partial} \\
& \mathcal{B}
\end{align*}
$$

over the space of $(n-2)$-boundaries,

$$
\begin{equation*}
\mathcal{B}=\partial \mathcal{D}_{n-1}^{\text {int }}(M) \subset \mathcal{D}_{n-2}^{\text {int }}(M) \tag{1.31}
\end{equation*}
$$

The spaces $\mathcal{E}_{\partial \xi}$ are the fibers of the bundle, the spaces of relative $(n-1)$-cycles,

$$
\begin{equation*}
\mathcal{E}_{\partial \xi}=\partial^{-1}(\partial \xi)=\left\{\xi^{\prime} \in \mathcal{D}_{n-1}^{\text {int }}(M): \partial \xi^{\prime}=\partial \xi\right\} \tag{1.32}
\end{equation*}
$$

The special fiber $\mathcal{E}_{0}=\mathcal{D}_{n-1}^{\text {int }}(M)_{0}$ is the space of integral ( $n-1$ )-cycles.
We will see that the geometry of currents in each $\mathcal{E}_{\partial \xi}$ is analogous to the geometry of currents in a Riemann surface. I will call such spaces "quasi Riemann surfaces".

### 1.7 Disclaiming rigor

I want to do quantum field theory on the space of extended objects, so I need calculus and tensor analysis on $\mathcal{D}_{n-1}^{\text {int }}(M)$. But the spaces $\mathcal{D}_{k}^{\text {int }}(M)$ are not, as far as I can tell, differentiable manifolds. On the other hand, there is a construction of currents - and flat currents and integral currents - in any complete metric space [5]. So a calculus of currents in $\mathcal{D}_{n-1}^{\mathrm{int}}(M)$ is available, and thus a calculus of differential forms as the duals of currents. The spaces $\mathcal{D}_{k}^{\text {int }}(M)$ have nice properties that lend themselves to geometric analysis - each is a normed abelian group that is generated as an abelian group by an arbitrarily small $\epsilon$-ball around 0 , and each is embedded in a normed vector space (of flat currents). The special case $\mathcal{D}_{n-1}^{\mathrm{int}}(M)$ has even nicer properties which will be described below.

It may be that the mathematical basis for calculus and tensor analysis on the spaces $\mathcal{D}_{k}^{\text {int }}(M)$ already exists, but I cannot tell - mathematical analysis is not exactly my cup of tea. I will proceed naively, without trying for mathematical rigor, blithely optimistic that rigor can be achieved eventually. I will explore the possibilities of achieving the project of constructing quantum field theories of extended objects from 2d quantum field theories to the point where I can formulate a more or less well-posed mathematical conjecture on which the program can be based. The mathematical conjecture is that there is a classification of equivalence classes of "quasi Riemann surfaces" that is analogous to and extends the classification of ordinary Riemann surfaces. I have almost no evidence for the conjecture. Its appeal is that it makes feasible the construction of quantum field theories of extended objects from 2d quantum field theories in the simplest and most direct fashion that I can imagine. So I call it a "wishful" conjecture.

### 1.8 Constructing a QFT of extended objects from a 2d QFT

If the conjecture holds, then there will be an essentially unique map from the space $\mathcal{E}_{\partial \xi}$ of integral relative $(n-1)$-cycles in the space-time $M$ to the space of integral relative 0 -cycles in a two-dimensional space $\Sigma$. The space $\Sigma$ will be a Riemann surface or something akin to one. For example, when the space-time is $M=S^{d}=\mathbb{R}^{d} \cup\{\infty\}$, then $\Sigma$ will be the Riemann sphere $S^{2}=\mathbb{C} \cup\{\infty\}$. A quantum field $\Phi(\xi)$ on the space of extended objects will correspond to a 2 d observable in the 2 d quantum field theory on $\Sigma$ located on the integral 0 -current in $\Sigma$ that corresponds to $\xi$. The correlation functions of the extended object fields will be given by the correlation functions in the 2 d quantum field theory on $\Sigma$. In this way, the quantum field theory of extended objects will be constructed from the 2 d quantum field theory.

## 2 Geometry of the space $\mathcal{D}_{k}^{\text {int }}(M)$ of integral $k$-currents in $M$

### 2.1 The principal fiber bundle $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M) \subset \mathcal{D}_{k-1}^{\text {int }}(M)$

The space $\mathcal{D}_{k}^{\text {int }}(M)$ is a fiber bundle

over the space of integral $(k-1)$-boundaries. The special fiber $\mathcal{D}_{k}^{\text {int }}(M)_{0}$ over $0 \in \mathcal{D}_{k-1}^{\text {int }}(M)$ is the space of integral $k$-cycles in $M$, the space of integral $k$-currents without boundary,

$$
\begin{equation*}
\mathcal{D}_{k}^{\mathrm{int}}(M)_{0}=\left\{\xi \in \mathcal{D}_{k}^{\mathrm{int}}(M), \partial \xi=0\right\} . \tag{2.2}
\end{equation*}
$$

$\mathcal{D}_{k}^{\text {int }}(M)_{0}$ is closed under addition, thus an abelian group. The other fibers $\mathcal{D}_{k}^{\text {int }}(M)_{\xi_{0}}=$ $\partial^{-1}\left(\xi_{0}\right), \xi_{0} \neq 0$, are the spaces of integral relative $k$-cycles in $M$. That is, if $\xi_{1}$ is a $k$-current in the fiber over $\xi_{0}$, then every $\xi$ in the same fiber differs from $\xi_{1}$ by a $k$-cycle,

$$
\begin{equation*}
\partial\left(\xi-\xi_{1}\right)=0 . \tag{2.3}
\end{equation*}
$$

Each fiber is isomorphic to the space of integral $k$-cycles $\mathcal{D}_{k}^{\text {int }}(M)_{0}$, but not in a canonical way. The isomorphism depends on the choice of $\xi_{1}$ in the fiber. The abelian group $\mathcal{D}_{k}^{\text {int }}(M)_{0}$ acts by addition on $\mathcal{D}_{k}^{\text {int }}(M)$, preserving the fibers, acting transitively and faithfully on each fiber. So the fiber bundle (2.1) is a principle fiber bundle with structure group the abelian group $\mathcal{D}_{k}^{\text {int }}(M)_{0}$ of integral $k$-cycles.

### 2.2 The local metric geometry of $\mathcal{D}_{k}^{\text {int }}(M)$ is the same at every point

Translation in $\mathcal{D}_{k}^{\mathrm{int}}(M)$ as abelian group takes any point to any other. In particular, translation by $\xi$ takes 0 to $\xi$, for any $\xi \in \mathcal{D}_{k}^{\text {int }}(M)$. Translation by $\xi$ preserves the fibers
of the bundle, taking the fiber over $\partial \xi^{\prime}$ to the fiber over $\partial\left(\xi^{\prime}+\xi\right)$. Translation by $\xi$ takes the $\epsilon$-ball $B_{\epsilon}$ around 0 to the $\epsilon$-ball $\xi+B_{\epsilon}$ around $\xi$. So the local metric geometry of the principal fiber bundle is the same at every point.

### 2.3 The tangent spaces of the fibers $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ all equal $\mathcal{V}_{k+1} \subset \mathcal{D}_{k+1}^{\text {flat }}(M)$

The space $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ of integral relative $k$-cycles is the fiber of the principal fiber bundle $\mathcal{D}_{k}^{\text {int }}(M)$ over the $(k-1)$-boundary $\partial \xi$. A tangent vector $\dot{\xi}$ in $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ is a vertical tangent vector in the principal fiber bundle. By translation in the abelian group, the vertical tangent space is the same at every point in the bundle,

$$
\begin{equation*}
T_{\xi}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{\partial \xi}\right)=T_{0}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{0}\right) \tag{2.4}
\end{equation*}
$$

equal to the tangent space to the fiber at the distinguished point $0 \in \mathcal{D}_{k}^{\text {int }}(M)$.
Suppose $\dot{\xi}$ is the tangent vector at $\epsilon=0$ to an infinitesimal curve $\epsilon \mapsto \xi(\epsilon)$ in the fiber $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$. Then $\xi(\epsilon)-\xi(0)$ is an infinitesimally small integral $k$-cycle. There is a unique integral $(k+1)$-current $\xi_{S}(\epsilon)$ of minimal mass solving

$$
\begin{equation*}
\partial \xi_{S}(\epsilon)=\xi(\epsilon)-\xi(0) \tag{2.5}
\end{equation*}
$$

It satisfies

$$
\begin{equation*}
\|\xi(\epsilon)-\xi(0)\|_{f l a t}=M_{k+1}\left(\xi_{S}(\epsilon)\right)=\left\|\xi_{S}(\epsilon)\right\|_{\text {flat }} \tag{2.6}
\end{equation*}
$$

This minimal integral $(k+1)$-current $\xi_{S}(\epsilon)$ can be thought of as the secant between $\xi(0)$ and $\xi(\epsilon)$. The vertical tangent vector is represented by the flat $(k+1)$-current

$$
\begin{equation*}
\dot{\xi}=\lim _{\epsilon \rightarrow 0} \frac{\xi_{S}(\epsilon)}{\epsilon} . \tag{2.7}
\end{equation*}
$$

Equation (2.6) implies that the $(k+1)$-current $\dot{\xi}$ faithfully represents the tangent vector to the curve $\xi(\epsilon)$, i.e., the map from vertical tangent vectors to flat $(k+1)$-currents is injective. So the tangent spaces to the fibers are all equal to a certain subspace of the flat $(k+1)$-currents,

$$
\begin{equation*}
T_{\xi}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{\partial \xi}\right)=T_{0}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{0}\right)=\mathcal{V}_{k+1} \subset \mathcal{D}_{k+1}^{\text {flat }}(M) \tag{2.8}
\end{equation*}
$$

The question then is: exactly what subspace $\mathcal{V}_{k+1}$ of flat $(k+1)$-currents is formed by the tangent vectors in the space of integral relative $k$-cycles?

The easiest examples of vertical tangent vectors are the delta-function ( $k+1$ )-currents. Work in coordinates $x^{\mu}$ on $M$ and let $\hat{e}_{\mu_{1}}, \ldots, \hat{e}_{\mu_{k+1}}$ be unit tangent vectors at $x_{0} \in$ $M$ along $k+1$ different axes. Let $\xi_{S}(\epsilon)$ be the $(k+1)$-current representing the $(k+1)$ parallelotope with vertex at $x_{0}$ and edges $\epsilon^{\frac{1}{k+1}} \hat{e}_{\mu_{1}}, \ldots, \epsilon^{\frac{1}{k+1}} \hat{e}_{\mu_{k+1}}$. The tangent vector at $\xi(0)$ to the curve $\xi(\epsilon)=\xi(0)+\partial \xi_{S}(\epsilon)$ is the flat ( $k+1$ )-current

$$
\begin{equation*}
\dot{\xi}=\lim _{\epsilon \rightarrow 0} \frac{\xi_{S}(\epsilon)}{\epsilon}=\delta^{d}\left(x-x_{0}\right) \hat{e}_{\mu_{1}} \wedge \cdots \wedge \hat{e}_{\mu_{k+1}} \tag{2.9}
\end{equation*}
$$

which is supported at the point $x_{0}$. The value of a $(k+1)$-form $\omega$ on this tangent vector is

$$
\begin{equation*}
\omega(\dot{\xi})=\omega\left(x_{0}\right)_{\mu_{1} \cdots \mu_{k+1}} . \tag{2.10}
\end{equation*}
$$

So, naively, there are at least enough tangent vectors to detect all ( $k+1$ )-forms.
The vector space $\mathcal{V}_{k+1}$ is the Gromov-Hausdorff tangent space. Let $B_{\epsilon}\left(\mathcal{D}_{k}^{\text {int }}(M)_{0}, 0\right)$ be the $\epsilon$-ball around 0 in the space $\mathcal{D}_{k}^{\text {int }}(M)_{0}$ of integral $k$-cycles. The secant map described above is a map

$$
\begin{equation*}
S: B_{\epsilon}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{0}, 0\right) \rightarrow B_{\epsilon}\left(\mathcal{D}_{k+1}^{\mathrm{int}}(M), 0\right) \subset \mathcal{D}_{k+1}^{\text {flat }}(M) \tag{2.11}
\end{equation*}
$$

The image of $S$ lies in the vector space $\mathcal{D}_{k+1}^{f l a t}(M)$, so it makes sense to construct the tangent space $\mathcal{V}_{k+1}$ by dilating the image of $S$ within that vector space. The unit ball in $\mathcal{V}_{k+1}$ is

$$
\begin{equation*}
B_{1}\left(\mathcal{V}_{k+1}, 0\right)=\lim _{\epsilon \rightarrow 0} \frac{1}{\epsilon} S\left(B_{\epsilon}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{0}, 0\right)\right) \tag{2.12}
\end{equation*}
$$

where the limit is taken in the Gromov-Hausdorff metric on metric spaces. A tangent vector, an element of $\mathcal{V}_{k+1}$ is a flat $(k+1)$-current in $M$ whose support set is the same as the support set of an infinitesimally small integral $(k+1)$-current. Naively, these should be the support sets of the integral $k^{\prime}$-currents with $k^{\prime}<k+1$.

## 3 The Hodge *-operator on the tangent spaces $T_{\xi}\left(\mathcal{E}_{\partial \xi}\right)=\mathcal{V}_{n} \subset$ $\mathcal{D}_{n}^{\text {flat }}(M)$

As above, the tangent spaces to the fibers $\mathcal{E}_{\partial \xi}=\mathcal{D}_{n-1}^{\text {int }}(M)_{\partial \xi}$ in the bundle of extended objects $\mathcal{E} \rightarrow \mathcal{B}$ are all equal to the vector space $\mathcal{V}_{n} \subset \mathcal{D}_{n}^{\text {flat }}(M)$. The crucial question is: does the Hodge $*$-operator act on $\mathcal{V}_{n}$ ? The Hodge $*$-operator obviously acts on the tangent vectors that are linear combinations of the delta-function $n$-currents,

$$
\begin{equation*}
\dot{\xi}=\delta^{d}\left(x-x_{0}\right) \hat{e}_{\mu_{1}} \wedge \cdots \wedge \hat{e}_{\mu_{n}} \mapsto * \dot{\xi}=\delta^{d}\left(x-x_{0}\right) \epsilon_{\mu_{1} \ldots \mu_{n}}{ }^{\nu_{1} \ldots \nu_{n}}\left(x_{0}\right) \hat{e}_{\nu_{1}} \wedge \cdots \wedge \hat{e}_{\nu_{n}}, \tag{3.1}
\end{equation*}
$$

since any $n$-vector can multiply the delta-function. But consider a tangent vector $\dot{\xi}$ whose support set is at the opposite extreme, the support of an integral current of dimension $n-1$. For simplicity, suppose $d=4, n=2$, and suppose space-time is euclidean $\mathbb{R}^{4}$. Let $\xi_{S}(\epsilon)$ be the 2-current

$$
\begin{equation*}
\xi_{S}(\epsilon)=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \theta_{[0, \epsilon]}\left(x^{4}\right) \hat{e}_{3} \wedge \hat{e}_{4}, \tag{3.2}
\end{equation*}
$$

representing a $1 \times \epsilon$ rectangle in the $3-4$ plane. Here $\theta_{[a, b]}$ is the characteristic function of the interval $[a, b] \subset \mathbb{R}$. Then $\xi(\epsilon)=\partial \xi_{S}(\epsilon)$ is the 1-current representing the boundary of the rectangle. The tangent vector to the curve $\xi(\epsilon)$ is the flat 2-current

$$
\begin{equation*}
\dot{\xi}=\lim _{\epsilon \rightarrow 0} \frac{\xi_{S}(\epsilon)}{\epsilon}=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \delta\left(x^{4}\right) \hat{e}_{3} \wedge \hat{e}_{4}, \tag{3.3}
\end{equation*}
$$

whose support is the interval $[0,1]$ in the 3 -axis. The *-operator acts on this tangent vector to give

$$
\begin{equation*}
* \dot{\xi}=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \delta\left(x^{4}\right) \hat{e}_{1} \wedge \hat{e}_{2} . \tag{3.4}
\end{equation*}
$$

It is clear that the 2 -current $* \dot{\xi}$ cannot be the tangent vector to a curve of singular 1 -currents, to a curve of naive one dimensional objects.

Appendix A contains the construction of a curve of integral 1-currents that has the flat 2 -current $* \dot{\xi}$ of (3.4) as tangent vector. So this $* \dot{\xi}$ does lie in the tangent space $\mathcal{V}_{n}$. The construction depends essentially on the metric completion of the space of integral currents. The possibility of this construction was the main motivation for taking the extended objects to be the integral $(n-1)$-currents in the general case. Appendix A goes on to explain how this example might serve as the germ of a rigorous proof that the Hodge $*$-operator takes all of $\mathcal{V}_{n}$ to itself, for any manifold $M$ of any dimension $d=2 n$. Essentially, the construction of Appendix A gives a basis for showing that the vertical tangent space $\mathcal{V}_{n}$ consists of all flat $n$-currents supported on integral ( $n-1$ )-currents. Then, since $*$ acts continuously on flat $n$-currents and does not change their supports, $*$ would act on $\mathcal{V}_{n}$.

I will assume that the Hodge $*$-operator does act on $\mathcal{V}_{n}$ and thus on all the tangent spaces of the $\mathcal{E}_{\partial \xi}$. The whole enterprise rests on this assumption, so it is especially urgent that the mathematical question be settled one way or the other.

## 4 Currents in $\mathcal{D}_{k}^{\text {int }}(M)$

We will suppose that [5], which constructs the space of currents in a complete metric space, gives a calculus of currents in $\mathcal{D}_{k}^{\text {int }}(M)$ that includes

- spaces $\mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)\right)$ of integral $j$-currents in $\mathcal{D}_{k}^{\text {int }}(M)$ contained in the spaces $\mathcal{D}_{j}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right)$ of complex currents, with properties analogous to those of the currents in $M$, and
- spaces of $j$-forms on $\mathcal{D}_{k}^{\text {int }}(M)$ dual to the spaces of $j$-currents in $\mathcal{D}_{k}^{\text {int }}(M)$,
- natural linear maps

$$
\begin{align*}
\partial_{*}: & \partial_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k-1}^{\mathrm{int}}(M)\right)  \tag{4.1}\\
\Pi_{*}: & \Pi_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j+k}^{\mathrm{int}}(M) \tag{4.2}
\end{align*}
$$

which satisfy

$$
\begin{equation*}
\partial\left(\Pi_{*}^{j, k} \eta\right)=\Pi_{*}^{j-1, k}(\partial \eta)+\Pi_{*}^{j, k-1}\left(\partial_{*}^{j, k} \eta\right), \quad j \geq 1 \tag{4.3}
\end{equation*}
$$

## $4.1 \quad \partial_{*}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k-1}^{\mathrm{int}}(M)\right)$

Composing with the boundary operator in $M, \partial: \mathcal{D}_{k}^{\text {int }}(M) \rightarrow \mathcal{D}_{k-1}^{\text {int }}(M)$, takes an integral $j$-current in $\mathcal{D}_{k}^{\text {int }}(M)$ to an integral $j$-current in $\mathcal{D}_{k-1}^{\text {int }}(M)$,

$$
\begin{equation*}
\partial_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k-1}^{\mathrm{int}}(M)\right) \tag{4.4}
\end{equation*}
$$

## $4.2 \quad \Pi_{*}: \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)\right) \rightarrow \mathcal{D}_{j+k}^{\text {int }}(M)$

There is natural map "pushing down" an integral $j$-current in $\mathcal{D}_{k}^{\text {int }}(M)$ to give an integral $(j+k)$-current in $M$,

$$
\begin{equation*}
\Pi_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j+k}^{\mathrm{int}}(M), \tag{4.5}
\end{equation*}
$$

based on the fact that a map from the simplex $\Delta^{j}$ to the space of maps from $\Delta^{k}$ to $M$ is a map from $\Delta^{j} \times \Delta^{k}$ to $M$, which is represented by an integral $(j+k)$-current in $M$ [6]. The pushdown operation extends to the vector space of (flat) currents in $\mathcal{D}_{k}^{\text {int }}(M)$,

$$
\begin{equation*}
\Pi_{*}^{j, k}: \mathcal{D}_{j}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)\right) \rightarrow \mathcal{D}_{j+k}(M) \tag{4.6}
\end{equation*}
$$

The interaction with the boundary operator is

$$
\begin{equation*}
\partial\left(\Pi_{*}^{j, k} \eta\right)=\Pi_{*}^{j-1, k}(\partial \eta)+\Pi_{*}^{j, k-1}\left(\partial_{*}^{j, k} \eta\right), \quad j \geq 1 \tag{4.7}
\end{equation*}
$$

which follows from $\partial\left(\Delta^{j} \times \Delta^{k}\right)=\left(\partial \Delta^{j}\right) \times \Delta^{k}+\Delta^{j} \times\left(\partial \Delta^{k}\right)$.
The pushdown operation $\Pi_{*}^{j, k}$ is translation invariant for $j \geq 1$,

$$
\begin{equation*}
\Pi_{*}^{j, k}\left(T_{\xi} \eta\right)=\Pi_{*}^{j, k}(\eta), \quad j \geq 1 \tag{4.8}
\end{equation*}
$$

where $T_{\xi}$ is translation by the integral $k$-current $\xi$. Roughly, the pushdown operation takes the $j$-current $\eta$ to the $(j+k)$-dimensional region swept out by the $j$-parameter family of $k$-currents. In the translated current $T_{\xi} \eta$, the $j$-parameter family of $k$-currents keeps $\xi$ constant, so nothing additional is swept out, for $j \geq 1$.

## $4.3 \quad \Pi_{*}: \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}\right) \rightarrow \mathcal{D}_{j+k}^{\text {int }}(M)$ commutes with $\partial$

Now restrict to the space $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ of integral relative $k$-cycles. For an integral $j$-current $\eta$ in the space $\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}$ of integral relative $k$-cycles,

$$
\begin{equation*}
\eta \in \mathcal{D}_{j}^{\text {int }}\left(\mathcal{D}_{k}^{\text {int }}(M)_{\partial \xi}\right), \quad j \geq 1 \tag{4.9}
\end{equation*}
$$

the composition with the boundary operator vanishes,

$$
\begin{equation*}
\partial_{*}^{j, k} \eta=0, \quad j \geq 1 \tag{4.10}
\end{equation*}
$$

because composing with the boundary operator takes the $j$-current $\eta$ to the single point $\partial \xi$ in $\mathcal{D}_{k-1}^{\text {int }}(M)$. Therefore the pushdown operation commutes with the boundary operator,

$$
\begin{equation*}
\partial\left(\Pi_{*}^{j, k} \eta\right)=\Pi_{*}^{j-1, k}(\partial \eta), \quad j \geq 1 \tag{4.11}
\end{equation*}
$$

## $4.4 \Pi_{*}$ is an isomorphism on the homology groups

The map

$$
\begin{equation*}
\Pi_{*}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{\partial \xi}\right) \rightarrow \mathcal{D}_{j+k}^{\mathrm{int}}(M) \tag{4.12}
\end{equation*}
$$

induces a map of homology groups

$$
\begin{equation*}
\Pi_{*}: H_{j}\left(\mathcal{D}_{k}^{\mathrm{int}}(M)_{\partial \xi}\right) \rightarrow H_{j+k}(M) \tag{4.13}
\end{equation*}
$$

which is an isomorphism [6].

### 4.5 Tangent vectors as infinitesimal 1-currents

Suppose $\epsilon \mapsto \xi(\epsilon)$ is an infinitesimal curve in $\mathcal{D}_{k}^{\text {int }}(M)$, at $\xi(0)=\xi$. It could be directed vertically, along a fiber, but need not be. The curve is a map from the interval $[0, \epsilon]$ to $\mathcal{D}_{k}^{\text {int }}(M)$, so it is represented by an infinitesimal integral 1 -current $\eta(\epsilon)$ in $\mathcal{D}_{k}^{\text {int }}(M)$ or, equivalently, as the flat 1 -current supported at $\xi$,

$$
\begin{equation*}
\dot{\xi}=\lim _{\epsilon \rightarrow 0} \frac{\eta(\epsilon)}{\epsilon} . \tag{4.14}
\end{equation*}
$$

The tangent space at $\xi$ is the space of flat 1 -currents supported at $\xi$, or, equivalently, the space of infinitesimal integral 1-currents at $\xi$.

This is just the usual idea of tangent vectors expressed in terms of currents. The boundary of $\eta(\epsilon)$ is the 0 -current

$$
\begin{equation*}
\partial \eta(\epsilon)=\delta_{\xi(\epsilon)}-\delta_{\xi(0)} . \tag{4.15}
\end{equation*}
$$

If $f$ is a function on $\mathcal{D}_{k}^{\text {int }}(M)$, i.e., a 0 -form, then the derivative of $f$ along the tangent vector at $\xi$ is

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{f\left(\delta_{\xi(\epsilon)}\right)-f\left(\delta_{\xi(0)}\right)}{\epsilon}=\lim _{\epsilon \rightarrow 0} \frac{f(\partial \eta(\epsilon))}{\epsilon}=f(\partial \dot{\xi})=d f(\dot{\xi}) . \tag{4.16}
\end{equation*}
$$

The pushdown map $\Pi_{*}^{1, k}$ takes the infinitesimal integral 1-current $\eta$ to an integral $(k+1)$-current in $M$, and the flat 1-current $\dot{\xi}$ to a flat $(k+1)$-current in $M$. If we restrict to vertical tangent vectors, the map $\Pi_{*}^{1, k}$ is injective on the vertical tangent vectors at $\xi$, giving the space $\mathcal{V}_{k+1}$ described previously.

From the same point of view, the $j$-vectors at $\xi$, i.e., the linear combinations of antisymmetric $j$-fold products of tangent vectors, are the infinitesimal integral $j$-currents at $\xi$, which are the flat $j$-currents supported at $\xi$. The pushdown map $\Pi_{*}^{j, k}$ takes an infinitesimal integral $j$-current to infinitesimal integral $(j+k)$-current in $M$.

### 4.6 The Hodge *-operator on 1-currents in $\mathcal{E}_{\partial \xi}$

Given our assumption that the Hodge $*$-operator acts on the vertical tangent spaces $\mathcal{V}_{n}$, it will act on the infinitesimal 1-currents in the fiber $\mathcal{E}_{\partial \xi}=\mathcal{D}_{n-1}^{\text {int }}(M)_{\partial \xi}$ and therefore on all 1-currents in $\mathcal{E}_{\partial \xi}$. Since $\Pi_{*}^{1, n-1}: \mathcal{D}_{1}\left(\mathcal{E}_{\partial \xi}\right) \rightarrow \mathcal{D}_{n}(M)$ identifies the infinitesimal 1-currents in the fibers with $\mathcal{V}_{n} \subset \mathcal{D}_{n}^{\text {flat }}(M)$,

$$
\begin{equation*}
* \Pi_{*}^{1, n-1}=\Pi_{*}^{1, n-1} * . \tag{4.17}
\end{equation*}
$$

## 5 An analog of a 2d conformal field theory on each $\mathcal{E}_{\partial \xi}$

## $5.1 n$-forms on space-time as 1 -forms on $\mathcal{E}_{\partial \xi}$

The $n$-forms $F$ and $F^{*}$ on $M$ pull up to 1 -forms $j$ and $j^{*}$ on each $\mathcal{E}_{\partial \xi}$,

$$
\begin{equation*}
j(\eta)=F\left(\Pi_{*} \eta\right), \quad j^{*}(\eta)=F^{*}\left(\Pi_{*} \eta\right), \quad \eta \in \mathcal{D}_{1}\left(\mathcal{E}_{\partial \xi}\right) \tag{5.1}
\end{equation*}
$$

Here $\eta$ is a 1 -current in the fiber $\mathcal{E}_{\partial \xi}$. Its pushdown $\Pi_{*} \eta$ is an $n$-current in $M$. Since Hodge $*$ commutes with the pushdown,

$$
\begin{equation*}
j^{*}=i^{-1} * j \tag{5.2}
\end{equation*}
$$

The 1-forms $j$ and $j^{*}$ on the fibers are closed,

$$
\begin{equation*}
d j=0, \quad d j^{*}=0 \tag{5.3}
\end{equation*}
$$

by the equations of motion on $F$ and $F^{*}$,

$$
\begin{equation*}
d j(\eta)=j(\partial \eta)=F\left(\Pi_{*} \partial \eta\right)=F\left(\partial \Pi_{*} \eta\right)=d F\left(\Pi_{*} \eta\right)=0 \tag{5.4}
\end{equation*}
$$

and similarly for $j^{*}$.
The 1 -forms $j$ and $j^{*}$ on the fibers of $\mathcal{E} \rightarrow \mathcal{B}$ are invariant under translations in the whole abelian group $\mathcal{E}$.

### 5.2 Scalar fields and vertex operators on $\mathcal{E}_{\partial \xi}$

On each $\mathcal{E}_{\partial \xi}$, integrate

$$
\begin{equation*}
j=d \phi, \quad j^{*}=d \phi^{*} . \tag{5.5}
\end{equation*}
$$

to get 0 -forms $\phi$ and $\phi^{*}$. Consider the 0 -forms as functions on the fibers, as scalar fields $\phi(\xi)$ and $\phi^{*}(\xi)$. Depend on context to distinguish scalar fields written as functions on the fiber from scalar fields written as linear functions on 0 -currents,

$$
\begin{equation*}
\phi(\xi)=\phi\left(\delta_{\xi}\right) . \tag{5.6}
\end{equation*}
$$

Define "vertex operators" on each fiber,

$$
\begin{equation*}
V_{p, p^{*}}(\xi)=e^{i p \phi(\xi)+i p^{*} \phi^{*}(\xi)} \tag{5.7}
\end{equation*}
$$

The general observable on a fiber $\mathcal{E}_{\partial \xi}$ is a product of vertex operators

$$
\begin{equation*}
V_{p_{1}, p_{1}^{*}}\left(\xi_{1}\right) \cdots V_{p_{N}, p_{N}^{*}}\left(\xi_{N}\right), \quad \partial \xi_{1}=\partial \xi_{2} \cdots=\partial \xi_{N}=\partial \xi \tag{5.8}
\end{equation*}
$$

On each $\mathcal{E}_{\partial \xi}$ there is a formal analog of the 2 d conformal field theory of a free 1 -form.

### 5.3 Global symmetry on $\mathcal{E}_{\partial \xi}$

Each of the scalar fields $\phi$ and $\phi^{*}$ is determined on each $\mathcal{E}_{\partial \xi}$ up to a constant of integration, except for the special fiber $\mathcal{E}_{0}$ where there is a natural normalization condition $\phi(0)=$ $\phi^{*}(0)=0$. So, on each non-special fiber $\mathcal{E}_{\partial \xi}, \partial \xi \neq 0$, the gauge symmetry of the space-time symmetry becomes a global symmetry, shifting $\phi$ and $\phi^{*}$ by constants $f$ and $f^{*}$,

$$
\begin{equation*}
\phi(\xi) \rightarrow \phi(\xi)+f(\partial \xi), \quad \phi^{*}(\xi) \rightarrow \phi^{*}(\xi)+f^{*}(\partial \xi) \tag{5.9}
\end{equation*}
$$

The constants $f$ and $f^{*}$ depend on the fiber $\mathcal{E}_{\partial \xi}$, so they are functions on the base space $\mathcal{B}=\partial \mathcal{D}_{n-1}^{\text {int }}(M)$.

The vertex operators transform under the symmetry group of the fiber as operators of charges $p, p^{*}$,

$$
\begin{equation*}
V_{p, p^{*}}(\xi) \rightarrow V_{p, p^{*}}(\xi) e^{i p f(\partial \xi)} e^{i p^{*} f^{*}(\partial \xi)} \tag{5.10}
\end{equation*}
$$

### 5.4 Space-time gauge symmetries are special collections of global symmetries

The product of the global symmetry groups of the fibers $\mathcal{E}_{\partial \xi}$ is considerably larger than the local gauge group in space-time. The space-time gauge potentials $A$ and $A^{*}$ pull up to scalar fields $\tilde{\phi}=\Pi^{*} A, \tilde{\phi}^{*}=\Pi^{*} A^{*}$ on $\mathcal{E}$ that, restricted to each fiber $\mathcal{E}_{\partial \xi}$, are solutions of (5.5). They are special solutions, characterized by the additional condition of additivity in $\mathcal{E}$,

$$
\begin{equation*}
\tilde{\phi}\left(\xi_{1}+\xi_{2}\right)=\tilde{\phi}\left(\xi_{1}\right)+\tilde{\phi}\left(\xi_{2}\right), \quad \tilde{\phi}^{*}\left(\xi_{1}+\xi_{2}\right)=\tilde{\phi}^{*}\left(\xi_{1}\right)+\tilde{\phi}^{*}\left(\xi_{2}\right) \tag{5.11}
\end{equation*}
$$

The collection of scalars $\phi, \phi^{*}$ on the fibers $\mathcal{E}_{\partial \xi}$ are not so constrained.
The space-time gauge symmetries $A \rightarrow A+d \tilde{f}, A^{*} \rightarrow A^{*}+d \tilde{f}^{*}$ are given by $(n-2)$ forms $\tilde{f}$ and $\tilde{f}^{*}$ on $M$ which pull up to 0 -forms on $\mathcal{B}=\partial \mathcal{D}_{n-1}^{\text {int }}(M)$ which give global symmetries in each $\mathcal{E}_{\partial \xi}$, as in (5.9), satisfying the additional additivity condition

$$
\begin{equation*}
\tilde{f}\left(\xi_{1}+\xi_{2}\right)=\tilde{f}\left(\xi_{1}\right)+\tilde{f}\left(\xi_{2}\right), \quad \tilde{f}^{*}\left(\xi_{1}+\xi_{2}\right)=\tilde{f}^{*}\left(\xi_{1}\right)+\tilde{f}^{*}\left(\xi_{2}\right) \tag{5.12}
\end{equation*}
$$

### 5.5 An analog of the $U(1) \times U(1)$ 2d gaussian model on each $\mathcal{E}_{\partial \xi}$

Identify

$$
\begin{equation*}
\phi(\xi) \sim \phi(\xi)+2 \pi R, \quad \phi^{*}(\xi) \sim \phi^{*}(\xi)+2 \pi R^{*} \tag{5.13}
\end{equation*}
$$

for positive real numbers $R, R^{*}$. The symmetry group of the theory on $\mathcal{E}_{\partial \xi}$ becomes the compact group $U(1) \times U(1)$,

$$
\begin{gather*}
f(\partial \xi) \sim f(\partial \xi)+2 \pi R, \quad f^{*}(\partial \xi) \sim f^{*}(\partial \xi)+2 \pi R^{*},  \tag{5.14}\\
U(1) \times U(1)=(\mathbb{R} / 2 \pi R \mathbb{Z}) \times\left(\mathbb{R} / 2 \pi R^{*} \mathbb{Z}\right) . \tag{5.15}
\end{gather*}
$$

The "momenta" of the vertex operators are quantized,

$$
\begin{equation*}
p=\frac{m}{R}, \quad p^{*}=\frac{m^{*}}{R^{*}}, \quad m, m^{*} \in \mathbb{Z} \tag{5.16}
\end{equation*}
$$

Now there is on each $\mathcal{E}_{\partial \xi}$ a formal analog of the 2 d conformal field theory of a free 1 -form with compact symmetry group $G=U(1) \times U(1)$. This 2 d conformal field theory is the 2 d gaussian model.

The compactification of the symmetry group on each fiber expresses the compactification of the global gauge group of the space-time $n$-form theory to $U(1) \times U(1)$.

## 6 Synopsis

The goal now is to flesh out the analogy between the 2d field theory and the field theory of extended objects on each of the fibers $\mathcal{E}_{\partial \xi}$, the end being to develop machinery to translate the construction of any 2 d quantum field theory into the actual construction of a quantum field theory of extended objects.

I can imagine replacing the analog of the 2d gaussian model on each fiber with analogs other 2 d quantum field theories by performing on each fiber the known constructions on
the 2 d gaussian model, including, for example, (1) constructing the twist fields of the $\mathbb{Z}_{2}$ orbifold of the 2d gaussian model, or (2) specializing to $R=R^{*}=1$ and constructing the $S U(2) \times S U(2)$ current algebra of the self-dual gaussian model, or (3) constructing the analog of 2 d conformal perturbation theory and then the analog of 2 d non-conformal perturbation theory. But what is really wanted is a general machine that implements the analogy for an arbitrary 2 d quantum field theory, by actually constructing a quantum field theory of extended objects from the data of the 2 d quantum field theory.

Sections 7-10 take up the quantum field theory of the free $n$-form. The quantization is expressed by the Schwinger-Dyson equations on the two-point correlation functions of the space-time $n$-forms and the gauge potentials. Section 7 develops the geometric structures on $M$ used in section 8 to write the Schwinger-Dyson equations in terms of currents. The geometric structures consist of a linear operator $J=\epsilon_{n} *$ on $n$-currents in $M$ satisfying $J^{2}=-1$, which is a small modification of the Hodge $*$-operator, and a skew-hermitian form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ on currents in $M$, which is a similarly small modification of the usual bilinear intersection form on currents. The small modifications are needed to make the properties of these geometric structures independent of the parity of $n$. The operator $J$ is imaginary for $n$ even, so $\mathcal{E}_{\partial \xi}$ must be complexified in order for $J$ to act on its tangent vectors. This is done in section 9. The complexification $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is then an almost-complex space. In section 10, the $J$-operator and the skew-hermitian form pulled up to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are used to write the Schwinger-Dyson equations for the fields of the free 1 -form quantum field theory on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The S-D equations on $\mathcal{E}_{\partial \xi}$ are formally identical to the S-D equations for the free 1 -form on a Riemann surface.

Section 11 points out the geometric resemblance of $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ to a Riemann surface. The linear operator $J$ acting on 1-currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ resembles resembles the linear operator $J$ acting on 1-currents in a Riemann surface that expresses the almost-complex structure of the Riemann surface. The push-down map $\Pi_{*}$ takes $k$-currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ to $(n-1+k)$-currents in $M$, so the pulled-up skew-hermitian form on currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}, \Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$, pairs a $k_{1-}$ current $\eta_{1}$ and a $k_{2}$-current $\eta_{2}$ when $k_{1}+k_{2}=2$, exactly as does the skew-hermitian intersection form on the currents in a Riemann surface.

Section 12 tries to capture the geometry of currents shared by the spaces $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ and by ordinary Riemann surfaces in the definition of a quasi Riemann surface. The quasi Riemann surfaces are to be the geometric settings for the general class of quantum field theories of extended objects. A quasi holomorphic curve is defined to be a morphism of quasi Riemann surfaces from an ordinary Riemann surface $\Sigma$ to one of the $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. A local q-h curve is a q-h curve where the Riemann surface $\Sigma$ is the open unit disk in the complex plane.

Section 13 describes how the free 1-form quantum field theory on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ pulls back along a quasi-holomorphic curve to give the 2d CFT of the free 1-form on the Riemann surface $\Sigma$. The q-h curves serve as 2 d probes of the extended objects. The local q-h curves probe the local structure of the extended objects.

In section 14, I propose a strong conjecture on the classification of quasi Riemann surfaces: that quasi Riemann surfaces are isomorphic iff they have the same homology data - the skew-hermitian intersection form and the complex structure on the integral
homology in the middle dimension. The conjecture identifies the spaces $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ of integral ( $n-1$ )-currents in $M$ with the space $\mathcal{D}^{\text {int }}(\Sigma)_{0}$ of integral 0 -currents in a two-dimensional space $\Sigma$ that has the same homology data as $M$. When $M$ has the homology data of a Riemann surface, the 2 d space $\Sigma$ will be a Riemann surface. For example, when $M$ is the conformal manifold $S^{d}=\mathbb{R}^{d} \cup\{\infty\}$, the 2 d space $\Sigma$ will be the Riemann sphere $S^{2}=\mathbb{C} \cup\{\infty\}$. The bundle $\mathcal{E} \rightarrow \mathcal{B}$ is replaced by a bundle $\mathcal{Q}(M) \rightarrow \mathcal{P} \mathcal{B}(M)$ of quasi Riemann surfaces over the integral projective space of $\mathcal{B}$.

Section 15 describes how the conjecture can give a means to construct the correlation functions of fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as correlation functions in the 2 d CFT on the two-dimensional space $\Sigma$. This will require extending the observables of the 2 d CFT from products of local fields over finite sets of points in $\Sigma$ - e.g., products of a finite set of vertex operators to products over integral 0-currents. Such an extended 2d conformal field theory (ECFT) transcribes directly from the quasi Riemann surface associated to the two-dimensional space $\Sigma$ to the quasi Riemann surfaces associated to the spaces $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$.

Section 16 is a formal discussion of the equivalence between perturbation theory in the space-time theory and perturbation theory in the 2d theory - first for the free $n$ form theory on $M$ and the free 1-form theory on $\Sigma$, then for perturbations in general. In the course of the discussion, a modicum of evidence for the conjecture is found. The discussion is formal in the sense that there are no distance scales, neither on $M$ nor on $\Sigma$. I hope that, eventually, a connection can be made between the cutoff scale in space-time and the cutoff 2 d scale, so that non-conformal quantum field theories of extended objects can be constructed from non-conformal 2d quantum field theories, both governed by the 2 d renormalization group.

Sections 17 and 18 return to the classical theory of the free $n$-form as a free 1 -form on $\mathcal{E}_{\partial \xi}$. Section 17 presents the local gauge symmetry in space-time as a specialization of local gauge symmetry in the bundle $\mathcal{E}_{\partial \xi} \rightarrow \mathcal{B}$. The local gauge transformations in the fibers $\mathcal{E}_{\partial \xi}$ are the global symmetries of the 2 d theory on $\mathcal{E}_{\partial \xi}$. The intent is to provide a prototype pattern for gauge symmetry in the general class of quantum field theories of extended objects. In particular, I hope that 2d quantum field theories with nonabelian global symmetry will give quantum field theories of extended objects in space-time with nonabelian local gauge symmetry in space-time. Section 18 takes up the multitude of the fibers $\mathcal{E}_{\partial \xi}$. The theory of extended objects should be a single entity woven from all the analog 2 d theories on the $\mathcal{E}_{\partial \xi}$. Again, the intent is to make a prototype argument that might be adapted to the general case.

Section 19 explores, incautiously, some consequences of the conjecture on quasi Riemann surfaces. First, to meet the technical requirements of a quasi Riemann surface, an ordinary Riemann surface $\Sigma$ needs to be modified slightly - "augmented" - to a space $\Sigma_{+}$. The group $\operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right)$of automorphisms of the quasi Riemann surface $\mathcal{Q}\left(\Sigma_{+}\right)$ associated to $\Sigma_{+}$is contained in a larger group $G\left(\Sigma_{+}\right)$which is essentially the group of automorphisms of the integral 1-currents in $\Sigma$. There is a universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ of quasi Riemann surfaces over the homogeneous space $\mathcal{P B}(0)=G\left(\Sigma_{+}\right) / \operatorname{Aut}(\mathcal{Q}(\Sigma))$. For every conformal manifold $M$ with the homology data of $\Sigma$, the bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ of quasi Riemann surfaces associated to $M$ is embedded in a natural way in the universal
bundle. The universal bundle of quasi Riemann surfaces becomes the natural setting for the quantum field theories of extended objects.

Section 20 lists the main mathematical questions that need to be resolved. Section 21 lists some of the further steps that might be taken, most of which require assuming that the mathematical questions are resolved favorably. Section 22 asks advice about the history of the ideas used and about references.

## 7 More on currents in $M$

### 7.1 Intersection of currents

The intersection form is the bilinear form on smooth currents

$$
\begin{equation*}
I_{M}\left(\xi_{1}, \xi_{2}\right)=\int_{M} d^{d} x \epsilon_{\mu_{1} \cdots \mu_{k_{1}} \nu_{1} \cdots \nu_{k_{2}}} \frac{1}{k_{1}!} \xi_{1}(x)^{\mu_{1} \cdots \mu_{k_{1}}} \frac{1}{k_{2}!} \xi_{2}(x)^{\nu_{1} \cdots \nu_{k_{2}}} \tag{7.1}
\end{equation*}
$$

for $k_{1}+k_{2}=d$, and zero for $k_{1}+k_{2} \neq d$. The intersection form extends to generic pairs of integral currents ("in general position"). On singular currents, the intersection form agrees with the intersection number of the corresponding singular chains. The intersection form on currents depends only on the orientation of $M$. It is independent of the conformal structure.

For $d=2 n$, the intersection form satisfies

$$
\begin{align*}
I_{M}\left(\xi_{1}, \xi_{2}\right) & =(-1)^{k_{1}} I_{M}\left(\xi_{2}, \xi_{1}\right) & &  \tag{7.2}\\
I_{M}\left(\partial \xi_{1}, \xi_{2}\right) & =(-1)^{k_{1}} I_{M}\left(\xi_{1}, \partial \xi_{2}\right) & &  \tag{7.3}\\
I_{M}\left(* \xi_{1}, \xi_{2}\right) & =(-1)^{n} I_{M}\left(\xi_{1}, * \xi_{2}\right), & & k_{1}=k_{2}=n  \tag{7.4}\\
I_{M}(\xi, * \xi) & >0, \quad \xi \neq 0, & & k_{1}=k_{2}=n \tag{7.5}
\end{align*}
$$

The last equation fixes the relation between the sign of $*$ and the orientation of $M$. The positive definite quadratic form $I_{M}(\xi, * \xi)$ is independent of the orientation.

### 7.2 The operator $J=\epsilon_{n} *$ on $n$-currents

The properties of the Hodge *-operator and of the intersection form $I_{M}\left(\xi_{1}, \xi_{2}\right)$ depend on the parity of $n$. In particular, the $*$-operator on $n$-currents satisfies $*^{2}=(-1)^{n}$. This is a problem if the spaces $\mathcal{E}_{\partial \xi}$ are to look like two dimensional spaces, for every $n$. Some small modifications are needed to make the properties uniform for all values of $n$, even and odd.

Define the $J$-operator acting on $n$-forms and on $n$-currents to be

$$
\begin{equation*}
J=\epsilon_{n} * \tag{7.6}
\end{equation*}
$$

where $\epsilon_{n}$ is a number satisfying

$$
\begin{equation*}
\epsilon_{n}^{2}=(-1)^{n-1} \tag{7.7}
\end{equation*}
$$

so that

$$
\begin{equation*}
J^{2}=-1 \tag{7.8}
\end{equation*}
$$

for very value of $n$. One possible choice of the numbers $\epsilon_{n}$ is

$$
\epsilon_{n}= \begin{cases}1, & n \text { odd }  \tag{7.9}\\ i, & n \text { even }\end{cases}
$$

For $n$ even, $J$ is imaginary, so this modification requires allowing currents to be complex. For the sake of uniformity in $n$, we will take the currents to be complex for all $n$. For $n$ odd, there will be a complex conjugation symmetry. For $n$ even, there will be a symmetry combining complex conjugation with reversal of orientation. Discussion of complex conjugation and reality conditions will be left until section D.

### 7.3 The chiral projection operators $P_{ \pm}$

Define the chiral projection operators acting on $n$-currents and on $n$-forms,

$$
\begin{equation*}
P_{ \pm}=\frac{1}{2}\left(1 \pm i^{-1} J\right) . \tag{7.10}
\end{equation*}
$$

$P_{+}$projects on the self-dual $n$-currents, $P_{-}$on the anti-self-dual $n$-currents.

### 7.4 The skew-hermitian intersection form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ on currents

Define the skew-hermitian intersection form on currents by

$$
\begin{equation*}
I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle=\epsilon_{n, k_{2}-n} I_{M}\left(\bar{\xi}_{1}, \xi_{2}\right), \tag{7.11}
\end{equation*}
$$

where the numbers $\epsilon_{n, k}$ are

$$
\begin{equation*}
\epsilon_{n, k}=(-1)^{n k+k(k+1) / 2} \epsilon_{n}^{-1} . \tag{7.12}
\end{equation*}
$$

The constants $\epsilon_{n, k}$ are chosen so that the skew-hermitian intersection form satisfies

$$
\begin{align*}
I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle & =-\overline{I_{M}\left\langle\bar{\xi}_{2}, \xi_{1}\right\rangle} & &  \tag{7.13}\\
I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle & =-I_{M}\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle & &  \tag{7.14}\\
I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle & =0, & & k_{1}+k_{2} \neq 2 n  \tag{7.15}\\
I_{M}\left\langle\overline{J \xi}_{1}, \xi_{2}\right\rangle & =-I_{M}\left\langle\bar{\xi}_{1}, J \xi_{2}\right\rangle, & & k_{1}=k_{2}=n  \tag{7.16}\\
I_{M}\langle\bar{\xi}, J \xi\rangle & >0, \quad \xi \neq 0, & & k_{1}=k_{2}=n . \tag{7.17}
\end{align*}
$$

The skew-hermitian intersection form on $n$-currents is block diagonal in the chiral decomposition,

$$
\begin{equation*}
I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle=I_{M}\left\langle\overline{P_{+} \xi_{1}}, P_{+} \xi_{2}\right\rangle+I_{M}\left\langle\overline{P_{-} \xi_{1}}, P_{-} \xi_{2}\right\rangle, \quad k_{1}=k_{2}=n \tag{7.18}
\end{equation*}
$$

## 8 Quantum field theory of the free $n$-form on $M$

The quantum field theory of the free $n$-form is described by its partition function and its two-point correlation functions. The two-point functions are determined by their Schwinger-Dyson equations, which will be written here in terms of the $J$-operator and the skew-intersection form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ on currents in $M$, with no mention of $n$. This means that $J$ and $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ encode all the geometric data needed to construct the quantum field theory.

The partition function has interesting dependence on the space-time manifold $M$ and its conformal structure. The derivatives of the partition function wrt the parameters describing $M$ can be derived from the two-point functions, so the S-D equations on the two-point functions completely determine the quantum theory. For now we are only interested in the quantum field theory on a fixed space-time. The partition function is left for later.

The Schwinger-Dyson equations for the $n$-form fields and the gauge potentials are obtained in Appendix B by deriving them in the free $n$-form theory on $\mathbb{R}^{d}$.

### 8.1 The chiral fields

Take the $n$-form field $F(x)$ to be a complex field $F=F_{1}+i F_{2}$ with the global $U(1)$ symmetry $F \rightarrow e^{i \alpha} F$. The reality condition $F=\bar{F}$ will be applied later.

The chiral $n$-form fields are

$$
\begin{equation*}
F_{ \pm}=P_{ \pm} F, \quad F_{ \pm}(\xi)=F\left(P_{ \pm} \xi\right) \tag{8.1}
\end{equation*}
$$

The chiral gauge potentials are the $(n-1)$-forms solving

$$
\begin{equation*}
d A_{ \pm}=F_{ \pm} . \tag{8.2}
\end{equation*}
$$

The euclidean adjoint fields are

$$
\begin{equation*}
F_{ \pm}^{\dagger}(\bar{\xi})=\overline{F_{\mp}(\xi)}, \quad A_{ \pm}^{\dagger}(\bar{\xi})=\overline{A_{\mp}(\xi)} . \tag{8.3}
\end{equation*}
$$

The euclidean adjoint fields are defined so that, on $\mathbb{R}^{d}$, the euclidean adjoint field is the Wick rotate of the Minkowski space adjoint field. This is worked out in Appendix B.

Use the index notation $F_{\alpha}, A_{\alpha}, \alpha= \pm$.

### 8.2 The Schwinger-Dyson equations in terms of $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$

The quantum fields $F_{\alpha}(x)$ are distributions that are smeared against smooth (complex) $n$-currents $\xi$ to give observables $F_{\alpha}(\xi)$. The $A_{\alpha}(x)$ are smeared against smooth $(n-1)$ currents. The two-point functions are determined by the Schwinger-Dyson equations (derived in Appendix (B),

$$
\begin{align*}
& \left\langle F_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{1}\right) F_{\beta}\left(\partial \xi_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\overline{\partial \xi}_{1}, \xi_{2}\right\rangle  \tag{8.4}\\
& \left\langle A_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{0}\right) F_{\beta}\left(\partial \xi_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\bar{\xi}_{0}, \xi_{2}\right\rangle \tag{8.5}
\end{align*}
$$

$$
\begin{equation*}
\gamma_{\overline{+}+}=1, \quad \gamma_{\bar{\mp}-}=0, \quad \gamma_{\Xi_{+}}=0, \quad \gamma_{\text {I- }}=-1 . \tag{8.6}
\end{equation*}
$$

Note that these S-D equations are consistent with $d A_{ \pm}=F_{ \pm}$, but they are not consistent with $F_{ \pm}=P_{ \pm} F$. The identities $F_{ \pm}=P_{ \pm} F$ hold only up to contact terms - they hold as field equations, but not as equations on the distributional correlation functions. The only ambiguity in the two-point functions is in the contact term in $\left\langle F_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{1}\right) F_{\beta}\left(\xi_{2}\right)\right\rangle$,

$$
\begin{equation*}
\left\langle F_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{1}\right) F_{\beta}\left(\xi_{2}\right)\right\rangle \rightarrow\left\langle F_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{1}\right) F_{\beta}\left(\xi_{2}\right)\right\rangle+2 \pi i\left(\Delta \gamma_{\bar{\alpha} \beta}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle \tag{8.7}
\end{equation*}
$$

Either $d A_{ \pm}=F_{ \pm}$or $F_{ \pm}=P_{ \pm} F$ can be satisfied in the distributional correlation functions, but not both.

The only geometric data used in the quantization of the free $n$-form on $M$ is the $J$ operator acting on $\mathcal{D}_{n}(M)$ and the skew-hermitian intersection form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ restricted to the subspace

$$
\begin{equation*}
\mathcal{D}_{n-1}(M) \oplus \mathcal{D}_{n}(M) \oplus \mathcal{D}_{n+1}(M) . \tag{8.8}
\end{equation*}
$$

The skew-hermitian intersection form depends only on the manifold structure of $M$. The $J$-operator depends on the conformal structure of $M$.

## $9 \quad \mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as an almost-complex space

For $n$ even, the operator $J=\epsilon_{n} *$ is imaginary, so we have to complexify $\mathcal{E}_{\partial \xi}$ in order to get a space where $J$ acts on the tangent vectors. There is no need to complexify when $n$ is odd, but we do so anyway for the sake of uniformity in $n$.

A straightforward, natural complexification is

$$
\begin{equation*}
\mathcal{E}_{\partial \xi}^{\mathbb{C}}=\mathcal{E}_{\partial \xi} \oplus i \partial \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{9.1}
\end{equation*}
$$

The tangent space at each point is the complex vector space $\mathcal{V}_{n} \oplus i \mathcal{V}_{n} . J$ acts on each tangent space and satisfies $J^{2}=-1$, so $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is an almost-complex space with almost complex structure $J$.

The pushdown maps $\Pi_{*}$ extend to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$, producing complex currents in $M$. The pushdown map $\Pi_{*}^{1, n-1}$ taking 1 -currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ to $n$-currents in $M$ is compatible with the $J$ operators on 1-currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ and on complex $n$-currents in $M$,

$$
\begin{equation*}
\Pi_{*}^{1, n-1} J=J \Pi_{*}^{1, n-1} \tag{9.2}
\end{equation*}
$$

The dual pull-up map $\Pi_{*}$ takes complex $n$-forms on $M$ to complex 1 -forms on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$, and is compatible with the $J$ operators on the forms,

$$
\begin{equation*}
J \Pi^{*}=\Pi^{*} J . \tag{9.3}
\end{equation*}
$$

## 10 The Schwinger-Dyson equations on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$

We pull the Schwinger-Dyson equations up from $M$ to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are the space-time fields pulled up from $M$,

$$
\begin{array}{rlrl}
j_{\alpha} & =\Pi^{*} F_{\alpha} & \phi_{\alpha} & =\Pi^{*} A_{\alpha} \\
j_{\alpha}(\eta) & =F_{\alpha}\left(\Pi_{*}^{1, n-1} \eta\right) & \phi_{\alpha}(\eta) & =A_{\alpha}\left(\Pi_{*}^{0, n-1} \eta\right) \\
j_{\alpha} & =P_{\alpha} j & d \phi_{\alpha} & =j_{\alpha} \\
j_{ \pm}^{\dagger}(\bar{\eta}) & =\overline{j_{\mp}(\eta)} & \phi_{ \pm}^{\dagger}(\bar{\eta}) & =\overline{\phi_{\mp}(\eta)} . \tag{10.4}
\end{array}
$$

The S-D equations (8.4 8.5) become

$$
\begin{align*}
& \left\langle j_{\bar{\alpha}}^{\dagger}\left(\bar{\eta}_{1}\right) j_{\beta}\left(\partial \eta_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} \Pi^{*} I_{M}\left\langle{\left.\overline{\partial \eta_{1}}, \eta_{2}\right\rangle}_{\left\langle\phi_{\bar{\alpha}}^{\dagger}\left(\bar{\eta}_{0}\right) j_{\beta}\left(\partial \eta_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} \Pi^{*} I_{M}\left\langle\bar{\eta}_{0}, \eta_{2}\right\rangle,} .\right. \tag{10.5}
\end{align*}
$$

where $\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ is the the skew-hermitian intersection form pulled up from $M$,

$$
\begin{equation*}
\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle=I_{M}\left\langle\overline{\Pi_{*} \eta_{1}}, \Pi_{*} \eta_{2}\right\rangle \tag{10.7}
\end{equation*}
$$

The S-D equations (10.5-10.6) on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are formally identical to the S-D equations (8.4 8.5) for the free 1 -form on a Riemann surface $\Sigma$, with $J$ taking the place of the almost complex structure of $\Sigma$ and $\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ taking the place of the skew-hermitian intersection form $I_{\Sigma}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ of $\Sigma$.

The analogy between the field theory on $\mathcal{E}_{\partial \xi}$ and the 2d field theory now holds on the quantum level. We have a quantum field theory on each $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ that has the same form as the 2 d quantum field theory.

## 11 Quasi two-dimensionality of $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$

The only geometric data used in the quantization on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is the $J$-operator acting on $\mathcal{D}_{1}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right)$ and the skew-hermitian form $\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ on

$$
\begin{equation*}
\mathcal{D}_{0}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right) \oplus \mathcal{D}_{1}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right) \oplus \mathcal{D}_{2}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right) \tag{11.1}
\end{equation*}
$$

$\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ vanishes unless the pushed-down currents intersect in $M$, i.e., unless

$$
\begin{equation*}
\left(k_{1}+n-1\right)+\left(k_{2}+n-1\right)=d \tag{11.2}
\end{equation*}
$$

which is

$$
\begin{equation*}
k_{1}+k_{2}=2, \tag{11.3}
\end{equation*}
$$

just as for the skew-hermitian intersection form of a two-dimensional manifold. The $J$-operator on 1-currents and the skew-hermitian form $\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ have exactly the properties (7.8, 7.13, 7.17) of the $J$-operator and the skew-hermitian intersection form of
a Riemann surface. The strict positivity property (7.17) will hold if we mod out by the null space of the skew-hermitian form on the 1-currents.

This structure - spaces of 0 -currents, 1 -currents, and 2 -currents with a $J$-operator and a skew-hermitian form - is to be the geometric setting for quantum field theory of extended objects.

## 12 Quasi Riemann surfaces

I will try to formulate an abstract definition of quasi Riemann surface that encompasses the geometry of currents on ordinary Riemann surfaces and also of currents on the spaces $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The idea is to include in the definition all the properties that apply both to the currents in a Riemann surface and to the currents in the spaces $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The definition will be not be entirely precise. I will mention some of the properties, but probably not all of them. And I will remark one key technical gap.

Then I will conjecture: (1) that quasi Riemann surfaces are classified up to isomorphism by the homology data - the homology along with the skew-hermitian form and the $J$-operator in the middle dimension, and (2) that every isomorphism class contains a two-dimensional conformal space, which is a Riemann surface when the homology data is appropriate. The homology of a connected Riemann surface $\Sigma$ is $H_{1}(\Sigma)$. The middle homology of $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is $H_{n}(M)$. For example, if $M=S^{2 n}$ then each $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is isomorphic to $S^{2}$ as a quasi Riemann surface, if the conjecture is true.

Further, I will suppose that every 2d conformal field theory (2d CFT) can be installed in a natural way on any two-dimensional quasi Riemann surface $\Sigma$, in particular on the quasi Riemann surface corresponding to an ordinary Riemann surface, e.g., $S^{2}$. This means extending the observables from products of ordinary local quantum fields over a finite collection of points in the two-dimensional space to products of local fields over an integral 0 -current in the two-dimensional space. The integral 0 -currents are the extended objects in the two-dimensional space. The extension of a 2d CFT from Riemann surfaces to the two-dimensional quasi Riemann surfaces might be called a 2d extended conformal field theory (2d ECFT).

The mathematical conjecture will then allow the 2d ECFT to be installed on each of the $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$, via an isomorphism of quasi Riemann surfaces between $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ and the corresponding twodimensional quasi Riemann surface. This will give a conformal field theory of extended objects in $M$ for every 2d CFT.

### 12.1 Definition

A quasi Riemann surface is to consist of abelian groups $\mathcal{Q}_{0}^{\text {int }}, \mathcal{Q}_{1}^{\text {int }}, \mathcal{Q}_{2}^{\text {int }}$. Each $\mathcal{Q}_{k}^{\text {int }}$ is contained in the corresponding complex vector space $\mathcal{Q}_{k}=\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{Q}_{k}^{\text {int }}$. The $\mathcal{Q}_{k}^{\text {int }}$ are complete metric spaces. There is a boundary operator $\partial$ and an augmented de Rham
complex

with

$$
\begin{equation*}
\mathcal{Q}_{-1}^{\text {int }}=\mathbb{Z}, \quad \mathcal{Q}_{-1}=\mathbb{C}, \quad \mathcal{Q}_{3}^{\text {int }}=\mathbb{Z} \oplus i \mathbb{Z}, \quad \mathcal{Q}_{3}=\mathbb{C} \tag{12.2}
\end{equation*}
$$

Call $\mathcal{Q}_{k}$ the $k$-space of the quasi Riemann surface, and call its elements the $k$-currents.
The imaginary parts are to cope with manifolds $M$ of dimension $d=2 n$ with $n$ even. If only $n$ odd were considered, then all the imaginary parts could be dropped.

There are pushdown maps

$$
\begin{equation*}
\Pi_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{Q}_{k}^{\mathrm{int}}\right) \rightarrow \mathcal{Q}_{j+k}^{\mathrm{int}} \tag{12.3}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
\partial \Pi_{*}^{j, k}=\Pi_{*}^{j-1, k} \partial+\Pi_{*}^{j, k-1} \partial_{*} . \tag{12.4}
\end{equation*}
$$

The induced maps on the homology groups of the augmented de Rham complex of integral currents $\mathcal{D}_{j}^{\text {int }}\left(\left(\mathcal{Q}_{k}^{\text {int }}\right)_{0}\right)$ on the space of integral $k$-cycles,

$$
\begin{equation*}
\Pi_{*}^{j, k}: H_{j}\left(\left(\mathcal{Q}_{k}^{\text {int }}\right)_{0}\right) \rightarrow H_{j+k}\left(\mathcal{Q}^{\text {int }}\right), \tag{12.5}
\end{equation*}
$$

should be isomorphisms.
There is a linear operator $J$ on $\mathcal{Q}_{1}$ and there is a nondegenerate skew-hermitian form $I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ on $\oplus_{k} \mathcal{Q}_{k}$. They satisfy

$$
\begin{equation*}
J^{2}=-1 \tag{12.6}
\end{equation*}
$$

and

$$
\begin{align*}
I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle & =-\overline{I_{\mathcal{Q}}\left\langle\bar{\eta}_{2}, \eta_{1}\right\rangle} & &  \tag{12.7}\\
I_{\mathcal{Q}}\left\langle\overline{\partial \eta_{1}}, \eta_{2}\right\rangle & =-I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \partial \eta_{2}\right\rangle & &  \tag{12.8}\\
I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle & =0, & & k_{1}+k_{2} \neq 2  \tag{12.9}\\
I_{\mathcal{Q}}\left\langle\overline{J \eta_{1}}, \eta_{2}\right\rangle & =-I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, J \eta_{2}\right\rangle, & & k_{1}=k_{2}=1  \tag{12.10}\\
I_{\mathcal{Q}}\langle\bar{\eta}, J \eta\rangle & >0, \quad \eta \neq 0, & & k_{1}=k_{2}=1  \tag{12.11}\\
I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle & =-\bar{\eta}_{1} \eta_{2}, & & k_{1}=-1, k_{2}=3 \tag{12.12}
\end{align*}
$$

More properties that might be included in the definition of quasi Riemann surface are discussed below, in section 12.5 ,

We will say that a quasi Riemann surface is connected when

$$
\begin{equation*}
H_{0}\left(\mathcal{Q}^{\text {int }}\right)=H_{2}\left(\mathcal{Q}^{\text {int }}\right)=0 \tag{12.13}
\end{equation*}
$$

So all the homology of the augmented de Rham complex is in the middle dimension, which is dimension $1, H_{2}\left(\mathcal{Q}^{\text {int }}\right)$. The quasi Riemann surface associated to a Riemann surface $\Sigma$ will be connected iff $\Sigma$ is connected. The quasi Riemann surfaces associated to a manifold $M$ will be connected iff $H_{n-1}(M)=0$. To avoid (minor) complications, I will assume connectedness. So Riemann surfaces $\Sigma$ will be assumed connected. And $H_{n-1}(M)=0$ will be assumed for space-time manifolds $M$.

Note that $\mathcal{Q}_{1}$ decomposes as a Hilbert space into three orthonormal subspaces

$$
\begin{equation*}
\mathcal{Q}_{1}=\left(\partial \mathcal{Q}_{2}\right) \oplus \mathcal{Q}_{1, H} \oplus\left(J \partial \mathcal{Q}_{2}\right) \tag{12.14}
\end{equation*}
$$

where $\mathcal{Q}_{1, H}$ is the space of harmonic 1-currents,

$$
\begin{equation*}
\mathcal{Q}_{1, H}=(\operatorname{Ker} \partial) \cap(J \operatorname{Ker} \partial), \quad J \mathcal{Q}_{1, H}=\mathcal{Q}_{1, H}, \quad \operatorname{Ker} \partial=\left(\partial \mathcal{Q}_{2}\right) \oplus \mathcal{Q}_{1, H} \tag{12.15}
\end{equation*}
$$

## 12.2 $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as a quasi Riemann surface

To interpret $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ as a quasi Riemann surface, let $\mathcal{Q}_{k}$ be the $k$-currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ modulo the null spaces of the skew-hermitian form $\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$,

$$
\begin{equation*}
\mathcal{Q}_{k}^{\text {int }}=\mathcal{D}_{k}^{\text {int }}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right) / \mathcal{N}_{k}^{\text {int }}, \quad \mathcal{Q}_{k}=\mathcal{D}_{k}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right) / \mathcal{N}_{k}, \quad k=0,1,2,3 \tag{12.16}
\end{equation*}
$$

The skew-hermitian form is

$$
\begin{equation*}
I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle=\Pi^{*} I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle=I_{M}\left\langle\overline{\Pi_{*} \eta_{1}}, \Pi_{*} \eta_{2}\right\rangle, \tag{12.17}
\end{equation*}
$$

which is manifestly nondegenerate. Take $J=\epsilon_{n} *$ to be the $J$-operator already defined on $\mathcal{D}_{1}^{\text {int }}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right)$. The skew-hermitian form factors through the pushdown maps $\Pi_{*}(M)$ of $M$ (so written to distinguish them from the pushdown maps of $\mathcal{Q}$ ), so the spaces $\mathcal{Q}_{k}^{\text {int }}$ are spaces of currents in $M$ modulo the appropriate null spaces,

$$
\begin{align*}
\mathcal{Q}_{0}^{\mathrm{int}} & =\underset{k \in \mathbb{Z}}{\oplus} \mathcal{D}_{n-1}^{\mathrm{int}}(M)_{k \partial \xi} \oplus i \partial \mathcal{D}_{n}^{\mathrm{int}}(M)  \tag{12.18}\\
\mathcal{Q}_{1}^{\mathrm{int}} & =\mathcal{D}_{n}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n}^{\mathrm{int}}(M)  \tag{12.19}\\
\mathcal{Q}_{2}^{\mathrm{int}} & =\left(\mathcal{D}_{n+1}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n+1}^{\mathrm{int}}(M)\right) / \mathcal{N}_{n+1}^{\mathrm{int}}(M, \partial \xi)  \tag{12.20}\\
\mathcal{Q}_{3}^{\mathrm{int}} & =\left(\mathcal{D}_{n+2}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n+2}^{\mathrm{int}}(M)\right) / \mathcal{N}_{n+2}^{\mathrm{int}}(M, \partial \xi)  \tag{12.21}\\
\mathcal{Q}_{0} & =\mathcal{D}_{n-1}(M)_{\mathrm{C} \partial \xi}  \tag{12.22}\\
\mathcal{Q}_{1} & =\mathcal{D}_{n}(M)  \tag{12.23}\\
\mathcal{Q}_{2} & =\mathcal{D}_{n+1}(M) / \mathcal{N}_{n+1}(M, \partial \xi)  \tag{12.24}\\
\mathcal{Q}_{3} & =\mathcal{D}_{n+2}(M) / \mathcal{N}_{n+2}(M, \partial \xi) \tag{12.25}
\end{align*}
$$

where

$$
\begin{align*}
& \mathcal{N}_{n+1}^{\text {int }}(M, \partial \xi)=\left\{\xi_{2} \in \mathcal{D}_{n+1}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n+1}^{\mathrm{int}}(M): I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle=0, \forall \xi_{1} \in \mathcal{Q}_{0}^{\mathrm{int}}\right\}  \tag{12.26}\\
& \mathcal{N}_{n+2}^{\mathrm{int}}(M, \partial \xi)=\left\{\xi_{2} \in \mathcal{D}_{n+2}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n+2}^{\mathrm{int}}(M): I_{M}\left\langle\overline{\partial \xi}, \xi_{2}\right\rangle=0\right\}  \tag{12.27}\\
& \mathcal{N}_{n+1}(M, \partial \xi)=\left\{\xi_{2} \in \mathcal{D}_{n+1}(M): I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle=0, \forall \xi_{1} \in \mathcal{D}_{n-1}(M)_{\mathbb{C} \xi \xi}\right\}  \tag{12.28}\\
& \mathcal{N}_{n+2}(M, \partial \xi)=\left\{\xi_{2} \in \mathcal{D}_{n+1}(M): I_{M}\left\langle\overline{\partial \xi}, \xi_{2}\right\rangle=0\right\} \tag{12.29}
\end{align*}
$$

The elements of $\mathcal{D}_{n-1}^{\text {int }}(M)_{\partial \xi}$ are of the form $\xi_{1}=\xi+\xi_{0}$ for arbitrary integral ( $n-1$ )-cycle $\xi_{0}$. Therefore

$$
\begin{equation*}
\mathcal{N}_{n+1}^{\text {int }}(M, \partial \xi)=\partial \mathcal{N}_{n+2}^{\mathrm{int}}(M, \partial \xi), \quad \mathcal{N}_{n+1}(M, \partial \xi)=\partial \mathcal{N}_{n+2}(M, \partial \xi) \tag{12.30}
\end{equation*}
$$

$\mathcal{N}_{n+2}^{\text {int }}(M, \partial \xi)$ is the subgroup of (imaginary) integral $(n+2)$-currents that do not intersect the $(n-2)$-boundary $\partial \xi . \quad \mathcal{N}_{n+1}^{\mathrm{int}}(M, \partial \xi)$ is the subgroup of (imaginary) integral $(n+1)$ boundaries that do not link the $(n-2)$-boundary $\partial \xi$.

The boundary operator acts on $\eta \in \mathcal{Q}_{0}^{\text {int }}$ by

$$
\begin{equation*}
\partial_{M} \eta=(\partial \eta) \partial \xi \tag{12.31}
\end{equation*}
$$

where the lhs is the boundary operator on currents in $M$ acting on $\eta$ considered as an integral ( $n-1$ )-current in $M$.

The problem is to define $1 \in \mathcal{Q}_{3}^{\text {int }}$. It must satisfy

$$
\begin{equation*}
I_{\mathcal{Q}}\langle 1,1\rangle=-1 \tag{12.32}
\end{equation*}
$$

where, in the skew-hermitian form, on the left is $1 \in \mathcal{Q}_{-1}^{\text {int }}$ and on the right is $1 \in \mathcal{Q}_{3}^{\text {int }}$. So $1 \in \mathcal{Q}_{3}^{\text {int }}$ must be represented by an $(n+2)$-current in $M$

$$
\begin{equation*}
\xi_{3} \in \mathcal{D}_{n+2}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n+2}^{\mathrm{int}}(M) \tag{12.33}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
I_{M}\left\langle\overline{\partial \xi}, \xi_{3}\right\rangle=-1 \tag{12.34}
\end{equation*}
$$

The definition of the skew-hermitian intersection form in $M$ in terms of the ordinary intersection form given in section 7.4 says

$$
\begin{equation*}
I_{M}\left\langle\overline{\partial \xi}, \xi_{3}\right\rangle=\epsilon_{n, 2} I_{M}\left(\overline{\partial \xi}, \xi_{3}\right)=I_{M}\left(\overline{\partial \xi},-\epsilon_{n}^{-1} \xi_{3}\right), \tag{12.35}
\end{equation*}
$$

so $\xi_{3}$ must satisfy

$$
\begin{equation*}
I_{M}\left(\bar{\partial} \xi, \epsilon_{n}^{-1} \xi_{3}\right)=1 \tag{12.36}
\end{equation*}
$$

The existence of such a current $\xi_{3}$ is a constraint on $\partial \xi$. There must exist an integral $(n+2)$-cycle $\epsilon_{n}^{-1} \xi_{3}$ which has intersection number 1 with $\partial \xi$. I believe that this is equivalent to the condition that $\partial \xi$ is irreducible, i.e., that

$$
\begin{equation*}
\partial \xi \neq k \partial \xi^{\prime}, \quad \forall k \in \mathbb{Z}, \partial \xi^{\prime} \in \partial \mathcal{D}_{n-1}^{\operatorname{int}}(M), \quad k \neq \pm 1 \tag{12.37}
\end{equation*}
$$

but I do not have a proof. Assuming this to be true, then $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ gives a quasi Riemann surface iff $\partial \xi$ is irreducible.

### 12.3 The bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ of quasi Riemann surfaces

$\mathcal{E}_{ \pm \partial \xi}^{\mathbb{C}}$ give the same quasi Riemann surface, so we can associate the quasi-Riemann surface to the integral line $\mathbb{Z} \partial \xi \subset \mathcal{B}$. The integral lines $\mathbb{Z} \partial \xi$ are the maximal abelian group homorphisms $\mathbb{Z} \rightarrow \mathcal{B}$. Write $\mathcal{P B}(M)$ for the space of integral lines in $\mathcal{B}$,

$$
\begin{equation*}
\mathcal{P B}(M)=\{\mathbb{Z} \partial \xi: \partial \xi \in \mathcal{B}, \partial \xi \text { irreducible }\} \tag{12.38}
\end{equation*}
$$

We might call $\mathcal{P B}(M)$ the integral projective space of $\mathcal{B}$.
There is a quasi Riemann surface $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ for every integral line $\mathbb{Z} \partial \xi \in \mathcal{P B}(M)$. Let

$$
\begin{equation*}
\mathcal{Q}(M)=\bigcup_{\mathbb{Z} \partial \xi \in \mathcal{P B}(M)} \mathcal{Q}(M)_{\mathbb{Z} \partial \xi} \tag{12.39}
\end{equation*}
$$

so

$$
\begin{equation*}
\mathcal{Q}(M) \rightarrow \mathcal{P B}(M) \tag{12.40}
\end{equation*}
$$

is a bundle of quasi Riemann surfaces.
The 0 -space of the quasi Riemann surface $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ is

$$
\begin{equation*}
\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 0}^{\operatorname{int}}=\underset{k \in \mathbb{Z}}{\oplus} \mathcal{D}_{n-1}^{\operatorname{int}}(M)_{k \partial \xi} \oplus i \partial \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{12.41}
\end{equation*}
$$

within which the space of 0 -cycles, $\operatorname{Ker} \partial$, is

$$
\begin{equation*}
\left(\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 0}^{\mathrm{int}}\right)_{0}=\mathcal{D}_{n-1}^{\mathrm{int}}(M)_{0} \oplus i \partial \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{12.42}
\end{equation*}
$$

Note that the same $\left(\mathcal{Q}_{\mathbb{Z} \partial \xi, 0}\right)_{0}$ occurs in every fiber $\mathcal{Q}_{\mathbb{Z} \partial \xi}$.
The sum of the 0 -spaces of all the fibers $\mathcal{Q}(M)_{\mathbb{Z} a \xi}$, taken as subgroups within the space of $(n-1)$-currents in $M$, is the full space of extended objects,

$$
\begin{equation*}
\sum_{\mathbb{Z} \partial \xi \in \mathcal{P B}(M)} \mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 0}^{\mathrm{int}}=\mathcal{D}_{n-1}^{\mathrm{int}}(M) \oplus i \partial \mathcal{D}_{n}^{\text {int }}(M)=\mathcal{E}^{\mathbb{C}} \tag{12.43}
\end{equation*}
$$

### 12.4 A Riemann surface $\Sigma$ as a quasi Riemann surface

Now we want to associate a quasi Riemann surface $\mathcal{Q}(\Sigma)$ to every ordinary Riemann surface $\Sigma$. The integral spaces $\mathcal{Q}(\Sigma)_{k}^{\text {int }}$ are complexifications of the integral $k$-currents in $\Sigma$, and the $\mathcal{Q}(\Sigma)_{k}$ are the complex $k$-currents in $\Sigma$,

$$
\begin{align*}
\mathcal{Q}(\Sigma)_{0}^{\text {int }} & =\mathcal{D}_{0}^{\text {int }}(\Sigma) \oplus i \partial \mathcal{D}_{1}^{\text {int }}(\Sigma) & &  \tag{12.44}\\
\mathcal{Q}(\Sigma)_{k}^{\text {int }} & =\mathcal{D}_{k}^{\text {int }}(\Sigma) \oplus i \mathcal{D}_{k}^{\mathrm{int}}(\Sigma), & & k=1,2  \tag{12.45}\\
\mathcal{Q}(\Sigma)_{k} & =\mathcal{D}_{k}(\Sigma), & & k=1,2,3 . \tag{12.46}
\end{align*}
$$

Again, the imaginary parts are needed to cope with manifolds $M$ of dimension $d=2 n$ with $n$ even.

The boundary operator acts on $\mathcal{Q}(\Sigma)_{0}^{\text {int }}$ by

$$
\begin{equation*}
\partial: \eta \mapsto \int_{\eta} 1 \tag{12.47}
\end{equation*}
$$

and on $\mathcal{Q}(\Sigma)_{3}$ by

$$
\begin{equation*}
\partial: 1 \mapsto \Sigma \tag{12.48}
\end{equation*}
$$

The $J$-operator is $\epsilon_{1} *$ with $\epsilon_{1}^{2}=1$. The skew-hermitian form is $I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle=I_{\Sigma}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$. Condition (12.12) expresses the extension of $I_{\Sigma}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ to $\eta_{1} \in \mathcal{D}_{-1}(\Sigma), \eta_{2} \in \mathcal{D}_{3}(\Sigma)$, by

$$
\begin{equation*}
I_{\Sigma}\langle\overline{1}, 1\rangle=I_{\Sigma}\left\langle\bar{\delta}_{z}, 1\right\rangle=-I_{\Sigma}\left\langle\bar{\delta}_{z}, \partial 1\right\rangle=-I_{\Sigma}\left\langle\bar{\delta}_{z}, \Sigma\right\rangle=-1 \tag{12.49}
\end{equation*}
$$

One significant technical issue will be left unresolved. The pushdown maps

$$
\begin{equation*}
\Pi_{*}^{j, k}: \mathcal{D}_{j}^{\mathrm{int}}\left(\mathcal{Q}(\Sigma)_{k}^{\mathrm{int}}\right) \rightarrow \mathcal{Q}(\Sigma)_{3}^{\mathrm{int}}, \quad j+k=3 \tag{12.50}
\end{equation*}
$$

must be identically zero, as the definition of $\mathcal{Q}(\Sigma)$ presently stands. Somehow, the spaces of currents $\mathcal{D}_{j}^{\text {int }}\left(\mathcal{Q}(\Sigma)_{k}^{\mathrm{int}}\right)$ will have to be augmented to reflect the augmentation of the de Rham complex of currents in $\Sigma$. This is discussed further in section 19.3 below.

### 12.5 Definition (more)

The abstract definition of quasi Riemann surface should require all properties that are satisfied by the concrete quasi Riemann surfaces $\mathcal{Q}(\Sigma)$ and $\mathcal{Q}_{\mathbb{Z} \partial \xi}$. The list of properties might include

1. Each of the complete metric spaces $\mathcal{Q}_{k}^{\text {int }}$ should be generated as an abelian group by an arbitrarily small neighborhood of the identity.
2. The tangent spaces $T_{0} \mathcal{Q}_{k}^{\text {int }}$ should be dense subspaces of the vector spaces $\mathcal{Q}_{k}$.
3. The operator $J$ on $\mathcal{Q}_{1}$ should preserve the tangent space $T_{0} \mathcal{Q}_{0}^{\text {int }}$.
4. The skew-hermitian form $I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ should be defined on a dense subspace of $\overline{\mathcal{Q}}_{k} \otimes$ $\mathcal{Q}_{2-k}$.
5. $I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ should be defined on a dense abelian subgroup of $\overline{\mathcal{Q}}_{k}^{\text {int }} \times \mathcal{Q}_{2-k}^{\text {int }}$, where it should take values in $\mathbb{Z} \oplus i \mathbb{Z}$.
6. The maps $\Pi_{*}^{j, k}$ should be surjective (onto). For $\mathcal{Q}(\Sigma)$ and $j+k=3$, this is an unresolved issue.
7. $\Pi_{*}^{1,0}$ should be injective on the tangent space $T_{0} \mathcal{Q}_{0}^{\text {int }}$ and $\Pi_{*}^{2,0}$ should be injective on the space of 2 -vectors at $0 \in \mathcal{Q}_{0}^{\text {int }}$.
8. The maps $\Pi_{*}^{j, k}$ should satisfy compatibility conditions, perhaps such as that

$$
\begin{array}{ll} 
& \mathcal{D}_{1}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}\left(\mathcal{Q}_{0}^{\text {int }}\right)\right) \rightarrow \mathcal{D}_{2}^{\text {int }}\left(\mathcal{Q}_{0}^{\text {int }}\right) \rightarrow \mathcal{Q}_{2}^{\text {int }} \\
\text { and } & \mathcal{D}_{1}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}\left(\mathcal{Q}_{0}^{\text {int }}\right)\right) \rightarrow \mathcal{D}_{1}^{\text {int }}\left(\mathcal{Q}_{1}^{\text {int }}\right) \rightarrow \mathcal{Q}_{2}^{\text {int }} \tag{12.52}
\end{array}
$$

should give the same result.

### 12.6 Morphisms

A morphism $f: \mathcal{Q} \rightarrow \mathcal{Q}^{\prime}$ of quasi Riemann surfaces is a set of maps

$$
\begin{equation*}
f_{k}: \mathcal{Q}_{k}^{\text {int }} \rightarrow \mathcal{Q}_{k}^{\prime \text { int }} \tag{12.53}
\end{equation*}
$$

preserving all the structures and properties of quasi Riemann surfaces. The morphism is determined by $f_{0}: \mathcal{Q}_{0}^{\text {int }} \rightarrow \mathcal{Q}_{0}^{\text {int }}$. The quasi Riemann surface conditions are constraints on $f_{0}$. The map $f_{1}: \mathcal{Q}_{1}^{\text {int }} \rightarrow \mathcal{Q}_{1}^{\text {int }}$ is the derivative of $f_{0}$,

$$
\begin{equation*}
\partial f_{1}=f_{0} \partial \tag{12.54}
\end{equation*}
$$

Extended to a linear map $f_{1}: \mathcal{Q}_{1} \rightarrow \mathcal{Q}_{1}^{\prime}$, it should preserve the $J$-operators

$$
\begin{equation*}
f_{1} J=J f_{1} \tag{12.55}
\end{equation*}
$$

and it should be a partial unitary transformation with respect to the positive definite hermitian forms,

$$
\begin{equation*}
I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, J \eta_{2}\right\rangle=I_{\mathcal{Q}^{\prime}}\left\langle\overline{f_{1} \eta_{1}}, J f_{1} \eta_{2}\right\rangle \tag{12.56}
\end{equation*}
$$

In addition, $f_{1}$ should preserve the kernels of the pushdown operator,

$$
\begin{equation*}
f_{1}\left(\operatorname{Ker} \Pi_{*}^{1,1}\right) \subset \operatorname{Ker} \Pi_{*}^{1,1} \tag{12.57}
\end{equation*}
$$

Then $f_{2}: \mathcal{Q}_{2}^{\text {int }} \rightarrow \mathcal{Q}_{2}^{\text {int }}$ will be given by

$$
\begin{equation*}
\partial f_{2}=f_{1} \partial \tag{12.58}
\end{equation*}
$$

Continuity of $f_{1}$ should guarantee that the skew-hermitian forms are preserved in toto. In particular, continuity should ensure that

$$
\begin{equation*}
f_{2} \partial 1=\partial 1 \tag{12.59}
\end{equation*}
$$

Alternatively, a morphism is determined by $f_{1}: \mathcal{Q}_{1}^{\text {int }} \rightarrow \mathcal{Q}_{1}^{\prime \text { int }}$, subject to the constraints
M1 $f_{1}$ is a homorphism of abelian groups.
M2 $f_{1}$ is continuous.
M3 $\quad f_{1}(\operatorname{Ker} \partial) \subset \operatorname{Ker} \partial$
M4 $\quad f_{1}(\operatorname{Im} \partial) \subset \operatorname{Im} \partial$
M5 $\quad f_{1} J=J f_{1}$
M6 $\quad I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, J \eta_{2}\right\rangle=I_{\mathcal{Q}^{\prime}}\left\langle\overline{f_{1} \eta_{1}}, J f_{1} \eta_{2}\right\rangle$
M7 $f_{1 *}\left(\operatorname{Ker} \Pi_{*}^{1,1}\right) \subset \operatorname{Ker} \Pi_{*}^{1,1}$.

The last condition is essential. All the quasi Riemann surfaces $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ have the same $\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\text {int }}\right)_{1}=\mathcal{D}_{n}^{\text {int }}(M)+i \mathcal{D}_{n}^{\text {int }}(M)$. The subgroups Ker $\partial$ and $\operatorname{Im} \partial$ in $\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\text {int }}\right)_{1}$ are the same. They have the same $J$ and $I_{\mathcal{Q}}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ on $\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{1}$. Only $\operatorname{Ker} \Pi_{*}^{1,1}$ depends nontrivially on $\mathbb{Z} \partial \xi$. Let

$$
\begin{equation*}
\mathcal{W}_{1,1}^{\text {int }}=\left\{\eta \in \mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\mathrm{int}}\right)_{1}\right): \Pi_{*}^{1,1} \eta \in \partial\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\mathrm{int}}\right)_{3}\right\} \tag{12.60}
\end{equation*}
$$

Then $\operatorname{Ker} \Pi_{*}^{1,1} \subset \mathcal{W}_{1,1}^{\text {int }}$ and

$$
\begin{equation*}
\mathcal{W}_{1,1}^{\mathrm{int}} / \operatorname{Ker} \Pi_{*}^{1,1}=\mathbb{Z} . \tag{12.61}
\end{equation*}
$$

$\mathcal{W}_{1,1}^{\text {int }}$ is the same for all $\mathbb{Z} \partial \xi$. What distinguishes - and characterizes - the different $\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\text {int }}\right)_{1}$ is the way that $\operatorname{Ker} \Pi_{*}^{1,1}$ sits inside $\mathcal{W}_{1,1}^{\text {int }}$.

The linear operator $f_{1}$ on $\mathcal{Q}_{1}=\left(\partial \mathcal{Q}_{2}\right) \oplus \mathcal{Q}_{1, H} \oplus\left(J \partial \mathcal{Q}_{2}\right)$ is determined by its action on $\partial \mathcal{Q}_{2}$ and its action on $\mathcal{Q}_{1, H}$, which is the complex homology. The action on $J \partial \mathcal{Q}_{2}$ then follows because $f_{1}$ commutes with $J$.

### 12.7 Isomorphisms and automorphisms

Write $\operatorname{Iso}\left(\mathcal{Q}, \mathcal{Q}^{\prime}\right)$ for the isomorphisms between two quasi Riemann surface. Write $\operatorname{Aut}(\mathcal{Q})$ for the group of automomorphisms of $\mathcal{Q}$.

If $f: \mathcal{Q} \rightarrow \mathcal{Q}^{\prime}$ is an isomorphism, then $f_{1}: \mathcal{Q}_{1} \rightarrow \mathcal{Q}_{1}^{\prime}$ is unitary. Conversely, unitarity of $f_{1}$ should imply that a morphism $f$ is an isomorphism.

For a Riemann surface $\Sigma$, the group $\operatorname{Conf}(\Sigma)$ of conformal symmetries of $\Sigma$ is a subgroup of $\operatorname{Aut}(\mathcal{Q}(\Sigma))$. For example, $\operatorname{Aut}\left(\mathcal{Q}\left(S^{2}\right)\right)$ contains $\operatorname{PSL}(2, \mathbb{C})$ as a subgroup.

For a conformal manifold $M$, the group $\operatorname{Conf}(M, \mathbb{Z} \partial \xi)$ of conformal symmetries of $M$ that preserve $\pm \partial \xi$ is a subgroup of $\operatorname{Aut}\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)$. For example, if $M=S^{d}$ and $\partial \xi$ is an $(n-2)$-sphere in $M$, then $\operatorname{Aut}\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)$ contains $S(O(n+2, \mathbb{R}) \times O(n-1, \mathbb{R}))$ as a subgroup.

### 12.8 Morphisms and homology

A morphism $f: \mathcal{Q} \rightarrow \mathcal{Q}^{\prime}$ maps the homology group $H_{1}\left(\mathcal{Q}^{\text {int }}\right)$ into the homology group $H_{1}\left(\mathcal{Q}^{\prime \text { int }}\right)$. The homology map is injective because the skew-hermitian form is preserved. The free parts of the homology groups are lattices in the real homology groups $H_{1}(\mathcal{Q})$ and $H_{1}\left(\mathcal{Q}^{\prime}\right)$, which are complex Hilbert spaces, the complex structures given by the respective $J$-operators and the positive definite hermitian inner product given by the skew-hermitian forms combined with the $J$-operators. The map of homology groups preserves this structure. I do not know how to deal with the torsion subgroup of the homology, so I will just disregard the possibility of torsion.

Call the homology of a quasi Riemann surface, along with the action of $J$ and the skew-hermitian form, the homology data. An isomorphism of quasi Riemann surfaces gives an isomorphism of homology data, so two quasi Riemann surfaces are isomorphic only if they have isomorphic middle homology data. I will conjecture shortly that this should be 'if and only if', that quasi Riemann surfaces are classified by the homology data.

For connected quasi Riemann surfaces, all the homology is in the middle dimension. For the quasi Riemann surface $\mathcal{Q}(\Sigma)$ associated to an ordinary Riemann surface $\Sigma$, the
middle homology of the quasi Riemann surface is the middle homology of $\Sigma$, doubled,

$$
\begin{equation*}
H_{1}\left(\mathcal{Q}(\Sigma)^{\text {int }}\right)=H_{1}(\Sigma, \mathbb{Z}) \oplus i H_{1}(\Sigma, \mathbb{Z}) \tag{12.62}
\end{equation*}
$$

For a quasi Riemann surface $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ associated to a manifold $M$ of dimension $d=2 n$, the middle homology of the quasi Riemann surface is the middle homology of $M$, doubled,

$$
\begin{equation*}
H_{1}\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}^{\text {int }}\right)=H_{1}(M, \mathbb{Z}) \oplus i H_{1}(M, \mathbb{Z}) \tag{12.63}
\end{equation*}
$$

Again, the imaginary parts can be dropped if we limit ourselves to $n$ odd.

### 12.9 Quasi-holomorphic curves

Given a Riemann surface $\Sigma$, define a quasi-holomorphic curve, or a quasi-holomorphic $\Sigma$-curve, to be a morphism of quasi Riemann surfaces from $\mathcal{Q}(\Sigma)$ to a $\mathcal{Q}_{\mathbb{Z} \partial \xi}$. Define a local quasi-holomorphic curve to be a q-h $\Sigma$-curve with $\Sigma$ conformally equivalent to the open unit disk.

Recall that

$$
\begin{equation*}
\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{0} \supset \partial^{-1}(1)=\mathcal{E}_{\partial \xi}^{\mathbb{C}}=\mathcal{D}_{n-1}^{\mathrm{int}}(M)_{\partial \xi} \oplus i \partial \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{12.64}
\end{equation*}
$$

The map $f_{0}: \mathcal{Q}(\Sigma)_{0} \rightarrow\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{0}$ is equivalent to the map

$$
\begin{equation*}
C: \Sigma \rightarrow \mathcal{E}_{\partial \xi}^{\mathbb{C}}, \quad C(z)=f_{0}\left(\delta_{z}\right) \tag{12.65}
\end{equation*}
$$

The map $f_{0}$ can be recovered from the map $C$ by

$$
\begin{equation*}
f_{0}: \sum_{i} n_{i} \delta_{z_{i}} \mapsto \sum_{i} n_{i} C\left(z_{i}\right) \tag{12.66}
\end{equation*}
$$

The map $f_{1}: \mathcal{Q}(\Sigma)_{1} \rightarrow\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{1}$ is the derivative of $C$,

$$
\begin{equation*}
f_{1}=C_{*}: \mathcal{D}_{1}^{\mathrm{int}}(\Sigma) \oplus i \mathcal{D}_{1}^{\mathrm{int}}(\Sigma) \rightarrow \mathcal{D}_{n}^{\mathrm{int}}(M) \oplus i \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{12.67}
\end{equation*}
$$

It preserves the almost-complex structures $J$ and the skew-hermitian forms on currents,

$$
\begin{gather*}
C_{*} J=J C_{*} \quad \text { on } \mathcal{D}_{1}(\Sigma),  \tag{12.68}\\
\Pi^{*} I_{M}\left\langle\bar{C}_{*} \eta_{1}, C_{*} \eta_{2}\right\rangle=I_{\Sigma}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle . \tag{12.69}
\end{gather*}
$$

A pseudo-holomorphic curve [7] is a map from a Riemann surface $\Sigma$ to an almost complex space that preserves the almost complex structures. So a quasi holomorphic curve is a pseudo-holomorphic curve in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ that preserves the skew-hermitian forms on currents in the middle dimension.

For $z$ a local complex coordinate on $\Sigma$, condition (12.68) takes the form

$$
\begin{equation*}
J \partial_{z} C=i \partial_{z} C, \quad J \partial_{\bar{z}} C=-i \partial_{\bar{z}} C . \tag{12.70}
\end{equation*}
$$

For $n$ odd, $J=\epsilon_{n} *$ is real, so it is consistent to impose the reality condition $\bar{C}=C$. Then $C: \Sigma \rightarrow \mathcal{E}_{\partial \xi}$ and we can forgo complexifying $\mathcal{E}_{\partial \xi}$. For $n$ even, $C$ must be complex.

A q-h curve $C$ sweeps out an integral $(n+1)$-current $\Pi_{*} C_{*} \Sigma$ in the space-time $M$. An $(n+1)$-current intersects an $(n-1)$-current at a 0 -current. A small local q-h curve will sweep out a small $(n+1)$-current in $M$, so a small local q-h curve sees the local structure of an extended object in the form of a 0 -current.

## 13 2d CFT on a quasi-holomorphic curve

Suppose $C$ is a quasi-holomorphic curve in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The 0 -form and 1 -form fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ pull back to give 0 -form and 1 -form fields on the Riemann surface $\Sigma$,

$$
\begin{align*}
C^{*} \phi_{ \pm}\left(\eta_{0}\right) & =\phi_{ \pm}\left(C_{*} \eta_{0}\right), & \eta_{0} \in \mathcal{D}_{0}(\Sigma)  \tag{13.1}\\
C^{*} j_{ \pm}\left(\eta_{1}\right) & =j_{ \pm}\left(C_{*} \eta_{1}\right), & \eta_{1} \in \mathcal{D}_{1}(\Sigma) . \tag{13.2}
\end{align*}
$$

The fields on $\Sigma$ satisfy the field equations of the 2d CFT,

$$
\begin{equation*}
d\left(C^{*} \phi_{ \pm}\right)=C^{*} j_{ \pm} . \tag{13.3}
\end{equation*}
$$

The 1-forms on $\Sigma$ are chiral by the quasi-holomorphic condition (12.68),

$$
\begin{equation*}
P_{+}\left(C^{*} j_{+}\right)=C^{*} j_{+}, \quad P_{-}\left(C^{*} j_{-}\right)=C^{*} j_{-} . \tag{13.4}
\end{equation*}
$$

Translated into the usual language of 2d CFT,

$$
\begin{align*}
\phi_{+}(z) & =C^{*} \phi_{+}\left(\delta_{z}\right), & \phi_{-}(\bar{z}) & =C^{*} \phi_{-}\left(\delta_{z}\right),  \tag{13.5}\\
j_{+}(z) & =\left(C^{*} j_{+}\right)^{z}(z), & j_{-}(\bar{z}) & =\left(C^{*} j_{-}\right)^{\bar{z}}(\bar{z}),  \tag{13.6}\\
\partial \phi_{+} & =j_{+}(z), & \partial \phi_{-} & =0,  \tag{13.7}\\
\bar{\partial} \phi_{+} & =0, & \bar{\partial} \phi_{-} & =j_{-}(z) . \tag{13.8}
\end{align*}
$$

We now have the classical fields of the 2d CFT on $\Sigma$.
To have the 2d quantum field theory on $\Sigma$, we need the correlation functions to satisfy the Schwinger-Dyson equations of the 2d CFT,

$$
\begin{align*}
& \left\langle\left(C^{*} j_{\bar{\alpha}}\right)^{\dagger}\left(\bar{\eta}_{1}\right) C^{*} j_{\beta}\left(\partial \eta_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{\Sigma}\left\langle{\left.\overline{\partial \eta_{1}}, \eta_{2}\right\rangle}_{\left\langle\left(C^{*} \phi_{\bar{\alpha}}\right)^{\dagger}\left(\bar{\eta}_{0}\right) C^{*} j_{\beta}\left(\partial \eta_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{\Sigma}\left\langle\bar{\eta}_{0}, \eta_{2}\right\rangle,} .\right. \tag{13.9}
\end{align*}
$$

which translates to

$$
\begin{equation*}
\bar{\partial}\left\langle\phi_{+}^{\dagger}(z) j_{+}(w)\right\rangle=2 \pi \delta^{2}(z-w), \quad \text { etc. } \tag{13.11}
\end{equation*}
$$

The second quasi-holomorphic condition (12.69) - that $C$ preserves the skew-hermitian forms - implies that the 2d S-D equations (13.9 13.10) follow from the S-D equations (10.5-10.6) on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. So we have the 2d CFT on the quasi-holomorphic curve.

The vertex operators $V(\xi)$ on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are exponentials of the $\phi_{ \pm}(\xi)$. They pull back to $\Sigma$ to the corresponding exponentials of $C^{*} \phi_{ \pm}$,

$$
\begin{equation*}
C^{*} V(\eta)=V\left(C_{*} \eta\right), \quad \eta \in \mathcal{D}_{0}^{\mathrm{int}}(\Sigma) \tag{13.12}
\end{equation*}
$$

In the usual language 2d CFT, the pulled back vertex operators are the local fields

$$
\begin{equation*}
V(z)=C^{*} V\left(\delta_{z}\right)=V(C(z)) \tag{13.13}
\end{equation*}
$$

The correlation functions of the 2-d fields,

$$
\begin{equation*}
\left\langle C^{*} V_{1}\left(\eta_{1}\right) \cdots C^{*} j_{ \pm}\left(\eta_{1}^{\prime}\right) \cdots\right\rangle=\left\langle V_{1}\left(C_{*} \eta_{1}\right) \cdots j_{ \pm}\left(C_{*} \eta_{1}^{\prime}\right) \cdots\right\rangle \tag{13.14}
\end{equation*}
$$

are the correlation functions of the 2 d CFT on $\Sigma$, since the local properties of the correlation functions are completely determined by the S-D equations.

For $n$ odd, the q-h curve $C$ can be taken real, a map $\Sigma \rightarrow \mathcal{E}_{\partial \xi}$. The real $n$-form field $F(x)$ on $M$ becomes the real 1-form $j$ on $\mathcal{E}_{\partial \xi}$. The 1-form field $C^{*} j$ on $\Sigma$ is then real. The 2 d CFT is the free theory of a real 1 -form. The 2 d CFT on $\Sigma$ is exactly what was called the analog theory on $\mathcal{E}_{\partial \xi}$. For $n$ even, the situation is more complicated. Even if $F(x)$ is a real $n$-form, $C^{*} j$ must be a complex 1 -form, because the q-h curve $C$ is necessarily complex. The fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ pulled back to $\Sigma$ comprise a subalgebra of the 2 d CFT of the free complex 1 -form. Consider the fields of the real $n$-form theory as a subalgebra in the complex $n$-form theory - the subalgebra generated by the vertex operators invariant under the complex conjugation symmetry $F \leftrightarrow \bar{F}$. On $\Sigma$, this becomes a different subalgebra of the 2d CFT of the free complex 1-form $C^{*} j$, the subalgebra generated by the vertex operators invariant under the combination of complex conjugating and reversing orientation $J \rightarrow-J, P_{+} \leftrightarrow P_{-}$. More detail is given in Appendix D.

## 14 A wishful conjecture

The project is to construct the CFT of extended objects in space-time from the 2d CFT. For every 2d CFT, there is to be a CFT of extended objects which, when pulled back to a quasi-holomorphic curve, gives the 2d CFT on the Riemann surface. Some theories of extended objects - for example, orbifolds of free $n$-form theories - might be constructed directly in terms of the skew-hermitian form on currents and the $J$-operator, so that their pull-backs to $q$-h curves are manifestly the corresponding 2 d theories. For the general case, however, a method is needed to construct the theory of extended objects from the 2d CFT. The data of the 2d CFT on the quasi-holomorphic curves must be enough to construct the correlation functions of the theory of extended objects.

The only way I can imagine realizing this project is by means of isomorphisms between the quasi Riemann surfaces $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ and the quasi Riemann surface $\mathcal{Q}(\Sigma)$. The fields and correlation functions on $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ will be constructed simply by pulling back the fields and correlation functions of the 2d theory on the Riemann surface $\Sigma$.

The only reason I can imagine for such isomorphisms to exist is if any two quasi Riemann surfaces with the same homology data are isomorphic. The conjecture is

The isomorphism classes of quasi Riemann surfaces are classified by their homology data.

Each isomorphism class contains a unique two-dimensional model. For each possible set of homology data there is a unique two-dimensional conformal space $\Sigma$ with that homology data, so that $\mathcal{Q}(\Sigma)$ belongs to the isomorphism class. The two-dimensional conformal space $\Sigma$ is an ordinary Riemann surface when the homology data allows.

I have no idea what the two-dimensional conformal space $\Sigma$ might be when the homology data is not that of a Riemann surface, much less how to construction the correlation functions of a 2d CFT on such a space. Prudence suggests limiting to space-times $M$ with homology data such that $\Sigma$ can be a Riemann surface. Actually, I will be more than delighted if the conjecture can be shown to hold for trivial homology data, so that it will apply to the basic case, $M=S^{d}, \Sigma=S^{2}$.

I do not know how to prove the conjecture. There might be a route via the local quasi-holomorphic curves. It might be supposed that

Every local quasi-holomorphic curve in $\mathcal{Q}(\Sigma)$ is given by a local neigborhood in $\Sigma$, up to automorphisms of $\mathcal{Q}(\Sigma)$.
A local quasi-holomorphic curve in a quasi Riemann surface $\mathcal{Q}$ is a rigid object in the sense that it has a unique "analytic" continuation to an isomorphism $\mathcal{Q}(\Sigma) \rightarrow \mathcal{Q}$ for some two-dimensional space $\Sigma$.

If the conjecture is true, then the group $\operatorname{Aut}(\mathcal{Q}(\Sigma))$ will be a very interesting object. It will naturally contain the group $\operatorname{Conf}(\Sigma)$ of conformal symmetries of $\Sigma$. It will be isomorphic to the groups $\operatorname{Aut}\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)$ for every conformal manifold $M$ with the same middle homology data as $\Sigma$, and every integral ( $n-1$ )-boundary $\partial \xi$ in $M$, so it will also contain the groups $\operatorname{Conf}(M, \mathbb{Z} \partial \xi)$ of conformal symmetries of $M$ that preserve $\pm \partial \xi$, for every such $M$ and $\partial \xi$.

## 15 Correlation functions from extended 2d CFT

Assuming the conjecture is true, the geometric isomorphism between each of the $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ and $\mathcal{Q}(\Sigma)$ can be used to construct a CFT on each of the $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ from any 2 d CFT on $\Sigma$. The observables on $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ will be the pull-backs under an isomorphism $f: \mathcal{Q}_{\mathbb{Z} \partial \xi} \rightarrow \mathcal{Q}(\Sigma)$,

$$
\begin{equation*}
f^{*} \Phi(\xi)=\Phi(f \xi) \tag{15.1}
\end{equation*}
$$

of the 2 d observables $\Phi(\eta)$ on $\mathcal{Q}(\Sigma)$. The correlation functions on $\mathcal{Q}_{\mathbb{Z} \partial \xi}$ will be given by the correlation functions on $\Sigma$,

$$
\begin{equation*}
\left\langle f^{*} \Phi(\xi) \cdots\right\rangle=\langle\Phi(f \xi) \cdots\rangle_{\Sigma} \tag{15.2}
\end{equation*}
$$

For this to work, the 2 d CFT on $\Sigma$ needs to be extended so that
The extended 2d observables $\Phi(\eta)$ are defined on the integral currents $\eta \in$ $\mathcal{Q}(\Sigma)_{k}^{\mathrm{int}}$. The automorphism group $\operatorname{Aut}(\mathcal{Q}(\Sigma))$ acts on the vector space of extended observables. The correlation functions of extended observables are invariant under $\operatorname{Aut}(\mathcal{Q}(\Sigma))$.

In ordinary 2d CFT, the observables are products of local fields over finite sets of distinct points,

$$
\begin{equation*}
\Phi\left(z_{1}, \ldots, z_{N}\right)=\varphi_{1}\left(z_{1}\right) \cdots \varphi_{n}\left(z_{N}\right) \tag{15.3}
\end{equation*}
$$

and linear combinations of such products. We can regard such an observable as living on the 0 -current $\eta=\sum_{i} \delta_{z_{i}}$. The problem is to extend observables to integral currents.

The vertex operators of the 2d gaussian model extend formally - classically - to the integral 0 -currents since the fields $\phi$ and $\phi^{*}$ are 0 -forms,

$$
\begin{equation*}
V_{p, p^{*}}(\eta)=e^{i p \phi(\eta)+i p^{*} \phi^{*}(\eta)} \tag{15.4}
\end{equation*}
$$

The regularization and renormalization of such extended vertex operators is still to be dealt with. From the start, I have been making an unspoken assumption, as a guiding hypothesis, that the vertex operators of the free $n$-form CFT can be constructed on the integral ( $n-1$ )-currents in the space-time $M$, as quantum fields. I have not actually carried out this construction. Now the proposal is to construct the extended vertex operators as observables in the extended 2d CFT, then transport them to the space-time $M$ via the conjectured isomorphism of quasi Riemann surfaces.

The construction of an extended 2d CFT from an ordinary 2d CFT is one of a number of further steps towards realizing the project that are listed in section 21 below.

## 16 Perturbation theory

The plan is to move, eventually, from conformal field theories of extended objects to non-conformal quantum field theories of extended objects, constructed from extended 2 d non-conformal quantum field theories. As a step in that direction, I consider perturbing a 2d CFT and constructing the corresponding perturbation of the CFT of extended objects in space-time. The discussion is formal. Nothing is said about a relation between the 2d cutoff scale and the space-time cutoff scale.

A perturbation of a 2 d CFT on a Riemann surface is given by integrating a quantum field that is a (1,1)-form on the Riemann surface. Such perturbations arise when the 2 d CFT depends on parameters, such as the parameter $R$ of the 2 d gaussian model, and also when the the 2d CFT can be perturbed to give a non-conformal 2d QFT. In the latter case, the integral will depend on the 2 d metric. It will have to be cut off and renormalized. The correspondence between variations of the 2d QFT and integrals of (1,1)-form fields is a manifestation of the action principle.

### 16.1 Varying the parameter $R$ of the gaussian model

The parameter $R$ in the theory of a free $n$-form $F$ on $M$ is varied by inserting in correlation functions the integral

$$
\begin{equation*}
\int_{M} * F \wedge F \tag{16.1}
\end{equation*}
$$

The $n$-form $F$ pulls up to the 1 -form $j=\Pi^{*} F$ on $\mathcal{E}_{\mathcal{\partial} \xi}^{\mathbb{C}}$. Suppose $\Sigma$ is a compact Riemann surface without boundary, and $C$ is a quasi-holomorphic $\Sigma$-curve. Then $C^{*} j$ is a free 1 -form on $\Sigma$. On the Riemann surface $\Sigma$, we have the 2 d gaussian model with parameter $R$. The parameter $R$ in the 2 d gaussian model is varied by inserting in the correlation
functions on $\Sigma$ the integral

$$
\begin{equation*}
\int_{\Sigma} *\left(C^{*} j\right) \wedge\left(C^{*} j\right)=\int_{\Sigma} C^{*}(* j \wedge j) \tag{16.2}
\end{equation*}
$$

which is the integral of the $(1,1)$-form $* j \wedge j$ pulled back from $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ to $\Sigma$.
The insertions (16.1) and (16.2) must give the same result. They can be written

$$
\begin{equation*}
\int_{M} * F \wedge F=I_{M}^{-1}\langle\overline{J F}, F\rangle, \quad \int_{\Sigma} *\left(C^{*} j\right) \wedge\left(C^{*} j\right)=I_{\Sigma}^{-1}\left\langle\overline{J C^{*}} j, C^{*} j\right\rangle \tag{16.3}
\end{equation*}
$$

where $I_{M}^{-1}\left\langle\bar{\omega}_{1}, \omega_{2}\right\rangle$ is the skew-hermitian bilinear form on $n$-forms that is the inverse of the skew-hermitian bilinear form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$ on $n$-currents in $M$, and $I_{\Sigma}^{-1}\left\langle\bar{\omega}_{1}, \omega_{2}\right\rangle$ is the inverse of $I_{\Sigma}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ on 1-currents in $\Sigma$.

The identity between the two integrals (16.1) and (16.2) for the variation of $R$ is now

$$
\begin{equation*}
I_{M}^{-1}\langle\overline{J F}, F\rangle=I_{\Sigma}^{-1}\left\langle\overline{J C^{*} \Pi^{*} F}, C^{*} \Pi^{*} F\right\rangle=\Pi_{*} C_{*}\left(I_{\Sigma}^{-1}\right)\langle\overline{J F}, F\rangle \tag{16.4}
\end{equation*}
$$

which will hold if and only if

$$
\begin{equation*}
I_{M}^{-1}=\Pi_{*} C_{*}\left(I_{\Sigma}^{-1}\right) \tag{16.5}
\end{equation*}
$$

Combined with the quasi-holomorphic condition on $C$,

$$
\begin{equation*}
I_{\Sigma}=C^{*} \Pi^{*} I_{M} \tag{16.6}
\end{equation*}
$$

equation (16.5) is equivalent to the unitarity of

$$
\begin{equation*}
\Pi_{*} C_{*}: \mathcal{D}_{1}(\Sigma) \rightarrow \mathcal{D}_{n}(M) . \tag{16.7}
\end{equation*}
$$

$\Pi_{*} C_{*}$ is unitary iff $C$ is an isomorphism of quasi Riemann surfaces. Therefore, if the conjecture holds, then the variation of $R$ in the 2 d CFT on $\Sigma$ is equivalent to the variation of $R$ in the $n$-form theory on $M$.

This argument for the unitarity of $\Pi_{*} C_{*}$ is based on the structure of the $n$-form CFT on $M$ and the 1 -form 2 d CFT on $\Sigma$. These are both free quantum field theories, so it should be possible to translate the argument into purely mathematical terms, amounting to a proof that, for $\Sigma$ a compact Riemann surface without boundary, any morphism of quasi Riemann surfaces $\mathcal{Q}(\Sigma) \rightarrow \mathcal{Q}_{\mathbb{Z} a \xi}$ must be an isomorphism.

### 16.2 Perturbing by a general (1, 1)-form

Suppose that the conjecture holds. Then perturbing the 2 d QFT on $\Sigma$ is equivalent to perturbing the QFT of extended objects in $M$. The 2d perturbation is given by integrating a ( 1,1 )-form over $\Sigma$. The perturbation of the QFT in $M$ should be given by integrating an $(n, n)$-form over $M$.

Suppose $f: \mathcal{Q}(\Sigma) \rightarrow \mathcal{Q}_{\mathbb{Z} \partial \xi}$ is an isomorphism. Suppose $\Phi_{\Sigma}$ is a $(1,1)$-form on $\Sigma$. Represent $\Phi_{\Sigma}$ as

$$
\begin{equation*}
\Phi_{\Sigma}=\sum_{a, b} c_{a, b} \overline{J w}_{a} \wedge w_{b} \tag{16.8}
\end{equation*}
$$

for some collection of 1 -forms $w_{a}$ on $\Sigma$ and some constants $c_{a, b}$. Then

$$
\begin{equation*}
\int_{\Sigma} \Phi_{\Sigma}=\sum_{a, b} c_{a, b} I_{\Sigma}^{-1}\left\langle\bar{w}_{a}, w_{b}\right\rangle \tag{16.9}
\end{equation*}
$$

The linear operator

$$
\begin{equation*}
f_{1}: \mathcal{Q}(\Sigma)_{1} \rightarrow\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{1}, \quad \mathcal{Q}(\Sigma)_{1}=\mathcal{D}_{1}(\Sigma), \quad\left(\mathcal{Q}_{\mathbb{Z} \partial \xi}\right)_{1}=\mathcal{D}_{n}(M) \tag{16.10}
\end{equation*}
$$

is invertible, so we can construct the $n$-forms $W_{a}$ on $M$,

$$
\begin{equation*}
W_{a}=\left(f_{1}^{*}\right)^{-1} w_{a} \tag{16.11}
\end{equation*}
$$

From the $W_{a}$ make a $(n, n)$-form on $M$,

$$
\begin{equation*}
\Phi_{M}=\sum_{a, b} c_{a, b} \overline{J W}_{a} \wedge W_{b} \tag{16.12}
\end{equation*}
$$

Then

$$
\begin{equation*}
\int_{M} \Phi_{M}=\sum_{a, b} c_{a, b} I_{M}^{-1}\left\langle\bar{W}_{a}, W_{b}\right\rangle=\sum_{a, b} c_{a, b} I_{\Sigma}^{-1}\left\langle\bar{w}_{a}, w_{b}\right\rangle=\int_{\Sigma} \Phi_{\Sigma} \tag{16.13}
\end{equation*}
$$

So the perturbation by the (1,1)-form $\Phi_{\Sigma}$ on $\Sigma$ and the perturbation by the $(n, n)$-form $\Phi_{M}$ on $M$ will have the same effect.

The construction of $\Phi_{M}$ from $\Phi_{\Sigma}$ is ambiguous. $\Phi_{M}$ is only determined up to an $(n, n)$-form that integrates to zero on $M$. Such $(n, n)$-forms are the redundant fields the perturbations that only reparametrize the fields of the QFT, taking the QFT to an equivalent QFT. Thus the 2d perturbation $\Phi_{\Sigma}$ maps to a class of equivalent perturbations $\Phi_{M}$ of the space-time QFT.

## 17 Gauge symmetry of the classical free $n$-form

This section and the next go back to the classical field theory of a free $n$-form in the space-time $M$ regarded as a collection of free 1-form theories on the fibers of the bundle $\mathcal{E} \rightarrow \mathcal{B}$ of integral ( $n-1$ )-currents in $M$ over the integral ( $n-2$ )-boundaries. This section expresses the space-time gauge symmetry of the theory in terms of a gauge symmetry over $\mathcal{B}$. The gauge symmetry group is the global symmetry group of the analog 2 d theory. One message is that generalization from the 2d gaussian model to 2 d quantum field theories with nonabelian global symmetry groups might yield space-time theories with nonabelian gauge symmetry. In section 19 below, I will try to translate the structure described here for the classical theory to the quantum theory on the bundle of quasi Riemann surfaces.

The $n$-form theory has local $U(1) \times U(1)$ gauge symmetries

$$
\begin{equation*}
A \mapsto A+d f, \quad A^{*} \mapsto A^{*}+d f^{*} \tag{17.1}
\end{equation*}
$$

given by pairs $f, f^{*}$ of $(n-2)$-forms on the space-time $M$. The 1 -form theory on each $\mathcal{E}_{\partial \xi}$ has a global $U(1) \times U(1)$ symmetry. We have seen that local gauge transformations on $M$ give global symmetry transformations in each $\mathcal{E}_{\partial \xi}$. If we are to build a space-time QFT from 2d QFT on the $\mathcal{E}_{\partial \xi}$, we will need to go in the opposite direction, constructing the local gauge symmetries in space-time from the global symmetries in the $\mathcal{E}_{\partial \xi}$.

### 17.1 The local gauge transformations over $\mathcal{B}$

In the 2 d gaussian model on a Riemann surface $\Sigma$, write $T_{\Sigma}$ for the space of solutions $\phi$, $\phi^{*}$ of

$$
\begin{equation*}
j=d \phi, \quad j^{*}=d \phi^{*} . \tag{17.2}
\end{equation*}
$$

Then $T_{\Sigma}=S^{1} \times S^{1}$. The global symmetry group $G=U(1) \times U(1)$ of the 2 d gaussian model acts by

$$
\begin{equation*}
\phi \mapsto \phi+g, \quad \phi^{*} \mapsto \phi^{*}+g^{*}, \tag{17.3}
\end{equation*}
$$

so $G$ acts on $T_{\Sigma}$ as a principal homogeneous space.
Think of $T_{\Sigma}$ as the space of equivalent 2 d theories built from the 1 -forms $j, j^{*}$. The local fields, such as the vertex operators $V_{p, p^{*}}=e^{i p \phi+i p^{*} \phi^{*}}$, transform in representations of $G$ which can be represented as functions on $T_{\Sigma}$. We can think of the action of $G$ on the local fields as factoring through its action on $T_{\Sigma}$.

On each non-special fiber $\mathcal{E}_{\partial \xi}$ of $\mathcal{E} \rightarrow \mathcal{B}$ there is space of theories all isomorphic to the (formal analog of the) 2 d gaussian model. On a given $\mathcal{E}_{\partial \xi}$, the space of isomorphic theories is the space of solutions $\phi(\xi)$ and $\phi^{*}(\xi)$ on $\mathcal{E}_{\partial \xi}$ - the space of integration constants for $\phi$ and $\phi^{*}$ on $\mathcal{E}_{\partial \xi}$. Write $\mathcal{T}_{\partial \xi}$ for the space of isomorphic theories on $\mathcal{E}_{\partial \xi}$. The space $\mathcal{T}_{0}$ of theories on the special fiber $\mathcal{E}_{0}$ is a single point. Collectively, the isomorphism classes of theories form a fiber bundle $\mathcal{T} \rightarrow \mathcal{B}$.

The isomorphism class of theories $\mathcal{T}_{\partial \xi}$ has symmetry group $\mathcal{G}_{\partial \xi}$ which is isomorphic, for $\partial \xi \neq 0$, to the global symmetry group $G=U(1) \times U(1)$. The group $\mathcal{G}_{\partial \xi}$ acts on the isomorphism class $\mathcal{T}_{\partial \xi}$ by

$$
\begin{equation*}
\phi(\xi) \mapsto \phi(\xi)+g(\partial \xi), \quad \phi^{*}(\xi) \mapsto \phi^{*}(\xi)+g^{*}(\partial \xi) \tag{17.4}
\end{equation*}
$$

The symmetry group $\mathcal{G}_{0}$ of the special fiber is the trivial group. Note that $g(\partial \xi)$ and $g^{*}(\partial \xi)$ are 0 -forms (functions) on $\mathcal{B}$. They are not necessarily additive functions on the abelian group $\mathcal{B}$, which is to say that they do not necessarily come from $(n-2)$-forms on $M$.

The isomorphism class $\mathcal{T}_{\partial \xi}$ is a principal homogeneous space for the group $\mathcal{G}_{\partial \xi}$. Collectively, the symmetry groups $\mathcal{G}_{\partial \xi}$ form a fiber bundle of groups $\mathcal{G} \rightarrow \mathcal{B}$. The sections of $\mathcal{G} \rightarrow \mathcal{B}$ are the local gauge transformations. This construction of local gauge symmetry over $\mathcal{B}$ would make sense for any 2 d CFT (or QFT), with any global 2 d symmetry group $G$, abelian or nonabelian.

Let $\mathcal{F}_{\partial \xi}=\mathbf{I s o}\left(T_{\Sigma}, \mathcal{T}_{\partial \xi}\right)$ be the space of equivalences, taking the scalars $\phi, \phi^{*}$ on $\Sigma$ to the scalars $\phi, \phi^{*}$ on $\mathcal{E}_{\partial \xi}$. The space $\mathcal{F}_{\partial \xi}$ is a principal homogeneous space both for $G$ acting on the right, and for $\mathcal{G}_{\partial \xi}$ acting on the left. Collectively, the $\mathcal{F}_{\partial \xi}$ form the fiber bundle of
frames $\mathcal{F} \rightarrow \mathcal{B}$, a principal fiber bundle with structure group $G$. If $V$ is a representation of $G$, the fields of charge $V$ on the $\mathcal{E}_{\partial \xi}$ live in the vector bundle $\mathcal{F} \times{ }_{G} V \rightarrow \mathcal{B}$.

### 17.2 The reconstruction of the space-time gauge potentials

The group of gauge transformations of the 1-form theory over $\mathcal{B}$ is much bigger than the group of gauge transformations of the $n$-form theory in space-time. I will attempt to argue now that there exists a partial gauge fixing in the 1 -form theory over $\mathcal{B}$ that reduces the large gauge group over $\mathcal{B}$ to the smaller space-time gauge group. The argument seems to depend crucially on the global symmetry group $G=U(1) \times U(1)$ being abelian, in concert with the abelian group structure of $\mathcal{B}$ and of $\mathcal{E}$. Also, the argument is only classical. So the argument very well might not have any general significance. For 2d theories with nonabelian global symmetry groups $G$, there will be the nonabelian local gauge symmetry over $\mathcal{B}$, but perhaps without any reduction to ordinary local nonabelian gauge symmetry in space-time.

Suppose $\phi(\xi), \phi^{*}(\xi)$ are solutions of $d \phi=j, d \phi^{*}=j^{*}$ in each fiber. The goal is to make a gauge transformation

$$
\begin{equation*}
\tilde{\phi}(\xi)=\phi(\xi)+\tilde{g}(\partial \xi), \quad \tilde{\phi}^{*}(\xi)=\phi^{*}(\xi)+\tilde{g}^{*}(\partial \xi) \tag{17.5}
\end{equation*}
$$

so that $\tilde{\phi}$ and $\tilde{\phi}^{*}$ are additive,

$$
\begin{equation*}
\tilde{\phi}\left(\xi_{1}\right)+\tilde{\phi}\left(\xi_{2}\right)-\tilde{\phi}\left(\xi_{1}+\xi_{2}\right)=0, \quad \tilde{\phi}^{*}\left(\xi_{1}\right)+\tilde{\phi}^{*}\left(\xi_{2}\right)-\tilde{\phi}^{*}\left(\xi_{1}+\xi_{2}\right)=0 \tag{17.6}
\end{equation*}
$$

Then $\tilde{\phi}(\xi)$ and $\tilde{\phi}^{*}(\xi)$ will be determined by their values on infinitesimal $\xi$, so will correspond to ( $n-1$ )-forms $A, A^{*}$ on space-time. The remaining gauge invariance will be given by additive functions $g(\partial \xi), g^{*}(\partial \xi)$, corresponding to $(n-2)$-forms $A, A^{*}$ on space-time, which is the space-time gauge invariance of the $n$-form theory.

Define

$$
\begin{align*}
C\left(\xi_{1}, \xi_{2}\right) & =\phi\left(\xi_{1}\right)+\phi\left(\xi_{2}\right)-\phi\left(\xi_{1}+\xi_{2}\right)  \tag{17.7}\\
C^{*}\left(\xi_{1}, \xi_{2}\right) & =\phi^{*}\left(\xi_{1}\right)+\phi^{*}\left(\xi_{2}\right)-\phi^{*}\left(\xi_{1}+\xi_{2}\right) \tag{17.8}
\end{align*}
$$

$C\left(\xi_{1}, \xi_{2}\right)$ and $C^{*}\left(\xi_{1}, \xi_{2}\right)$ actually depend only on $\partial \xi_{1}$ and $\partial \xi_{2}$, because, if $\partial \xi_{1}^{\prime}=\partial \xi_{1}$, then

$$
\begin{align*}
\tilde{C}\left(\xi_{1}^{\prime}, \xi_{2}\right)-\tilde{C}\left(\xi_{1}, \xi_{2}\right) & =\left[\phi\left(\xi_{1}^{\prime}\right)-\phi\left(\xi_{1}\right)\right]-\left[\phi\left(\xi_{1}^{\prime}+\xi_{2}\right)-\phi\left(\xi_{1}+\xi_{2}\right)\right]  \tag{17.9}\\
& =j\left(\partial\left(\xi_{1}^{\prime}-\xi_{1}\right)\right)-j\left(\partial\left(\xi_{1}^{\prime}+\xi_{2}-\xi_{1}-\xi_{2}\right)\right)  \tag{17.10}\\
& =0 \tag{17.11}
\end{align*}
$$

and similarly for $\partial \xi_{2}^{\prime}=\partial \xi_{2}$ and for $C^{*}$ in place of $C$. So we can write

$$
\begin{equation*}
C\left(\xi_{1}, \xi_{2}\right)=c\left(\partial \xi_{1}, \partial \xi_{2}\right), \quad C^{*}\left(\xi_{1}, \xi_{2}\right)=c^{*}\left(\partial \xi_{1}, \partial \xi_{2}\right) \tag{17.12}
\end{equation*}
$$

To get the additive condition (17.6), we need the gauge transformation (17.5) to solve

$$
\begin{align*}
c\left(\partial \xi_{1}, \partial \xi_{2}\right) & =\tilde{g}\left(\partial \xi_{1}+\partial \xi_{2}\right)-\tilde{g}\left(\partial \xi_{2}\right)-\tilde{g}\left(\partial \xi_{2}\right)  \tag{17.13}\\
c^{*}\left(\partial \xi_{1}, \partial \xi_{2}\right) & =\tilde{g}^{*}\left(\partial \xi_{1}+\partial \xi_{2}\right)-\tilde{g}^{*}\left(\partial \xi_{2}\right)-\tilde{g}^{*}\left(\partial \xi_{2}\right) \tag{17.14}
\end{align*}
$$

for $\tilde{g}$ and $\tilde{g}^{*}$. This is a problem in group cohomology. The pair $c, c^{*}$ is a 2 -cochain on the abelian group $\mathcal{B}$ with coefficients in the group $G=U(1) \times U(1)$. Equations (17.13-17.14) are solvable iff $c, c^{*}$ is trivial in the group cohomology. But $c\left(\partial \xi_{1}, \partial \xi_{2}\right)$ and $c^{*}\left(\partial \xi_{1}, \partial \xi_{2}\right)$ are symmetric in their arguments, so (17.13-17.14) can be solved if the symmetric 2-cocycles are trivial in the group cohomology of $\mathcal{B}$. I believe that this is the case, because $\mathcal{B}$ and $G$ are abelian.

## 18 Connecting the $\mathcal{E}_{\partial \xi}$

The extended objects in the free $n$-form theory are to be described by fields on $\mathcal{E}$. But the analog 2 d theories on the fibers of $\mathcal{E} \rightarrow \mathcal{B}$ would seem to give only correlation functions of fields on the same fiber $\mathcal{E}_{\partial \xi}$. What about correlation functions between fields on different fibers $\mathcal{E}_{\partial \xi}$ ? Such correlation functions will be non-zero only on gauge invariant observables on the individual fibers - only on observables invariant under the global symmetry of the fiber. So we need to understand the correlation functions of products of invariant observables on different fibers.

Here I argue that the invariant observables are the same on every fiber, so correlation functions of products of invariant observables should be calculable in any one fiber. The argument is specific to the free $n$-form theory and is only classical. The argument uses a natural connection in the bundle $\mathcal{E} \rightarrow \mathcal{B}$, described in the next section. My hope is that there might be a parallel argument in the general quantum case, using some analogous natural geometric structure in the bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ of quasi Riemann surfaces.

### 18.1 The natural connection in $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M)$

Recall that $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M)$ is a principal fiber bundle for the additive abelian group $\mathcal{D}_{k}^{\text {int }}(M)_{0}$ of integral $k$-cycles. A connection in $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M)$ is a collection of linear maps lifting tangent vectors $v_{0}$ in the base at $\partial \xi$ to tangent vectors $v$ in the total space at $\xi$, one such linear map for every point $\xi$ in the total space. These linear maps must be compatible with the bundle structure, i.e., must satisfy $\partial_{*} v=v_{0}$ and must be invariant under translations of $\xi$ in the fiber over $\partial \xi$.

A tangent vector in the base $\partial \mathcal{D}_{k}^{\text {int }}(M)$ is a perturbation by an infinitesimal element $v_{0}$ in $\partial \mathcal{D}_{k}^{\text {int }}(M)$. We have seen that there is a uniquely determined minimal infinitesimal element $v \in \mathcal{D}_{k}^{\text {int }}(M)$ satisfying $\partial v=v_{0}$. This $v$ is the lift of $v_{0}$.

Equivalently, $v_{0}$ corresponds to an infinitesimal integral 1-current $\eta_{0}$ in the base at $\partial \xi$ by

$$
\begin{equation*}
\Pi_{*}^{0, k-1} \partial \eta_{0}=v_{0} . \tag{18.1}
\end{equation*}
$$

The pushdown of $\eta_{0}$,

$$
\begin{equation*}
v=\Pi_{*}^{1, k-1} \eta_{0} \in \mathcal{D}_{k}^{\mathrm{int}}(M) \tag{18.2}
\end{equation*}
$$

corresponds to an infinitesimal integral 1-current $\eta$ in the total space $\mathcal{D}_{k}^{\text {int }}(M)$ at $\xi$, by

$$
\begin{equation*}
\Pi_{*}^{0, k} \partial \eta=v \tag{18.3}
\end{equation*}
$$

with $\eta$ a lift of $\eta_{0}$,

$$
\begin{equation*}
\partial_{*} \eta=\eta_{0} . \tag{18.4}
\end{equation*}
$$

The lift $v_{0} \mapsto v$, or $\eta_{0} \mapsto \eta$, is translation invariant in the fiber, since the construction of the lift makes no mention of $\xi$. In fact, the construction is translation invariant in the total space $\mathcal{D}_{k}^{\text {int }}(M)$ of the bundle. The lift $v_{0} \mapsto v$ thus is a natural, translation invariant connection in the principal fiber bundle $\mathcal{D}_{k}^{\text {int }}(M) \xrightarrow{\partial} \partial \mathcal{D}_{k}^{\text {int }}(M)$, a natural splitting of the tangent space

$$
\begin{equation*}
T_{\xi} \mathcal{D}_{k}^{\mathrm{int}}(M)=\mathcal{V}_{k+1}(M) \oplus \mathcal{V}_{k}(M) \tag{18.5}
\end{equation*}
$$

where the first summand is the vertical subspace and the second summand is the horizontal subspace, the tangent space in the base $\partial \mathcal{D}_{k}(M)$.

Write $D^{\text {nat }}$ for the covariant derivative of the natural connection. The curvature tensor $\left(D^{n a t}\right)^{2}$ is a 2 -form on the base with values in the translation invariant vertical vector fields. To calculate the curvature, consider an infinitesimal 2 -current $\eta_{2}$ in the base at $\partial \xi$. Use the natural connection to lift the 1-current $\eta_{0}=\partial \eta_{2}$ to a 1-current $\eta$ in the total space at $\xi$. Then $\partial \eta$ is an infinitesimal 0 -current in the fiber. This is is the monodromy around $\partial \eta_{2}$, which is the curvature tensor acting on $\eta_{2}$ to give the vertical tangent vector

$$
\begin{equation*}
v_{c}=\Pi_{*}^{0, k} \partial \eta=\Pi_{*}^{1, k-1}\left(\eta_{0}\right)=\Pi_{*}^{1, k-1}\left(\partial \eta_{2}\right)=\partial \Pi_{*}^{2, k-1} \eta_{2} . \tag{18.6}
\end{equation*}
$$

We can write the curvature as

$$
\begin{equation*}
\left(D^{n a t}\right)^{2} \eta_{2}=\Pi_{*}^{2, k-1} \eta_{2} \tag{18.7}
\end{equation*}
$$

which is an infinitesimal element in $\mathcal{D}_{k+1}^{\mathrm{int}}(M)$, whose boundary

$$
\begin{equation*}
v_{c}=\partial \Pi_{*}^{2, k-1} \eta_{2} \tag{18.8}
\end{equation*}
$$

is an infinitesimal vertical perturbation in the fiber.

### 18.2 The connection in the bundle of theories over $\mathcal{B}$

Specializing to $k=n-1$, we have a natural connection in $\mathcal{E} \rightarrow \mathcal{B}$, with covariant derivative $D^{n a t}$. The natural connection in $\mathcal{E} \rightarrow \mathcal{B}$ combines with the 0 -forms $j, j^{*}$ to give a connection in the bundle of theories $\mathcal{T} \rightarrow \mathcal{B}$, as follows. An infinitesimal motion in $\mathcal{B}$, from $\partial \xi$ to $\partial \xi^{\prime}$, lifts to a translation from the fiber $\mathcal{E}_{\partial \xi}$ to the fiber $\mathcal{E}_{\partial \xi^{\prime}}$, taking 0 -currents on the first fiber to 0 -currents on the second, and pulling 0 -forms back from the second to the first. The 1 -forms $j$ and $j^{*}$ are translation invariant, so the natural connection pulls back solutions of $d \phi=j, d \phi^{*}=j^{*}$ on $\mathcal{E}_{\partial \xi^{\prime}}$ to solutions on $\mathcal{E}_{\partial \xi}$. This is the connection in $\mathcal{T} \rightarrow \mathcal{B}$.

Suppose $\phi, \phi^{*}$ is a local section of $\mathcal{T} \rightarrow \mathcal{B}$. That is, $\phi(\xi)$ and $\phi^{*}(\xi)$ are solutions in the fibers over some neighborhood in $\mathcal{B}$, a choice of integration constants in each fiber. The covariant derivative $D^{n a t} \phi(\xi)$, or $D^{n a t} \phi^{*}(\xi)$, is, on each fiber, the difference of two solutions, thus is a constant on each fiber,

$$
\begin{equation*}
D^{n a t} \phi(\xi)=D \phi(\partial \xi), \quad D^{n a t} \phi^{*}(\xi)=D \phi^{*}(\partial \xi) \tag{18.9}
\end{equation*}
$$

giving the covariant derivative $D$ of the connection in $\mathcal{T} \rightarrow \mathcal{B}$.
The curvature tensor of the connection in $\mathcal{T} \rightarrow \mathcal{B}$ is a 2 -form on the base $\mathcal{B}$ with values in the invariant vertical vector fields - the pair of 2 -forms on $\mathcal{B}$

$$
\begin{align*}
\left(D^{n a t}\right)^{2} \phi & =(d \phi)\left(D^{n a t}\right)^{2}=j\left(D^{n a t}\right)^{2}  \tag{18.10}\\
\left(D^{n a t}\right)^{2} \phi^{*} & =\left(d \phi^{*}\right)\left(D^{n a t}\right)^{2}=j^{*}\left(D^{n a t}\right)^{2} . \tag{18.11}
\end{align*}
$$

Here, $d \phi$ and $d \phi^{*}$ are the vertical derivatives in $\mathcal{E}$. When the 1 -forms $j$ and $j^{*}$ on $\mathcal{E}_{\partial \xi}$ come from $n$-forms $F$ and $F^{*}$ on $M$,

$$
\begin{equation*}
j=F \Pi_{*}^{1, n-1}, \quad j^{*}=F^{*} \Pi_{*}^{1, n-1} \tag{18.12}
\end{equation*}
$$

then

$$
\begin{align*}
\left(D^{n a t}\right)^{2} \phi & =F \Pi_{*}^{1, n-1}\left(D^{n a t}\right)^{2}=F \Pi_{*}^{2, n-2}  \tag{18.13}\\
\left(D^{n a t}\right)^{2} \phi^{*} & =F^{*} \Pi_{*}^{1, n-1}\left(D^{n a t}\right)^{2}=F^{*} \Pi_{*}^{2, n-2} \tag{18.14}
\end{align*}
$$

We might think of the bundle $\mathcal{T} \rightarrow \mathcal{B}$ with connection $D$ as an alternate representation of the 1 -form fields $j$ and $j^{*}$, or of the $n$-form fields $F$ and $F^{*}$.

### 18.3 Transport of observables between fibers $\mathcal{E}_{\partial \xi}$

A vertex operator $e^{i p \phi(\xi)+i p^{*} \phi^{*}(\xi)}$ transforms under the symmetry group $\mathcal{G}_{\partial \xi}$ in the onedimensional representation labelled by the charges $p, p^{*}$. Equivalently, the vertex operator belongs to the one-dimensional vector space associated to the space $\mathcal{T}_{\partial \xi}$ of theories on the fiber and the one-dimensional representation of the global symmetry group $G=$ $U(1) \times U(1)$ labelled by $p, p^{*}$. In general, the observables (5.8) on each fiber belong to vector bundles associated to the $G$-bundle which is the fiber product

$$
\begin{equation*}
\mathcal{E} \times_{\mathcal{B}} \mathcal{T} \rightarrow \mathcal{B}, \quad\left(\mathcal{E} \times_{\mathcal{B}} \mathcal{T}\right)_{\partial \xi}=\mathcal{E}_{\partial \xi} \times \mathcal{T}_{\partial \xi} . \tag{18.15}
\end{equation*}
$$

The natural connection in $\mathcal{E}$ combines with the connection in $\mathcal{T}$ to give a connection in the fiber product, which determines a parallel transport of observables from fiber to fiber. The observables are invariant under this parallel transport.

One way to see this is by choosing a local section of $\mathcal{E} \rightarrow \mathcal{B}$. This is a choice $\xi_{1}$ of a relative ( $n-1$ )-cycle in each fiber over a neighborhood $\mathcal{N}$ in $\mathcal{B}$. The choice of the $\xi_{1}$ trivializes $\mathcal{E}$ as $\mathcal{N} \times \mathcal{E}_{0}$ over $\mathcal{N}$,

$$
\begin{equation*}
\partial \xi, \xi \longleftrightarrow \partial \xi, \xi-\xi_{1}(\partial \xi) \tag{18.16}
\end{equation*}
$$

This also trivializes $\mathcal{T}$ over $\mathcal{N}$ by singling out the local section given by the normalization condition

$$
\begin{equation*}
\phi_{1}\left(\xi_{1}\right)=\phi_{1}^{*}\left(\xi_{1}\right)=0 . \tag{18.17}
\end{equation*}
$$

Now the covariant derivative for the natural connection in $\mathcal{E}$ is

$$
\begin{equation*}
D^{n a t}=d+A^{n a t} \tag{18.18}
\end{equation*}
$$

where

$$
\begin{equation*}
A^{n a t}=D^{n a t} \xi_{1} \tag{18.19}
\end{equation*}
$$

is a 1 -form on $\mathcal{B}$ with values in the vertical tangent space $\mathcal{V}_{n}$. The covariant derivative in $\mathcal{T}$ is

$$
\begin{array}{cl}
D \phi=(d+A) \phi, & D \phi^{*}=\left(d+A^{*}\right) \phi^{*} \\
A=D^{n a t} \phi_{1}=j A^{n a t}, & A^{*}=D^{n a t} \phi_{1}^{*}=(* j) A^{n a t} \tag{18.21}
\end{array}
$$

The total covariant derivative in the fiber product, acting on an observable of charges $p$, $p^{*}$ is

$$
\begin{equation*}
D^{t o t}=d+A^{n a t} d_{V}-i p A-i p^{*} A^{*} \tag{18.22}
\end{equation*}
$$

where $d_{V}$ is the vertical derivative in the trivialization of $\mathcal{E}$.
The covariant constancy of the observables implies that the observables are annihilated by the curvature tensor $\left(D^{t o t}\right)^{2}$. This is just $j=d \phi, * j=d \phi^{*}$. Thus the gauge structure - the bundle $\mathcal{T} \rightarrow \mathcal{B}$ and its connection $D$ - encodes the classical theory of extended objects of the free $n$-form.

The charged observables are associated to nontrivial representations of the symmetry group $G$ and live in vector bundles associated to the fiber product bundle $\mathcal{E} \times_{\mathcal{B}} \mathcal{T}$. The connection that transports the charged observables between fibers is dynamical. But neutral observables - observables that are $G$-invariant, that are associated to the trivial $G$ representation - do not see the gauge bundle $\mathcal{T}$. The neutral observables are transported from fiber to fiber by the natural connection in $\mathcal{E}$, which is independent of the dynamics, i.e., kinematical. The neutral observables are, essentially, the products of vertex operators with zero total charges,

$$
\begin{equation*}
e^{i p_{1} \phi\left(\xi_{1}\right)+i p_{1}^{*} \phi^{*}\left(\xi_{1}\right)} \cdots e^{i p_{N} \phi\left(\xi_{N}\right)+i p_{N}^{*} \phi^{*}\left(\xi_{N}\right)}, \quad \sum p_{i}=\sum p_{i}^{*}=0 \tag{18.23}
\end{equation*}
$$

Assume that the gauge symmetry is unbroken, so that only the gauge invariant observables have non-zero expectation values. Gauge invariant observables are products of neutral observables on fibers. Neutral observables can be transported from fiber to fiber, independent of the dynamics. So expectation values of gauge invariant observables can be calculated entirely on any single fiber.

This argument might seem to imply that all calculations can be done in a single fiber, for example in the distinguished fiber over $0 \in \mathcal{B}$. But this is too extreme. Certainly, in a 2 d quantum field theory there is interesting information to obtain about the charged observables. In the space $\mathcal{E}$ of extended objects, the algebra of charged observables will depend on the fiber. On the other hand, products of charged observables from different fibers will always have zero expectation values. So the theory of extended objects can be considered as the collection of theories on the fibers.

## 19 Explorations

In this section, for simplicity, the imaginary parts of the quasi Riemann surfaces are omitted, and all quasi Riemann surfaces are assumed connected. And the conjecture is presumed to hold.

### 19.1 The $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$

As described in section 12.2, each of the quasi Riemann surfaces $\mathcal{Q}=\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ shares a common core,

$$
\begin{equation*}
\mathcal{Q}_{1}^{\mathrm{int}}=\mathcal{D}_{n}^{\mathrm{int}}(M), \quad\left(\mathcal{Q}_{0}^{\mathrm{int}}\right)_{0}=\mathcal{D}_{n-1}^{\mathrm{int}}(M)_{0}=\partial \mathcal{D}_{n}^{\mathrm{int}}(M) \tag{19.1}
\end{equation*}
$$

The last identity expresses the connectedness of the quasi Riemann surfaces,

$$
\begin{equation*}
H_{0}(\mathcal{Q})=H_{n-1}(M)=0 \tag{19.2}
\end{equation*}
$$

Consider the restriction of $\Pi_{*}^{1,1}$ to the space of integral 1-cycles in the space of integral 1-cycles of $\mathcal{Q}$,

$$
\begin{equation*}
\Pi_{*}^{1,1}: \mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0} \rightarrow \mathcal{Q}_{2}^{\text {int }} \tag{19.3}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
\partial \Pi_{*}^{1,1}=0, \tag{19.4}
\end{equation*}
$$

so

$$
\begin{equation*}
\Pi_{*}^{1,1} \mathcal{D}_{1}^{\mathrm{int}}\left(\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)_{0}\right)_{0}=\left(\mathcal{Q}_{2}\right)_{0}=\partial \mathcal{Q}_{3} \simeq \mathbb{Z} \tag{19.5}
\end{equation*}
$$

Define

$$
\begin{equation*}
\Lambda(\mathbb{Z} \partial \xi)^{\mathrm{int}}=\operatorname{Ker} \Pi_{*}^{1,1} \subset \mathcal{D}_{1}^{\mathrm{int}}\left(\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)_{0}\right)_{0} \tag{19.6}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0} / \Lambda(\mathbb{Z} \partial \xi)^{\text {int }}=\left(\mathcal{Q}_{2}\right)_{0}=\partial \mathcal{Q}_{3} \simeq \mathbb{Z} \tag{19.7}
\end{equation*}
$$

The subgroup $\Lambda(\mathbb{Z} \partial \xi)^{\text {int }}$ is a co-line in $\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0}$, where a co-line is defined to be a minimal subgroup among those whose quotient is $\mathbb{Z}$.

The map $\Pi_{*}^{1,1}$ acts in two stages

$$
\begin{equation*}
\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0}=\mathcal{D}_{1}^{\text {int }}\left(\mathcal{D}_{n}^{\text {int }}(M)_{0}\right)_{0} \rightarrow \mathcal{D}_{n+1}^{\text {int }}(M)_{0} \rightarrow\left(\mathcal{Q}_{2}\right)_{0} \tag{19.8}
\end{equation*}
$$

Write the first stage

$$
\begin{equation*}
\Pi_{*}^{1,1}(M): \mathcal{D}_{1}^{\mathrm{int}}\left(\mathcal{D}_{n}^{\mathrm{int}}(M)_{0}\right)_{0} \rightarrow \mathcal{D}_{n+1}^{\mathrm{int}}(M)_{0} \tag{19.9}
\end{equation*}
$$

and define the subgroup

$$
\begin{equation*}
\Lambda_{M}^{\mathrm{int}}=\operatorname{Ker} \Pi_{*}^{1,1}(M) \subset \mathcal{D}_{1}^{\mathrm{int}}\left(\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)_{0}\right)_{0} \tag{19.10}
\end{equation*}
$$

Then

$$
\begin{equation*}
\Lambda_{M}^{\mathrm{int}} \subset \Lambda(\mathbb{Z} \partial \xi)^{\mathrm{int}} \tag{19.11}
\end{equation*}
$$

In fact, any co-line in $\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0}$ that contains $\Lambda_{M}^{\text {int }}$ is $\Lambda(\mathbb{Z} \partial \xi)^{\text {int }}$ for some $\mathbb{Z} \partial \xi \in \mathcal{P B}(M)$,

$$
\begin{equation*}
\mathcal{P} \mathcal{B}(M)=\left\{\text { co-lines } \Lambda^{\text {int }}: \Lambda_{M}^{\text {int }} \subset \Lambda^{\text {int }} \subset \mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0}\right\} \tag{19.12}
\end{equation*}
$$

### 19.2 Reconstruct $\mathcal{Q}$ from $\mathcal{Q}_{1}$

All of the considerations of the previous section up to the introduction of $\Lambda_{M}$ apply to any quasi Riemann surface $\mathcal{Q}$. Every quasi Rieman surface $\mathcal{Q}$ has a distinguished co-line

$$
\begin{equation*}
\Lambda^{\mathrm{int}}=\operatorname{Ker} \Pi_{*}^{1,1} \subset \mathcal{D}_{1}^{\mathrm{int}}\left(\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)_{0}\right)_{0} \tag{19.13}
\end{equation*}
$$

Conversely, every co-line in $\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)_{0}$ corresponds to a quasi Riemann surface. That is, any quasi Riemann surface $\mathcal{Q}$ can be reconstructed from the data on its 1 -subspace $\mathcal{Q}_{1}^{\text {int }}$. The core of the quasi Riemann surface is $\mathcal{Q}_{1}^{\text {int }}$, with the skew-hermitian form and the $J$-operator, and with the two subgroups of boundaries and cycles,

$$
\begin{equation*}
\partial \mathcal{Q}_{2}^{\text {int }} \subset\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0} \subset \mathcal{Q}_{1}^{\text {int }} \tag{19.14}
\end{equation*}
$$

From the core data we reconstruct

$$
\begin{equation*}
\left(\mathcal{Q}_{0}^{\text {int }}\right)_{0}=\mathcal{Q}_{1}^{\text {int }} /\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0} \tag{19.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{Q}_{2}^{\text {int }} /\left(\mathcal{Q}_{2}^{\text {int }}\right)_{0}=\partial \mathcal{Q}_{2}^{\text {int }} \tag{19.16}
\end{equation*}
$$

Only one line's worth of data is left to specify in order to reconstruct all of $\mathcal{Q}$.
Define

$$
\begin{equation*}
\mathcal{Z}\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)=\mathcal{D}_{1}^{\mathrm{int}}\left(\left(\mathcal{Q}_{1}^{\mathrm{int}}\right)_{0}\right)_{0} \tag{19.17}
\end{equation*}
$$

The last piece of data is a co-line

$$
\begin{equation*}
\Lambda^{\mathrm{int}} \subset \mathcal{Z}\left(\mathcal{Q}_{1}^{\mathrm{int}}\right) \tag{19.18}
\end{equation*}
$$

Now we reconstruct

$$
\begin{equation*}
\left(\mathcal{Q}_{2}^{\text {int }}\right)_{0}=\mathcal{Z}\left(\mathcal{Q}_{1}^{\text {int }}\right) / \Lambda^{\text {int }} \tag{19.19}
\end{equation*}
$$

and we finish the construction of $\mathcal{Q}_{0}^{\text {int }}$ using duality under the skew-hermitian intersection form. The map $\Pi_{*}^{1,1}$ is given by

$$
\begin{equation*}
\Pi_{*}^{1,1}: \mathcal{Z}\left(\mathcal{Q}_{1}^{\mathrm{int}}\right) \rightarrow \mathcal{Z}\left(\mathcal{Q}_{1}^{\mathrm{int}}\right) / \Lambda^{\mathrm{int}}=\mathcal{Q}_{2}^{\mathrm{int}} \tag{19.20}
\end{equation*}
$$

So a quasi Riemann surface $\mathcal{Q}$ is specified by the 1 -space $\mathcal{Q}_{1}^{\text {int }}$ along with the core data

- the skew-hermitian form on $\mathcal{Q}_{1}^{\text {int }}$,
- the $J$-operator on $\mathcal{Q}_{1}$,
- the subgroups $\partial \mathcal{Q}_{2}^{\text {int }} \subset\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0} \subset \mathcal{Q}_{1}^{\text {int }}$,
plus
- a co-line $\Lambda^{\text {int }} \subset \mathcal{Z}\left(\mathcal{Q}_{1}^{\text {int }}\right)$.


### 19.3 Augment $\Sigma$

Now we want to use the conjecture to equate the $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ with the quasi Riemann surface $\mathcal{Q}(\Sigma)$ associated to a two-dimensional conformal space $\Sigma$. We want to do this in a way that preserves the commonality of the cores of all the $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$.

First, however, we have to construct $\mathcal{Q}(\Sigma)$. As pointed out in section 12.4, there is a flaw in the straightforward construction of a quasi-Riemann surface

$$
\begin{equation*}
0 \leftarrow \mathcal{Q}_{-1}^{\text {int }} \stackrel{\partial}{\leftarrow} \mathcal{Q}_{0}^{\text {int }} \stackrel{\partial}{\leftarrow} \mathcal{Q}_{1}^{\text {int }} \stackrel{\partial}{\leftarrow} \mathcal{Q}_{2}^{\text {int }} \stackrel{\partial}{\leftarrow} \mathcal{Q}_{3}^{\text {int }} \leftarrow 0 \tag{19.21}
\end{equation*}
$$

from an ordinary Riemann surface $\Sigma$, as the augmented de Rham complex

$$
\begin{equation*}
0 \leftarrow \mathbb{Z} \stackrel{\partial}{\leftarrow} \mathcal{D}_{0}^{\text {int }}(\Sigma) \stackrel{\partial}{\leftarrow} \mathcal{D}_{1}^{\text {int }}(\Sigma) \stackrel{\partial}{\leftarrow} \mathcal{D}_{2}^{\text {int }}(\Sigma) \stackrel{\partial}{\leftarrow} \mathbb{Z} \leftarrow 0 \tag{19.22}
\end{equation*}
$$

The problem is that we need the maps $\Pi_{*}^{j, k}$ on the currents in the $\left(\mathcal{Q}_{k}^{\text {int }}\right)_{0}$ to induce isomorphisms of homology,

$$
\begin{equation*}
H_{2}\left(\left(\mathcal{Q}_{0}^{\text {int }}\right)_{0}\right)=H_{1}\left(\left(\mathcal{Q}_{1}^{\text {int }}\right)_{0}\right)=H_{2}\left(\mathcal{Q}^{\text {int }}\right)=0 \tag{19.23}
\end{equation*}
$$

However, the homology groups $H_{j}\left(\mathcal{D}_{k}^{\text {int }}(\Sigma)_{0}\right)$ are isomorphic to the homology groups $H_{j+k}(\Sigma)$ of the un-augmented de Rham complex,

$$
\begin{equation*}
H_{2}\left(\mathcal{D}_{0}^{\text {int }}(\Sigma)_{0}\right)=H_{1}\left(\mathcal{D}_{1}^{\operatorname{int}}(\Sigma)_{0}\right)=H_{2}(\Sigma)=\mathbb{Z} \tag{19.24}
\end{equation*}
$$

That is, there exist

$$
\begin{equation*}
\mu_{1,1} \in \mathcal{D}_{1}^{\mathrm{int}}\left(\mathcal{D}_{1}^{\mathrm{int}}(\Sigma)_{0}\right)_{0}, \quad \mu_{2,0} \in \mathcal{D}_{2}^{\mathrm{int}}\left(\mathcal{D}_{0}^{\mathrm{int}}(\Sigma)_{0}\right)_{0} \tag{19.25}
\end{equation*}
$$

with

$$
\begin{equation*}
\Pi_{*}^{1,1} \mu_{1,1}=\Sigma, \quad \Pi_{*}^{2,0} \mu_{2,0}=\Sigma \tag{19.26}
\end{equation*}
$$

so neither $\mu_{1,1}$ nor $\mu_{2,0}$ can be a boundary. They generate the non-trivial homology groups.
We need to construct an "augmentation", $\Sigma_{+}$, of $\Sigma$ such that the ordinary de Rham complex of $\Sigma_{+}$is

$$
\begin{equation*}
0 \leftarrow \mathcal{D}_{-1}^{\mathrm{int}}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{0}^{\mathrm{int}}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{1}^{\mathrm{int}}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{2}^{\mathrm{int}}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{3}^{\mathrm{int}}\left(\Sigma_{+}\right) \leftarrow 0, \tag{19.27}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{D}_{-1}^{\mathrm{int}}\left(\Sigma_{+}\right) \simeq \mathbb{Z}, \quad \mathcal{D}_{3}^{\text {int }}\left(\Sigma_{+}\right) \simeq \mathbb{Z} \tag{19.28}
\end{equation*}
$$

and homology

$$
\begin{equation*}
H_{1}\left(\Sigma_{+}\right)=H_{1}(\Sigma), \quad H_{k}\left(\Sigma_{+}\right)=0, \quad k \neq 1 . \tag{19.29}
\end{equation*}
$$

In some sense, we want to make $\Sigma_{+}$from $\Sigma$ by adding three new integral currents,

$$
\begin{align*}
\mathcal{D}_{3}^{\text {int }}\left(\Sigma_{+}\right) & =\mathbb{Z} \eta_{3}  \tag{19.30}\\
\mathcal{D}_{2}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}\left(\Sigma_{+}\right)_{0}\right) & =\mathcal{D}_{2}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}(\Sigma)_{0}\right) \oplus \mathbb{Z} \mu_{2,1}  \tag{19.31}\\
\mathcal{D}_{3}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}\left(\Sigma_{+}\right)_{0}\right) & =\mathcal{D}_{3}^{\text {int }}\left(\mathcal{D}_{0}^{\text {int }}(\Sigma)_{0}\right) \oplus \mathbb{Z} \mu_{3,0} \tag{19.32}
\end{align*}
$$

satisfying

$$
\begin{gather*}
\partial \eta_{3}=\Sigma, \quad \partial \mu_{2,1}=\mu_{1,1}, \quad \partial \mu_{3,0}=\mu_{2,0}  \tag{19.33}\\
\Pi_{*}^{2,1} \mu_{2,1}=\eta_{3}, \quad \Pi_{*}^{3,0} \mu_{3,0}=\eta_{3} . \tag{19.34}
\end{gather*}
$$

Now $\Sigma, \mu_{1,1}$, and $\mu_{2,0}$ are boundaries. The bothersome homology groups are killed. I will have to suppose that a precise construction of $\Sigma_{+}$can be made.

The de Rham complex of $\Sigma_{+}$now gives an actual quasi-Riemann surface $\mathcal{Q}\left(\Sigma_{+}\right)$. The defining co-line is

$$
\begin{equation*}
\Lambda\left(\Sigma_{+}\right)^{\mathrm{int}}=\operatorname{Ker} \Pi_{*}^{1,1}=\partial \mathcal{D}_{2}^{\mathrm{int}}\left(\mathcal{D}_{1}^{\mathrm{int}}(\Sigma)_{0}\right) \subset \partial \mathcal{D}_{2}^{\mathrm{int}}\left(\mathcal{D}_{1}^{\mathrm{int}}\left(\Sigma_{+}\right)_{0}\right)=\mathcal{D}_{1}^{\mathrm{int}}\left(\mathcal{D}_{1}^{\mathrm{int}}\left(\Sigma_{+}\right)_{0}\right)_{0} \tag{19.35}
\end{equation*}
$$

### 19.4 Isomorphisms of the $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ to $\mathcal{Q}\left(\Sigma_{+}\right)$

Pick a point $\mathbb{Z} \partial \xi \in \mathcal{P B}(M)$. Pick an isomorphism

$$
\begin{equation*}
f: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi} \rightarrow \mathcal{Q}\left(\Sigma_{+}\right) \tag{19.36}
\end{equation*}
$$

which is determined by its isomorphism of the core structures,

$$
\begin{equation*}
f_{1}: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 1}^{\mathrm{int}} \rightarrow \mathcal{Q}\left(\Sigma_{+}\right)_{1}^{\mathrm{int}} \tag{19.37}
\end{equation*}
$$

of the defining co-lines,

$$
\begin{equation*}
f_{1 *} \Lambda(\mathbb{Z} \partial \xi)^{\mathrm{int}}=\Lambda\left(\Sigma_{+}\right)^{\mathrm{int}} \tag{19.38}
\end{equation*}
$$

Now consider another point $\mathbb{Z} \partial \xi^{\prime} \in \mathcal{P B}(M)$. The core structures of $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}}$ are the same as those of $\mathcal{Q}(M)_{\mathbb{Z}} \xi$, so $f_{1}$ is also an isomorphism

$$
\begin{equation*}
f_{1}: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}, 1} \rightarrow \mathcal{Q}\left(\Sigma_{+}\right)_{1}, \tag{19.39}
\end{equation*}
$$

preserving the core structures. However, the defining co-lines of $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}}$ and $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ are different, so

$$
\begin{equation*}
f_{1 *} \Lambda\left(\mathbb{Z} \partial \xi^{\prime}\right)^{\mathrm{int}} \neq \Lambda\left(\Sigma_{+}\right)^{\mathrm{int}} \tag{19.40}
\end{equation*}
$$

so $f_{1}$ does not give an isomorphism of quasi Riemann surfaces between $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}}$ and $\mathcal{Q}\left(\Sigma_{+}\right)$. Its image is a quasi Riemann surface that has the same core as $\mathcal{Q}\left(\Sigma_{+}\right)$, but a different defining co-line.

### 19.5 The universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0)$ of quasi Riemann surfaces

Define

$$
\begin{equation*}
\mathcal{Z}=\mathcal{D}_{1}^{\text {int }}\left(\left(\mathcal{Q}\left(\Sigma_{+}\right)_{1}\right)_{0}\right)_{0} \tag{19.41}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{P B}(0)=\{\text { co-lines } \Lambda \subset \mathcal{Z}\} \tag{19.42}
\end{equation*}
$$

so

$$
\begin{equation*}
\Lambda\left(\Sigma_{+}\right)^{\mathrm{int}} \in \mathcal{P B}(0) \tag{19.43}
\end{equation*}
$$

is a distinguished point. Let $\mathcal{Q}(0)_{\Lambda}$ be the quasi Riemann surface with the same core structure as $\mathcal{Q}\left(\Sigma_{+}\right)$but with defining co-line $\Lambda$. Then

$$
\begin{equation*}
\mathcal{Q}(0)=\underset{\Lambda \in \mathcal{P} \mathcal{B}(0)}{\cup} \mathcal{Q}(0)_{\Lambda} \tag{19.44}
\end{equation*}
$$

forms a bundle of quasi Riemann surfaces

$$
\begin{equation*}
\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0) \tag{19.45}
\end{equation*}
$$

### 19.6 Embed $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ in the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$

Consider again the isomorphism $f: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi} \rightarrow \mathcal{Q}\left(\Sigma_{+}\right)$of section 19.4. Now

$$
\begin{equation*}
f_{1}: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}, 1}^{\operatorname{int}} \rightarrow \mathcal{Q}(0)_{\Lambda^{\prime}, 1}^{\mathrm{int}}, \quad \Lambda^{\prime}=f_{1 *} \Lambda\left(\mathbb{Z} \partial \xi^{\prime}\right)^{\mathrm{int}} \tag{19.46}
\end{equation*}
$$

preserves the defining co-lines, so gives an isomorphism

$$
\begin{equation*}
f: \mathcal{Q}(M)_{\mathbb{Z} \partial \xi^{\prime}} \rightarrow \mathcal{Q}(0)_{\Lambda^{\prime}} \tag{19.47}
\end{equation*}
$$

Thus we have the embedding

which is natural, up to the choice of the point $\mathbb{Z} \partial \xi \in \mathcal{P B}(M)$ to be mapped to $\Lambda\left(\Sigma_{+}\right)^{\text {int }} \in$ $\mathcal{P B}(0)$, and up to the choice of the isomorphism $f$.

In terms of the subgroup

$$
\begin{equation*}
f_{1 *} \Lambda_{M}^{\mathrm{int}} \subset \mathcal{Z} \tag{19.49}
\end{equation*}
$$

the embedding of $\mathcal{P B}(M)$ is

$$
\begin{equation*}
f_{1 *} \mathcal{P} \mathcal{B}(M)=\left\{\text { co-lines } \Lambda \subset \mathcal{Z}: f_{1 *} \Lambda_{M}^{\text {int }} \subset \Lambda\right\} \tag{19.50}
\end{equation*}
$$

and the embedding of $\mathcal{Q}(M)$ is the restriction of $\mathcal{Q}(0)$. So $M$ is characterized by the subgroup $f_{1 *} \Lambda_{M}^{\text {int }} \subset \mathcal{Z}$, again up to the choice of $\Lambda\left(\Sigma_{+}\right)^{\text {int }}$ and $f$.

### 19.7 Homogeneity of $\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0)$

Let

$$
\begin{equation*}
G\left(\Sigma_{+}\right)=\operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)_{1}^{\mathrm{int}}\right) \tag{19.51}
\end{equation*}
$$

be the group of automorphisms of the core structure of $\mathcal{Q}\left(\Sigma_{+}\right)$, the continuous abelian group automorphisms that preserve the skew-hermitian form, the $J$-operator, and the subspaces of cycles and boundaries.

Assuming that the conjecture holds, all of the quasi Riemann surfaces $\mathcal{Q}(0)_{\Lambda}$ in the universal bundle are isomorphic as quasi Riemann surfaces. Every isomorphism between
$\mathcal{Q}(0)_{\Lambda}$ and $\mathcal{Q}(0)_{\Lambda^{\prime}}$ is given by an element of $G\left(\Sigma_{+}\right)$. Therefore $G\left(\Sigma_{+}\right)$acts transitively on $\mathcal{P B}(0)$. The subgroup of $G\left(\Sigma_{+}\right)$that fixes $\Lambda$ is $\operatorname{Aut}\left(\mathcal{Q}(0)_{\Lambda}\right)$. Therefore $\mathcal{P B}(0)$ is the homogeneous space

$$
\begin{equation*}
\mathcal{P B}(0)=G\left(\Sigma_{+}\right) / \operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right) \tag{19.52}
\end{equation*}
$$

And the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ is the bundle on $G\left(\Sigma_{+}\right) / \operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right)$associated to the action of $\operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right)$on $\mathcal{Q}\left(\Sigma_{+}\right)$.

## 20 Mathematical questions

This section is simply a list of the main mathematical questions that the project depends on, with references to the relevant sections of the paper where some background can be found. Apart from these more or less specific questions, all of the mathematical arguments in the paper need tightening. For the most part, I have not specified topologies on vector spaces of currents, or domains of definition of forms on spaces of currents. I have supposed the context to determine what is appropriate.

Recall from section 1.1 that $M$ is a compact, real, oriented manifold of even dimension $d=2 n$, endowed with a conformal class of Riemannian metrics - or at least with a Hodge *-operator acting in the middle dimension, on $n$-forms.

Recall from section 1.5 that $\mathcal{D}_{k}^{\mathrm{int}}(M)$ is the space of integral $k$-currents in $M$.

### 20.1 Does the Hodge $*$-operator act on $T_{0} \mathcal{D}_{n-1}^{\text {int }}(M)_{0}$ ?

$\mathcal{D}_{n-1}^{\text {int }}(M)_{0}$ is the space of integral $(n-1)$-cycles in $M$. Its tangent space at 0 can be identified with a subspace of the flat $n$-currents in $M$,

$$
\begin{equation*}
T_{0} \mathcal{D}_{n-1}^{\text {int }}(M)_{0}=\mathcal{V}_{n} \subset \mathcal{D}_{n}^{\text {flat }}(M) \tag{20.1}
\end{equation*}
$$

The Hodge $*$-operator acts on the flat $n$-currents. Does it preserve $\mathcal{V}_{n}$ ?
Appendix A contains what I believe is the germ of a proof that the answer is 'yes'. I think that only some detail needs to be filled in.

This question was asked in section 3, based on definitions in section 2. The project depends on an affirmative answer.

The space $\mathcal{E}_{\partial \xi}=\mathcal{D}_{n-1}^{\mathrm{int}}(M)_{\partial \xi}$ is the fiber over $\partial \xi$ of the bundle

$$
\begin{equation*}
\mathcal{D}_{n-1}^{\mathrm{int}}(M) \xrightarrow{\partial} \partial \mathcal{D}_{n-1}^{\mathrm{int}}(M) \subset \mathcal{D}_{n-2}^{\mathrm{int}}(M) \tag{20.2}
\end{equation*}
$$

All of the vertical tangent spaces in the fibers are the same,

$$
\begin{equation*}
T_{\xi} \mathcal{E}_{\partial \xi}=T_{0} \mathcal{D}_{n-1}^{\mathrm{int}}(M)_{0}=\mathcal{V}_{n} \subset \mathcal{D}_{n}^{\text {flat }}(M) \tag{20.3}
\end{equation*}
$$

The action of the Hodge $*$-operator on the tangent spaces of $\mathcal{E}_{\partial \xi}$ is used in sections 7 and 9 to define an operator $J=\epsilon_{n} *, J^{2}=-1$, on the tangent spaces that makes $\mathcal{E}_{\partial \xi}$ - or its complexification $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ when $n$ is even - into an almost-complex space.

### 20.2 Are quasi Riemann surfaces classified by their homology data?

In section 7, the intersection form on currents in $M$ is modified slightly to become a skew-hermitian form $I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle$. In sections 11 and 12, the skew-intersection intersection form is pulled up to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The almost-complex structure $J$ and the skew-hermitian form on currents in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ have exactly the properties of the almost-complex structure and the skew-hermitian intersection form on currents in a Riemann surface. This structure is codified in the definition of quasi Riemann surface in section 12.

The quasi Riemann surface associated to an ordinary Riemann surface is written $\mathcal{Q}(\Sigma)$. The quasi Riemann surface associated to the $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are written $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$. They form a bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ over the integral projective space $\mathcal{P B}(M)$ of the space $\mathcal{B}$.

In section 14, a 'wishful" conjecture is proposed: that quasi Riemann surfaces are classified up to isomorphism by their homology data, and that each isomorphism class contains the quasi Riemann space $\mathcal{Q}(\Sigma)$ of a two dimensional conformal space $\Sigma$. The space $\Sigma$ is a Riemann surface when the homology data is that of a Riemann surface.

For the connected case, which is $H_{0}\left(\mathcal{D}_{n-1}^{\text {int }}(M)\right)=\mathbb{Z}$, the homology data consists of the integral homology group $H_{n}(M)$ in the middle dimension, with the skew-hermitian form derived from the intersection form and with the almost-complex structure.

If the conjecture should fail, there would still be the question, do quasi-holomorphic curves exist? A quasi-holomorphic curve would be a morphism of quasi Riemann surfaces from some $\mathcal{Q}(\Sigma)$ to $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$.

When the homology data is not that of a Riemann surface, the conjecture supposes a two-dimensional conformal space $\Sigma$ whose integral currents form a quasi Riemann surface with that homology data. What are these two-dimensional conformal spaces?

If the conjecture holds, is there any possibility, say in the basic case, $M=S^{d}, \Sigma=S^{2}$, of actually writing an isomorphism between $\mathcal{Q}(M)$ and $\mathcal{Q}(\Sigma)$.

If the conjecture holds, then the automorphism $\operatorname{group} \operatorname{Aut}(\mathcal{Q})$ of a quasi Riemann surface $\mathcal{Q}$ depends only on the homology data of $\mathcal{Q}$. What can be said about $\operatorname{Aut}(\mathcal{Q})$, beyond the elementary comments made in section 12.7?

If the conjecture holds, what can be said about the bundle of quasi Riemann surfaces $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ associated to $M$ ? It will have structure group $\operatorname{Aut}(\mathcal{Q}(\Sigma))$ where $\Sigma$ is the Riemann surface, or two-dimensional conformal space, with the same homology data as $M$. The homotopy groups of the bundle $\mathcal{E} \rightarrow \mathcal{B}$ are given by the homology groups of $M$ [6]. Presumably there are analogous results on the homotopy groups of the bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$.

### 20.3 Can a Riemann surface be augmented?

In section 12.4, it is pointed out that there is an obstacle to forming a quasi Riemann surface from the currents in a Riemann surface $\Sigma$. The homology groups $H_{j}$ of the spaces $\mathcal{D}_{k}^{\text {int }}(\Sigma)_{0}$ of integral cycles in $\Sigma$ are not isomorphic to the homology groups $H_{j+k}$ of the augmented de Rham complex of $\Sigma$.

In section [19, it is supposed that a Riemann surface $\Sigma$ - or, more generally, one of the conjectured two-dimensional conformal spaces - can be augmented to a space $\Sigma_{+}$
whose de Rham complex of integral currents is the augmented de Rham complex of the original space $\Sigma$, which has homology only in the middle dimension, dimension 1 ,

$$
\begin{equation*}
0 \leftarrow \mathcal{D}_{-1}^{\text {int }}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{0}^{\text {int }}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{1}^{\text {int }}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{2}^{\text {int }}\left(\Sigma_{+}\right) \stackrel{\partial}{\leftarrow} \mathcal{D}_{3}^{\text {int }}\left(\Sigma_{+}\right) \leftarrow 0 \tag{20.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{D}_{-1}^{\mathrm{int}}\left(\Sigma_{+}\right) \simeq \mathbb{Z}, \quad \mathcal{D}_{3}^{\mathrm{int}}\left(\Sigma_{+}\right) \simeq \mathbb{Z} \tag{20.5}
\end{equation*}
$$

and homology

$$
\begin{equation*}
H_{1}\left(\Sigma_{+}\right)=H_{1}(\Sigma), \quad H_{k}\left(\Sigma_{+}\right)=0, \quad k \neq 1 . \tag{20.6}
\end{equation*}
$$

The augmentation would kill the homology of $\Sigma$ in dimension 2 by adding an integral 3 -current $\eta_{3}$ whose boundary is $\Sigma, \partial \eta_{3}=\Sigma$. This would have to be done in such a way as to kill the homology groups $H_{1}\left(\mathcal{D}_{1}^{\text {int }}(\Sigma)_{0}\right)$ and $H_{2}\left(\mathcal{D}_{0}^{\text {int }}(\Sigma)_{0}\right)$.

Does a Riemann surface $\Sigma$ have such an augmentation $\Sigma_{+}$?

### 20.4 What can be said about the universal bundle of quasi Riemann surfaces?

The "explorations" of section 19 lead to a universal bundle of quasi Riemann surfaces $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ for each value of the homology data, where $\mathcal{P B}(0)$ is a homogeneous space,

$$
\begin{equation*}
\mathcal{P B}(0)=G\left(\Sigma_{+}\right) / \operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right), \tag{20.7}
\end{equation*}
$$

and $\mathcal{Q}(0)$ is the bundle over $\mathcal{P B}(0)$ associated to the action of $\operatorname{Aut}\left(\mathcal{Q}\left(\Sigma_{+}\right)\right)$on the quasi Riemann surface $\mathcal{Q}\left(\Sigma_{+}\right)$. The group $G\left(\Sigma_{+}\right)$is the group of automorphisms of $\mathcal{D}_{1}^{\text {int }}\left(\Sigma_{+}\right)$ - the continuous abelian group automorphisms that preserve the skew-hermitian form and the $J$-operator, and the subgroups of cycles and boundaries.

The homogeneous space $\mathcal{P B}(0)$ is also described as the space of integral co-lines in

$$
\begin{equation*}
\mathcal{Z}=\mathcal{D}_{1}^{\mathrm{int}}\left(\mathcal{D}_{1}^{\mathrm{int}}(\Sigma)_{0}\right)_{0} \tag{20.8}
\end{equation*}
$$

The conjecture leads to the association of a subgroup of $\mathcal{Z}$ to the manifold $M$, and identifies $\mathcal{P B}(M)$ with the subset of integral co-lines in $\mathcal{Z}$ that contain that subgroup. So $\mathcal{P B}(M)$ is embedded in $\mathcal{P B}(0)$, and the bundle of quasi Riemann surfaces $\mathcal{Q}(M) \rightarrow$ $\mathcal{P B}(M)$ is the restriction of $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$.

This is to be the universal setting for extended conformal quantum field theory and perhaps, eventually, extended non-conformal quantum field theory. Anything about its structure could be useful. In particular, the group of conformal symmetries of $M, \operatorname{Conf}(M)$, acts on $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$. Is $\operatorname{Conf}(M)$ a conjugacy class of subgroups of $G\left(\Sigma_{+}\right)$?

### 20.5 How much function theory can be done on a quasi Riemann surface?

This is a vague question. Two-dimensional conformal field theory uses a fair amount of the the theory of functions - and conformal tensors - on Riemann surfaces. How much of that can be done on a quasi Riemann surface? Extended conformal field theory on quasi Riemann surfaces will presumably have need of it. The Schwinger-Dyson equations (10.5-10.6) on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ would seem to be a starting point.

### 20.6 Are there mathematical applications?

This is presumptuous, considering the amount of conjecture and supposition. Still, if all can be made rigorous, there should be mathematical applications. The setting is rather general - a space $M$ that has forms and currents, of even dimension $d=2 n$, with a $*$ operator acting in the middle dimension. The association of the bundle $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ of quasi Riemann surfaces to each such $M$ should offer opportunities for constructing invariants of $M$ and for investigating the geometry of $M$. The embedding in a universal bundle of quasi Riemann surfaces might offer opportunities for studying the space of such spaces $M$.

## 21 Further steps

This section lists some possible further steps. The project depends on answers to at least the first three of the mathematical questions listed in section 20. However, it might be possible to make some progress - in particular, constructing extended conformal field theories on quasi Riemann surfaces - without resolving the conjecture on the equivalence of quasi Riemann surfaces. And it might be possible to take some additional formal steps assuming the conjecture, without having the actual isomorphisms in hand and without knowing the automorphism groups of the quasi Riemann surfaces.

### 21.1 Extended CFT on quasi Riemann surfaces

A 2d extended conformal field theory (2d ECFT) is an ordinary 2d CFT on Riemann surfaces $\Sigma$ extended to the quasi Riemann surfaces $\mathcal{Q}(\Sigma)$. As discussed in section [15, the observables are to be extended from products of local fields over finite sets of points in $\Sigma$ to products over integral 0 -currents in $\Sigma$. The extension should use only the geometric structures of the quasi Riemann surface, so that the resulting ECFT will be invariant under the automorphism group $\operatorname{Aut}(\mathcal{Q}(\Sigma))$ of the quasi Riemann surface.

One might start by trying to extend the 2 d gaussian model on $\Sigma=S^{2}$, where we have explicit expressions for the correlation functions of the ordinary vertex operators. The 0 -form fields $\phi_{ \pm}$are linear functionals on 0 -currents. Vertex operators $V_{i}(z)$, which are exponentials of the $\phi_{ \pm}$, give ordinary observables of the form

$$
\begin{equation*}
\Phi(\eta)=V_{1}\left(z_{1}\right) V_{2}\left(z_{2}\right) \cdots V_{N}\left(z_{N}\right), \quad \eta=\sum_{i} \delta_{z_{i}} . \tag{21.1}
\end{equation*}
$$

associated to the 0 -current $\eta$. The extension to $V(\eta)$ defined at arbitrary integral 0 currents $\eta$ will be an observable of a novel sort in the 2 d CFT. It will be a product of ordinary 2 d vertex operators at a fractal set of points in the Riemann surface $\Sigma$.

Then one might try to carry out on the extended 2 d gaussian model the usual constructions that are performed on the ordinary CFT, such as the $\mathbb{Z}_{2}$ orbifold and the $S U(2) \times S U(2)$ symmetry in the $R=1$ model.

In the end, we want a general construction of a 2d ECFT from an ordinary 2d CFT. The space of extended observables should be some kind of limit

$$
\begin{equation*}
\mathcal{H}=\lim _{N \rightarrow \infty} \underset{\left\{z_{1}, \ldots, z_{N}\right\}}{\oplus}\left(\underset{z \in\left\{z_{1}, \ldots, z_{N}\right\}}{\otimes} \mathcal{H}_{z}\right) \tag{21.2}
\end{equation*}
$$

where $\mathcal{H}_{z}$ is the vector space of local fields at the point $z \in \Sigma$, including the identity field, such that the extended observables have the form of functions $\Phi(\eta)$ on $\mathcal{D}_{0}^{\text {int }}(\Sigma)$. Among the questions that arise - what is the role of the abelian group structure of $\mathcal{D}_{0}^{\text {int }}(\Sigma)$ in the general ECFT?

It might be possible to describe the extended observables on $\mathcal{Q}(\Sigma)$ by their manifestations on the local q-h curves. The local q-h curves in $\mathcal{Q}(\Sigma)$ are given, conjecturally, by the local coordinate neighborhoods in $\Sigma$, up to automorphisms of $\mathcal{Q}(\Sigma)$. For each q-h curve $C$, an extended observable will appear as a state $\psi(C)$ in the state space of the radial quantization of the 2d CFT on the unit complex disk. The collection of states $\psi(C)$, subject to coherence conditions, might serve to characterize the extended observables.

The stress-energy tensor $T(z), \bar{T}(\bar{z})$ of the 2d CFT on the Riemann surface $\Sigma$ has to be lifted to $\mathcal{Q}(\Sigma)$. In the free $n$-form $/ 2 d$ gaussian model, they lift to fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$,

$$
\begin{equation*}
T_{++}(\xi)=-\frac{1}{2} j_{+}^{\dagger}(\bar{\xi}) j_{+}(\xi), \quad T_{--}(\xi)=-\frac{1}{2} j_{-}^{\dagger}(\bar{\xi}) j_{-}(\xi) \tag{21.3}
\end{equation*}
$$

where $j_{ \pm}$are the chiral 1-form fields on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. Pulled back along a q-h curve, $T_{++}$and $T_{--}$ become the analytic and anti-analytic stress-energy tensors on the unit disk. For each local q-h curve, the usual pair of Virasoro algebras will act on the states. Collectively, these should form a large Lie algebra acting on the space of extended observables. An operator product algebra of the extended observables should be built from a product of the operator product algebras on the q-h curves. And perhaps there should be an inner product on the space of extended observables, so that there is a representation by operators on Hilbert space. Eventually, there might be an axiomatic formulation of extended conformal field theory.

### 21.2 Gauge invariance

If the conjecture holds, the ECFT on $\mathcal{Q}(\Sigma)$ is to be transported by isomorphisms to each of the fibers of the bundle of quasi Riemann surfaces $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$. That bundle is embedded in the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$. The core of each fiber is the same. The 1-spaces in each fiber are the same,

$$
\begin{equation*}
\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 1}=\mathcal{D}_{n}^{\mathrm{int}}(M), \quad \mathcal{Q}(0)_{\lambda, 1}=\mathcal{D}_{1}^{\mathrm{int}}\left(\Sigma_{+}\right) \tag{21.4}
\end{equation*}
$$

The 0 -cycles in each fiber are the same,

$$
\begin{equation*}
\left(\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 0}^{\mathrm{int}}\right)_{0}=\mathcal{D}_{n-1}^{\mathrm{int}}(M)_{0}, \quad\left(\mathcal{Q}(0)_{\lambda, 0}^{\mathrm{int}}\right)_{0}=\mathcal{D}_{0}^{\mathrm{int}}\left(\Sigma_{+}\right)_{0} \tag{21.5}
\end{equation*}
$$

as are the dual spaces of 2-currents. For convenience, I am omitting the complexifications that are needed when $n$ is even.

The remaining data of the quasi Riemann surfaces, that distinguishes the fibers, is the orientation of the integral co-line in $\mathcal{Z}=\mathcal{D}_{1}^{\text {int }}\left(\mathcal{D}_{1}^{\text {int }}(\Sigma)_{0}\right)_{0}$ which determines the integral line of 2-cycles

$$
\begin{equation*}
\left(\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 2}^{\operatorname{int}}\right)_{0} \quad \text { or } \quad\left(\mathcal{Q}(0)_{\lambda, 2}^{\mathrm{int}}\right)_{0} \tag{21.6}
\end{equation*}
$$

and its dual line

$$
\begin{equation*}
\mathcal{Q}(M)_{\mathbb{Z} \xi \xi, 0}^{\mathrm{int}} /\left(\mathcal{Q}(M)_{\mathbb{Z} \partial \xi, 0}^{\mathrm{int}}\right)_{0} \quad \text { or } \quad \mathcal{Q}(0)_{\lambda, 0}^{\mathrm{int}} /\left(\mathcal{Q}(0)_{\lambda, 0}^{\mathrm{int}}\right)_{0} \tag{21.7}
\end{equation*}
$$

We might call the extended observables on the core of a quasi Riemann surface the core of the ECFT. The cores of the ECFTs on the fibers of $\mathcal{Q}(M) \rightarrow \mathcal{P} \mathcal{B}(M)$ must be all the same, because they are the same fields of extended objects in $M$. It is natural to require the same in the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$. For the 2 d gaussian model, the core ECFT is the algebra generated by the integrals of the 1 -forms $j_{ \pm}$over integral 1-cycles.

On each fiber, there will be some freedom in the extension of the core ECFT to an ECFT on the entire quasi Riemann surface. The is the local gauge symmetry over $\mathcal{P B}(M)$ or, universally, over $\mathcal{P B}(0)$. In the free $n$-form/ 2 d gaussian model, this is the freedom to choose the zero-modes of the 0 -form fields $\phi_{ \pm}$, as described in section 17 for the classical theory. We get a bundle of theories $\mathcal{T}(M) \rightarrow \mathcal{P B}(M)$ or, universally, $\mathcal{T}(0) \rightarrow \mathcal{P B}(0)$, which are the ECFTs extending the core ECFT.

It seems, then, that extended CFT will have additional structure beyond what is given in ordinary 2d CFT. There should always be a core ECFT and a set of extensions to the full quasi Riemann surface. Consider for example the 2 d gaussian model at $R=1$, where the two 1 -form fields $j_{ \pm}$of the $U(1) \times U(1)$ current algebra grow to the six 1-form fields of a $S U(2) \times S U(2)$ current algebra. The core ECFT of the 2d gaussian model is generated by the integrals of $j_{ \pm}$over 1-cycles. The space of extensions is a principal homogeneous space for the global symmetry group $U(1) \times U(1)$. But, at $R=1$, the core ECFT can be expanded to the algebra generated by integrals of the $S U(2) \times S U(2)$ 1-form fields over 1-cycles. The space of extensions will be a principal homogeneous space for the global symmetry group $S U(2) \times S U(2)$. Different ECFTs can be produced from the same ordinary CFT, depending on how the core of the ECFT is chosen.

Is the set of extensions always a homogeneous space for a group? That group would be the global symmetry group. Is the set of extensions always a principal homogeneous space for the global symmetry group? Some examples might clarify.

Is there some general method to gauge-fix over $\mathcal{P B}(M)$, partially, so that the remaining gauge symmetry is a local gauge symmetry in the space-time $M$, as described in section 17 for the classical $n$-form theory?

### 21.3 Connecting the $\mathcal{Q}(M)_{\partial \xi}$

We need a general picture of how the ECFTs on the fibers of $\mathcal{Q}(M) \rightarrow \mathcal{P B}(M)$ fit together to make one quantum field theory. Or perhaps we can fit together all the ECFTs on the fibers of the universal bundle $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ to form a single universal quantum field theory that specializes to any space-time $M$ with the appropriate homology data. The
conformal symmetry group of space-time, $\operatorname{Conf}(M)$, will act on $\mathcal{Q}(M) \rightarrow \mathcal{P} \mathcal{B}(M)$, and perhaps on $\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0)$, so we certainly need a single theory on which $\operatorname{Conf}(M)$ will be represented.

I am not sure how to adapt the argument of section 18 to the general case. In section 19, $\mathcal{P B}(0)$ is described as the homogeneous space $G\left(\Sigma_{+}\right) / \operatorname{Aut}(\mathcal{Q}(\Sigma))$, and $\mathcal{Q}(0) \rightarrow \mathcal{P B}(0)$ is described as an associated fiber bundle, with $G\left(\Sigma_{+}\right)$acting as a symmetry group on the fiber bundle. The action of $G\left(\Sigma_{+}\right)$should provide a natural connection in $\mathcal{Q}(0) \rightarrow \mathcal{P} \mathcal{B}(0)$. But I do not see that it fixes the common core of the fibers, so I do not see how it will give a connection in the bundle $\mathcal{T}(0) \rightarrow \mathcal{P B}(0)$ of theories. And I am not sure what the result should look like. The result of section 18 for the $n$-form theory is that gauge invariant observables in the bundle can be transported to any one single fiber. Perhaps a better statement would be that gauge invariant observables can be transported to the common core of all the fibers - that the non-zero correlation functions of the space-time theory are the correlation functions of the core ECFT.

### 21.4 The local fields in space-time

Given an extended CFT, there should be a general way to identify the local fields on the space-time $M$, acted on by the conformal symmetry group $\operatorname{Conf}(M)$. Taking $M=$ $S^{d}$, it should be possible to derive the spectrum of scaling dimensions from the data of the underlying 2d CFT. Perhaps the local fields can be identified from the extended observables on the infinitesimally small integral currents.

### 21.5 Partition functions and variation of conformal structure

If the conjecture holds, then it seems reasonable to suppose that not only the correlation functions but also the partition function of the space-time conformal field theory will be given by the corresponding 2d CFT. If in fact there does exist a two-dimensional conformal space $\Sigma$ for every set of homology data, the partition function of each theory will be a function of the homology data - or a section of a line bundle over the space of homology data. The identity between the partition functions of the two theories can be demonstrated by showing that they have the same variations with respect to the homology data, and then showing that they have the same behavior at transition singularities where the homology group changes.

In the two-dimensional conformal space, a variation of the homology data is given by an infinitesimal perturbation of the almost-complex structure $J$, which is expressed by a Beltrami differential, which is a (-1,1)-tensor $h_{\bar{z}}^{z}$. Somehow, if there is to be a twodimensional conformal space for every set of homology data, the usual conditions on the Beltrami differentials on a Riemann surface must be modified, because the homology data has more directions of variation than there are moduli of the Riemann surface. Perhaps the augmentations $\Sigma_{+}$of Riemann surfaces $\Sigma$ discussed in section 19.3 might have different moduli.

The variation of the partition function is given by the integral over $\Sigma$ of the (1,1)form $\left\langle T_{z z} h_{\bar{z}}^{z}\right\rangle d z d \bar{z}$ where $T_{z z}$ is the holomorphic (2,0)-component of the stress-energy
tensor. The considerations of section 16 might be used to equate this variation of the 2 d partition function with the variation of the space-time partition function with respect to the homology data of $M$.

### 21.6 Non-conformal extended quantum field theory

If extended CFTs in space-time can be built successfully from 2d CFTs, next will be to extend the correspondence to non-conformal quantum field theories by establishing an equivalence between renormalized perturbation theory around an extended CFT in spacetime and renormalized perturbation theory around the corresponding 2d CFT. Space-time will be $M=\mathbb{R}^{d}$. The corresponding two-dimensional space will be, presumably, $\Sigma=\mathbb{R}^{2}$. Perturbation theory in space-time will be renormalized with respect to the euclidean metric on $\mathbb{R}^{d}$ in order to construct a quantum field theory with euclidean symmetry, to be Wick-rotated to Minkowski space $\mathbb{R}^{1, d-1}$.

Some way is needed to transfer the euclidean metric on $\mathbb{R}^{d}$ to the two-dimensional space $\Sigma$. Perhaps the simplest route is to put a specific metric space structure on the quasi Riemann surface $\mathcal{Q}(M)_{\mathbb{Z} \partial \xi}$ using the euclidean metric on $\mathbb{R}^{d}$, then transfer the metric space structure to $\mathcal{Q}(\Sigma)$ via an isomorphism of quasi Riemann surfaces (assuming the conjecture). I do not know of any guarantee that the induced metric space structure on $\mathcal{Q}(\Sigma)$ would come from a Riemannian metric on $\Sigma$. So it would be necessary to investigate the renormalization of perturbations of an ECFT with respect to a metric space structure on $\mathcal{Q}(\Sigma)$. A very naive hope would be that this is the same as renormalization of perturbations of the underlying 2 d CFT on $\Sigma$ with respect to a metric on $\Sigma$, giving a correspondence between non-conformal 2 d quantum field theories on $\Sigma$ and non-conformal quantum field theories of extended objects in $\mathbb{R}^{d}$. Then one might wonder how the operator representation of the 2d QFT is related to the operator representation in the Minkowski space-time QFT, how the 2d S-matrix might be related to the space-time S-matrix, how 2d integrability might be manifested in the space-time QFT.

A fanciful prospect is an integrable, asymptotically free 2d QFT with nonabelian global symmetry corresponding to an asymptotically free 4d quantum field theory of extended objects with nonabelian local gauge symmetry.

## 22 Questions about history and references

I would appreciate advice on the history of ideas germane to this work, and on their proper citation. To be safe, I have cited the basic works on the free 2 -form in $\mathrm{d}=4$ dimensions [1]-3].

The proximate influences on the present work were

- G. Moore's frequent comments that a QFT cannot be understood without understanding its extended objects,
- N. Seiberg's September 23, 2014 Rutgers seminar on his paper with Gaiotto, Kapustin, and Willett on generalized charges in quantum field theory, Generalized
global symmetries [8].
- many conversations with S. Thomas and C. Keller about conformal field theory in dimensions $d>2$, and
- especially a comment of S. Thomas suggesting that nonabelian structure might be found in the extended objects, after the realization [9] that nonabelian operator algebras of self-dual $n$-forms were impossible in $d>2$ dimensions,
- a search of the World Wide Web for 'spaces of cycles', which yielded a number of pointers to geometric measure theory, especially Gromov's 2015 note Morse Spectra, Homology Measures, Spaces of Cycles and Parametric Packing Problems [10].

In statistical mechanics, the 2 d gaussian model as the free 1 -form with compact symmetry group $U(1) \times U(1)$ should perhaps be attributed to Mermin-Wagner-Hohenberg and Berezinskii-Kosterlitz-Thouless.

In string theory, the free 1-form in 2d was the basic tool of the world-sheet technology. The decomposition of the free 1-form into chiral components and the vertex operator representation of the "extended objects" were done in string theory. The compactification of the target to a circle of radius $R$ was the implementation of Kaluza-Klein theory.

I am not sure where the name 2d gaussian model came from. Free massless scalar field theory is known in statistical mechanics as the gaussian model. In 2d, the logarithmic infrared divergence forces compactification of the target space to a circle of radius $R$. Maybe I learned to call this the $2 d$ gaussian model from Leo Kadanoff in the early 1980s.

Who first noted the analogy between the free 1-form in 2d and Maxwell's theory in 4 d and the generalization to the free $n$-form in $d=2 n$ dimensions? The line integrals of gauge potentials in nonabelian gauge theory are called Wilson loop operators. Where was it noticed that these are the operators describing extended objects in the free $n$-form theory?

There is a large mathematics and physics literature on the free $n$-form theories in $d=2 n$ dimensions with $n$ odd, especially the theories of self-dual $n$-forms - in particular, Hopkins and Singer, Quadratic functions in geometry, topology, and M-theory [11] and Freed, Moore, and Graeme Segal, Heisenberg groups and noncommutative fluxes [12]. These papers are too far over my head for me to tell if they are germane to the present work. I do notice that the latter paper discusses the analogy between the free $n$-form theory and the 2 d gaussian model, and some of its calculations seem to resemble those on gauge invariance in section 17. In any case, if the present project works out, there will surely be points of contact.

## Geometric measure theory

I have cited what I understand to be the basic references for geometric measure theory [4-6], but my grasp of the subject is weak and superficial.

## Acknowledgments

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## Appendices

## A Construction of a path of integral currents a la Game of Thrones

This section gives a construction that should be the germ of a proof that the Hodge *operator acts on the tangent spaces of the space $\mathcal{D}_{n-1}^{\text {int }}(M)_{0}$ of integral ( $n-1$ )-cycles in a conformal manifold $M$ of dimension $d=2 n$.

As in section 3 above, we consider the path of singular 1-cycles in $\mathbb{R}^{4}$,

$$
\begin{equation*}
\xi(\epsilon)=\partial\left[\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \theta_{[0, \epsilon]}\left(x^{4}\right) \hat{e}_{3} \wedge \hat{e}_{4}\right] \tag{A.1}
\end{equation*}
$$

where $\theta_{[a, b]}$ is the characteristic function of the interval $[a, b] \subset \mathbb{R}$. The 1-cycles $\xi(\epsilon)$ are the boundaries of rectangles in the 3-4 plane in $\mathbb{R}^{4}$, shrinking to the interval $[0,1]$ in the 3 -axis. The tangent vector to the path $\xi(\epsilon)$ at $\epsilon=0$ is the flat 2 -current

$$
\begin{equation*}
\dot{\xi}=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \delta\left(x^{4}\right) \hat{e}_{3} \wedge \hat{e}_{4} . \tag{A.2}
\end{equation*}
$$

The Hodge $*$-operator on $\dot{\xi}$ is

$$
\begin{equation*}
* \dot{\xi}=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \delta\left(x^{4}\right) \hat{e}_{1} \wedge \hat{e}_{2} \tag{A.3}
\end{equation*}
$$

The flat 2 -current $* \dot{\xi}$ is not tangent to any path of singular 1-cycles.
Here, a path $\xi_{1}^{i n t}(\epsilon)$ of integral 1-cycles is constructed whose tangent vector is

$$
\begin{equation*}
\dot{\xi}_{1}^{\mathrm{int}}=* \dot{\xi} . \tag{A.4}
\end{equation*}
$$

The rest of the proof might then go:

1. Use the rotations in $\mathbb{R}^{4}$ that leave the 3-axis fixed to get paths of integral cycles with tangent vectors of the form

$$
\begin{equation*}
\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \delta\left(x^{4}\right) t^{\mu \nu} \tag{A.5}
\end{equation*}
$$

for arbitrary 2 -vector $t^{\mu \nu}$.
2. Scale and rotate in $\mathbb{R}^{4}$ to get paths with tangent vectors $\theta_{I} t^{\mu \nu}$, where $\theta_{I}$ is the characteristic 0-current of any interval $I$ in $\mathbb{R}^{4}$, arbitrarily small, and where $t^{\mu \nu}$ is any 2 -vector. This space of flat 2 -currents is manifestly closed under the action of the Hodge $*$-operator.
3. Using coordinates in $M$, take limits of linear combinations of the paths constructed in the previous step to get paths of integral 1-currents with tangent vector any flat 2-current in $M$ that is supported on an integral 1-current in $M$.
4. Thus the tangent space at 0 of the space of integral 1 -cycles in $M$ is exactly the space of flat 2-currents in $M$ that are supported on the integral 1-currents in $M$.
5. The Hodge *-operator manifestly acts as a bounded operator on the space of flat 2-currents that are supported on the singular 1-crrents, so it acts on the tangent space.

Generalizing the construction of the path $\xi_{1}^{\text {int }}(\epsilon)$ to arbitrary $d=2 n$ will be straightforward, and the rest of the proof is the same as in $d=4$ dimensions.

The construction of the path $\xi_{1}^{\text {int }}(\epsilon)$ takes place within $\mathbb{R}^{3} \subset \mathbb{R}^{4}$. First, an integral 2 -current $\xi_{2}^{\text {int }}$ is constructed, supported in the unit cube $([0,1])^{3} \subset \mathbb{R}^{3}$. Then a path of integral 2 -currents $\xi_{2}^{\text {int }}(\epsilon)$ is constructed by scaling $\xi_{2}^{\text {int }}$ in the 1-2 plane, under

$$
\begin{equation*}
\left(x^{1}, x^{2}, x^{3}\right) \mapsto\left(\epsilon^{\frac{1}{2}} x^{1}, \epsilon^{\frac{1}{2}} x^{2}, x^{3}\right) \tag{A.6}
\end{equation*}
$$

As $\epsilon \rightarrow 0$, the original 2 -current $\xi_{2}^{\mathrm{int}}$ is squashed onto the interval $[0,1]$ in the 3 -axis. The construction is designed so that $\lim _{\epsilon \rightarrow 0} \xi_{2}^{\text {int }}(\epsilon)=* \dot{\xi}$. So the path of integral 1-cycles $\xi_{1}^{\text {int }}(\epsilon)=\partial \xi_{2}^{\text {int }}(\epsilon)$ has tangent vector $\dot{\xi}_{1}^{\text {int }}=* \dot{\xi}$.

The integral 2-current $\xi_{2}^{\text {int }}$ is a fractal. Its construction is illustrated in the attached animations GOT1.gif and GOT2.gif, which are also available at

```
http://www.physics.rutgers.edu/pages/friedan/GOT/.
```

Animated gifs can be viewed in most web browsers. The animated gifs were made with Sagemath [13] and ImageMagick [14]. The visualized construction bears a slight resemblance to a part of the title sequence animation of the television show Game of Thrones [15].

Define the 2 -current $S\left(y^{1}, y^{2}, y^{3} ; a\right)$ to be the square in the 1-2 plane with corner at $\vec{y}=\left(y^{1}, y^{2}, y^{3}\right)$ and side $a$,

$$
\begin{equation*}
S(\vec{y} ; a)=\theta_{\left[y_{1}, y_{1}+a\right]}\left(x^{1}\right) \theta_{\left[y_{2}, y_{2}+a\right]}\left(x^{2}\right) \delta\left(x^{3}-y^{3}\right) \hat{e}_{1} \wedge \hat{e}_{2} . \tag{A.7}
\end{equation*}
$$

Define an operator $R(b)$ on such squares that splits the square into 4 quadrants and lifts the quadrants in the 3 -direction, not lifting the first quadrant, lifting the second quadrant by $b / 4$, lifting the third quadrant by $2 b / 4$, and lifting the fourth quadrant by $3 b / 4$,

$$
\begin{equation*}
R(b) S(\vec{y} ; a)=\sum_{i=1}^{4} S\left(\overrightarrow{y_{i}} ; a / 2\right) \tag{A.8}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\vec{y}_{4}=\vec{y}+\frac{a}{2} \hat{e}_{2}+\frac{3 b}{4} \hat{e}_{3}, & \overrightarrow{y_{3}}=\vec{y}+\frac{a}{2}\left(\hat{e}_{1}+\hat{e}_{2}\right)+\frac{2 b}{4} \hat{e}_{3} \\
\overrightarrow{y_{1}}=\vec{y}, & \vec{y}_{2}=\vec{y}+\frac{a}{2} \hat{e}_{1}+\frac{b}{4} \hat{e}_{3} . \tag{A.10}
\end{array}
$$

Then extend $R(b)$ to sums of squares by linearity over the integers.
Let $\xi_{2,0}^{\text {int }}$ be the square in the $1-2$ plane of side 1 with corner at the origin,

$$
\begin{equation*}
\xi_{2,0}^{\mathrm{int}}=S(\overrightarrow{0} ; 1) \tag{A.11}
\end{equation*}
$$

Define a sequence of integral 2 -currents $\xi_{2,0}^{\mathrm{int}}, \xi_{2,1}^{\mathrm{int}}, \xi_{2,2}^{\mathrm{int}}, \ldots$ by

$$
\begin{equation*}
\xi_{2, k+1}^{\mathrm{int}}=R\left(4^{-k}\right) \xi_{2, k}^{\mathrm{int}} \tag{A.12}
\end{equation*}
$$

so $\xi_{2, k}^{\mathrm{int}}$ is a sum of $4^{k}$ squares in the 1-2 plane, each of area $4^{-k}$, at heights

$$
\begin{equation*}
x^{3}=j 4^{-k}, \quad j=0,1,2, \ldots, 4^{k}-1 . \tag{A.13}
\end{equation*}
$$

The total area is left unchanged by $R(b)$, so

$$
\begin{equation*}
\left\|\xi_{2, k}^{\mathrm{int}}\right\|_{\text {flat }}=1 \tag{A.14}
\end{equation*}
$$

The operator $R(b)$ acting on a square sweeps out a 3 -chain in the shape of a 4 -step spiral staircase with square steps of heights $0, b / 4,2 b / 4$, and $3 b / 4$. Let $S_{3}$ be the 3 -current representing the staircase and let $S_{2}$ be the 2-current representing the vertical sides of the staircase, so

$$
\begin{equation*}
R(b) S(\vec{y} ; a)-S(\vec{y}, a)=\partial S_{3}-S_{2} \tag{A.15}
\end{equation*}
$$

The volume of the staircase is

$$
\begin{equation*}
M_{3}\left(S_{3}\right)=\frac{b}{4}\left(\frac{a}{2}\right)^{2}+\frac{2 b}{4}\left(\frac{a}{2}\right)^{2}+\frac{3 b}{4}\left(\frac{a}{2}\right)^{2}=\frac{3}{8} b a^{2} \tag{A.16}
\end{equation*}
$$

The vertical sides consist of $24-6=18$ rectangles of vertical side $b / 4$ and horizontal side $a / 2$, so the area of the vertical sides is

$$
\begin{equation*}
M_{2}\left(S_{2}\right)=18 \frac{b}{4} \frac{a}{2}=\frac{9}{4} b a \tag{A.17}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\|R(b) S(\vec{y} ; a)-S(\vec{y}, a)\|_{f l a t} \leq M_{2}\left(S_{2}\right)+M_{3}\left(S_{3}\right)=\frac{9}{4} b a+\frac{3}{8} b a^{2} \tag{A.18}
\end{equation*}
$$

In $\xi_{2, k}^{\text {int }}$, there are $4^{k}$ squares, each with $a=2^{-k}$, so

$$
\begin{equation*}
\left\|\xi_{2, k+1}^{\mathrm{int}}-\xi_{2, k}^{\mathrm{int}}\right\|_{f l a t} \leq 4^{k}\left[\frac{9}{4} 4^{-k} 2^{-k}+\frac{3}{8} 4^{-k}\left(2^{-k}\right)^{2}\right]=\frac{9}{4} 2^{-k}+\frac{3}{8} 4^{-k} \tag{A.19}
\end{equation*}
$$

so $\xi_{2, k}^{\mathrm{int}}$ is a Cauchy sequence. The space of integral currents with bounded norm and bounded support is compact, so Cauchy sequences converge. Let

$$
\begin{equation*}
\xi_{2}^{\mathrm{int}}=\lim _{k \rightarrow \infty} \xi_{2, k}^{\mathrm{int}} \tag{A.20}
\end{equation*}
$$

Scale $\xi_{2}^{\mathrm{int}}$ and the $\xi_{2, k}^{\mathrm{int}}$ by $\epsilon^{\frac{1}{2}}$ in the 1-2 plane to get

$$
\begin{equation*}
\xi_{2}^{\mathrm{int}}(\epsilon)=\lim _{k \rightarrow \infty} \xi_{2, k}^{\mathrm{int}}(\epsilon) \tag{A.21}
\end{equation*}
$$

with

$$
\begin{equation*}
\left\|\xi_{2}^{\mathrm{int}}\right\|_{f l a t}=\left\|\xi_{2, k}^{\mathrm{int}}\right\|_{f l a t}=1 \tag{A.22}
\end{equation*}
$$

The integral 2-current $\xi_{2, k}^{\mathrm{int}}(\epsilon)$ consists of the square $\left[0, \epsilon^{\frac{1}{2}}\right] \times\left[0, \epsilon^{\frac{1}{2}}\right]$ in the 1-2 plane divided into a checkerboard of $4^{k}$ squares each of area $4^{-k} \epsilon$, each small square raised to one of the evenly distributed heights,

$$
\begin{equation*}
x^{3}=j 4^{-k}, \quad j=0,1,2, \ldots, 4^{k}-1 \tag{A.23}
\end{equation*}
$$

As $\epsilon \rightarrow 0$, each of the $4^{k}$ small squares is squashed onto the 3 -axis. For $\omega$ a smooth 2-form,

$$
\begin{equation*}
\int_{\xi_{2}^{\text {int }}(\epsilon)} \omega=\lim _{k \rightarrow \infty} \int_{\xi_{2, k}^{\operatorname{int}}(\epsilon)} \omega \approx \lim _{k \rightarrow \infty} \sum_{j=0}^{4^{k}-1} 4^{-k} \epsilon \omega_{12}\left(0,0,4^{-k} j\right) \approx \epsilon \int_{0}^{1} d x^{3} \omega_{12}\left(0,0, x^{3}\right) \tag{A.24}
\end{equation*}
$$

so

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{\xi_{2}^{\mathrm{int}}(\epsilon)}{\epsilon}=\delta\left(x^{1}\right) \delta\left(x^{2}\right) \theta_{[0,1]}\left(x^{3}\right) \hat{e}_{1} \wedge \hat{e}_{2}=* \dot{\xi} \tag{A.25}
\end{equation*}
$$

Therefore the tangent vector at $\epsilon=0$ to the path of integral 1-cycles

$$
\begin{equation*}
\xi_{1}^{i n t}(\epsilon)=\partial \xi_{2}^{\mathrm{int}}(\epsilon) \tag{A.26}
\end{equation*}
$$

is

$$
\begin{equation*}
\dot{\xi}_{1}^{\mathrm{int}}=* \dot{\xi} . \tag{A.27}
\end{equation*}
$$

## B The free complex $n$-form on euclidean $\mathbb{R}^{d}$

In this section, the Schwinger-Dyson equations are written for the chiral fields $F_{ \pm}(x)$ and $A_{ \pm}(x)$ of the free complex $n$-form quantum field theory on euclidean $\mathbb{R}^{d}, d=2 n$. The S-D equations on $M=\mathbb{R}^{d}$ determine the S-D equations on any manifold $M$, by dimensional analysis. There are some arbitrary choices: (1) the overall normalization of the two-point functions, and (2) the contact terms in the two-point functions. The overall normalization is fixed by matching to a standard convention in the 2 d theory. The contact terms are fixed by imposing symmetry.

The notation is as in sections 1.1, 1.2, and 7 ,

## B. 1 Adjoints of $F$ and $F^{*}$

Wick rotate to Minkowski space. Write $x^{i}, i=1, \ldots, d-1$ for the spatial coordinates. Write $x^{d}$ for euclidean time and $x^{0}$ for Minkowski space time, with $x^{d}=i x^{0}$.

The magnetic field (up to normalization) is $F_{i_{1} \ldots i_{n}}(x)$. The electric field (up to normalization) is $F_{i_{1} \ldots i_{n-1} 0}(x)=i F_{i_{1} \ldots i_{n-1} d}(x)$. For real $F$, the magnetic and electric fields are self-adjoint, so, for complex $F$,

$$
\begin{equation*}
F_{i_{1} \ldots i_{n}}^{\dagger}(x)=\bar{F}_{i_{1} \ldots i_{n}}(x), \quad F_{i_{1} \ldots i_{n-1} d}^{\dagger}(x)=-\bar{F}_{i_{1} \ldots i_{n-1} d}(x) . \tag{B.1}
\end{equation*}
$$

The Hodge *-operator acts by

$$
\begin{equation*}
* F_{\mu_{1} \ldots \mu_{n}}=\frac{1}{n!} \epsilon_{\mu_{1} \ldots \mu_{n}}{ }^{\nu_{1} \ldots \nu_{n}} F_{\nu_{1} \ldots \nu_{n}} \tag{B.2}
\end{equation*}
$$

so

$$
\begin{equation*}
* F_{i_{1} \ldots i_{n}}=\frac{1}{(n-1)!} \epsilon_{i_{1} \ldots i_{n}}{ }^{j_{1} \ldots j_{n-1} d} F_{j_{1} \ldots j_{n-1} d}, \quad * F_{i_{1} \ldots i_{n-1} d}=\frac{1}{n!} \epsilon_{i_{1} \ldots i_{n-1} d}{ }^{j_{1} \ldots j_{n}} F_{j_{1} \ldots j_{n}} \tag{B.3}
\end{equation*}
$$

so

$$
\begin{equation*}
\left(* F_{i_{1} \ldots i_{n}}\right)^{\dagger}=-* \bar{F}_{i_{1} \ldots i_{n}}, \quad\left(* F_{i_{1} \ldots i_{n-1} d}\right)^{\dagger}=* \bar{F}_{i_{1} \ldots i_{n-1} d} \tag{B.4}
\end{equation*}
$$

so, if we define the dual $n$-form

$$
\begin{equation*}
F^{*}=\delta_{n} i^{-1} * F, \quad \delta_{n}= \pm 1 \tag{B.5}
\end{equation*}
$$

then $F^{*}$ has the same self-adjointness properties as $F$. That is, in Minkowski space,

$$
\begin{equation*}
F^{\dagger}(x)=\bar{F}(x), \quad F^{* \dagger}(x)=\bar{F}^{*}(x)=-\bar{F}^{*}(x) \tag{B.6}
\end{equation*}
$$

The choice of $\delta_{n}= \pm 1$ will be left arbitrary in this section. In the body of the paper, $\delta_{n}=1$ is used.

## B. 2 Notation: adjoints of euclidean fields

Define the adjoint of a euclidean field to be the Wick-rotate of the adjoint of the Minkowski field:

$$
\begin{equation*}
F^{\dagger}(x)=\bar{F}(x), \quad F^{* \dagger}(x)=\bar{F}^{*}(x)=-\bar{F}^{*}(x) \tag{B.7}
\end{equation*}
$$

Reflection positivity of the euclidean correlation functions is then

$$
\begin{equation*}
\left\langle F^{\dagger}(R x) F(x)\right\rangle>0, \quad R x \neq x \tag{B.8}
\end{equation*}
$$

where $R$ is the reflection $x^{d} \rightarrow-x^{d}$.

## B. 3 The chiral fields $F_{ \pm}$and $A_{ \pm}$and their adjoints

The chiral components of $F$ and $A$ and their adjoints are

$$
\begin{align*}
& F_{ \pm}=\frac{1}{2}\left(1 \pm i^{-1} J\right) F=\frac{1}{2}\left(1 \pm i^{-1} \epsilon_{n} *\right) F=\frac{1}{2}\left(F \pm \epsilon_{n} \delta_{n} F^{*}\right),  \tag{B.9}\\
& A_{ \pm}=\frac{1}{2}\left(A \pm \epsilon_{n} \delta_{n} A^{*}\right)  \tag{B.10}\\
& F_{ \pm}^{\dagger}=\frac{1}{2}\left(F^{\dagger} \pm \bar{\epsilon}_{n} \delta_{n} F^{* \dagger}\right)=\frac{1}{2}\left(\bar{F} \mp \bar{\epsilon}_{n} \delta_{n} \bar{F}^{*}\right)=\bar{F}_{\mp},  \tag{B.11}\\
& A_{ \pm}^{\dagger}=\bar{A}_{\mp} . \tag{B.12}
\end{align*}
$$

## B. 4 The two-point functions and the Schwinger-Dyson equations

$F(x)$ has scaling dimension $n$, so its two-point functions are linear combinations of two invariants, one of which is a pure contact term. The non-contact invariant is

$$
\begin{gather*}
G\left(\bar{\xi}_{1}, \xi_{2}\right)=\int d^{n} x \int d^{n} y \frac{1}{n!} \bar{\xi}_{1}^{\mu_{1} \ldots \mu_{n}}(x) G(x, y)_{\mu_{1} \ldots \mu_{n} ; \nu_{1} \ldots \nu_{n}} \frac{1}{n!} \xi_{2}^{\nu_{1} \ldots \nu_{n}}(y) \\
G(x, y)_{\mu_{1} \ldots \mu_{n} ; \nu_{1} \ldots \nu_{n}}=\int \frac{d^{d} p}{(2 \pi)^{d}} e^{i p(x-y)} G(p)_{\mu_{1} \ldots \mu_{n} ; \nu_{1} \ldots \nu_{n}}  \tag{B.13}\\
G(p)_{\mu_{1} \ldots \mu_{n} ; \nu_{1} \ldots \nu_{n}}=\frac{1}{p^{2}} \frac{1}{(n-1)!} \operatorname{Alt}_{\mu} \operatorname{Alt}_{\nu}\left(p_{\mu_{1}} p_{\nu_{1}} \delta_{\mu_{2} \nu_{2}} \cdots \delta_{\mu_{n} \nu_{n}}\right) .
\end{gather*}
$$

The pure contact invariant is given in terms of the intersection form,

$$
\begin{equation*}
I\left(\bar{\xi}_{1}, * \xi_{2}\right)=\int d^{n} x \frac{1}{n!} \bar{\xi}_{1}^{\mu_{1} \ldots \mu_{n}}(x) \operatorname{Alt}_{\mu}\left(\delta_{\mu_{1} \nu_{1}} \cdots \delta_{\mu_{n} \nu_{n}}\right) \frac{1}{n!} \xi_{2}^{\nu_{1} \ldots \nu_{n}}(x) \tag{B.14}
\end{equation*}
$$

They satisfy

$$
\begin{gather*}
\left.\left.\overline{G\left(\bar{\xi}_{1}, \xi_{2}\right.}\right)=G\left(\bar{\xi}_{2}, \xi_{1}\right), \quad \overline{I_{M}\left(\bar{\xi}_{1}, * \xi_{2}\right.}\right)=I_{M}\left(\bar{\xi}_{2}, * \xi_{1}\right),  \tag{B.15}\\
G\left(\bar{\xi}_{1}, \partial \xi_{2}\right)=0, \quad G\left(\overline{\partial \xi_{1}}, \xi_{2}\right)=0,  \tag{B.16}\\
G\left(\bar{\xi}_{1}, \xi_{2}\right)+G\left(\bar{\xi}_{1}, * \xi_{2}\right)=I_{M}\left(\bar{\xi}_{1}, * \xi_{2}\right) . \tag{B.17}
\end{gather*}
$$

A simple way to derive the last equation, (B.17), is by calculating

$$
\begin{equation*}
G(p)_{1,2, \ldots, n ; 1,2, \ldots, n}=\frac{1}{p^{2}} \sum_{i=1}^{n} p_{i}^{2}, \quad G(p)_{n+1, n+2, \ldots, d ; n+1, n+2, \ldots, d}=\frac{1}{p^{2}} \sum_{i=n+1}^{d} p_{i}^{2} . \tag{B.18}
\end{equation*}
$$

Equation (B.17) is equivalent to

$$
\begin{equation*}
G\left(\overline{P_{+} \xi_{1}}, P_{+} \xi_{2}\right)=\frac{i}{2} I_{M}\left\langle\bar{\xi}_{1}, P_{+} \xi_{2}\right\rangle, \quad G\left(\overline{P_{-} \xi_{1}}, P_{-} \xi_{2}\right)=-\frac{i}{2} I_{M}\left\langle\bar{\xi}_{1}, P_{-} \xi_{2}\right\rangle \tag{B.19}
\end{equation*}
$$

The first S-D equations are imposed by fiat,

$$
\begin{equation*}
\left\langle F^{\dagger}\left(\bar{\xi}_{1}\right) F\left(\partial \xi_{2}\right)\right\rangle=0, \quad\left\langle F^{* \dagger}\left(\bar{\xi}_{1}\right) F^{*}\left(\partial \xi_{2}\right)\right\rangle=0 \tag{B.20}
\end{equation*}
$$

These determine two of the two-point functions,

$$
\begin{equation*}
\left\langle F^{\dagger}\left(\bar{\xi}_{1}\right) F\left(\xi_{2}\right)\right\rangle=B_{n} G\left(\bar{\xi}_{1}, \xi_{2}\right), \quad\left\langle F^{* \dagger}\left(\bar{\xi}_{1}\right) F^{*}\left(\xi_{2}\right)\right\rangle=B_{n} G\left(\bar{\xi}_{1}, \xi_{2}\right) \tag{B.21}
\end{equation*}
$$

where $B_{n}$ is a real constant. By (B.18), reflection positivity implies

$$
\begin{equation*}
B_{n}>0 . \tag{B.22}
\end{equation*}
$$

The remaining two-point functions have the form

$$
\begin{align*}
& \left\langle F^{* \dagger}\left(\bar{\xi}_{1}\right) F\left(\xi_{2}\right)\right\rangle=B_{n} i^{-1} \delta_{n}\left[G\left(\bar{\xi}_{1}, \xi_{2}\right)+b_{n} I_{M}\left(\bar{\xi}_{1}, \xi_{2}\right)\right]  \tag{B.23}\\
& \left\langle F^{\dagger}\left(\bar{\xi}_{1}\right) F^{*}\left(\xi_{2}\right)\right\rangle=B_{n} i^{-1} \delta_{n}\left[G\left(\bar{\xi}_{1}, * \xi_{2}\right)+\bar{b}_{n} I_{M}\left(\xi_{2}, \bar{\xi}_{1}\right)\right] \tag{B.24}
\end{align*}
$$

for some complex constant $b_{n}$. The two-point functions of the chiral components are then

$$
\begin{align*}
\left\langle F_{+}^{\dagger}\left(\bar{\xi}_{1}\right) F_{-}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[-\frac{i}{4}\left(1+b_{n}+\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.25}\\
\left\langle F_{-}^{\dagger}\left(\bar{\xi}_{1}\right) F_{+}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[\frac{i}{4}\left(1+b_{n}+\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.26}\\
\left\langle F_{+}^{\dagger}\left(\bar{\xi}_{1}\right) F_{+}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[G\left(\bar{P}_{-} \bar{\xi}_{1}, \xi_{2}\right)+\frac{i}{4}\left(1-b_{n}+\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.27}\\
& =B_{n}\left[G\left(\bar{\xi}_{1}, P_{+} \xi_{2}\right)-\frac{i}{4}\left(1+b_{n}-\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.28}\\
\left\langle F_{-}^{\dagger}\left(\bar{\xi}_{1}\right) F_{-}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[G\left(\bar{P}_{+} \xi_{1}, \xi_{2}\right)-\frac{i}{4}\left(1-b_{n}+\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.29}\\
& =B_{n}\left[G\left(\bar{\xi}_{1}, P_{-} \xi_{2}\right)+\frac{i}{4}\left(1+b_{n}-\bar{b}_{n}\right) I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right] \tag{B.30}
\end{align*}
$$

The most symmetric choice is $b_{n}=-\frac{1}{2}$, giving

$$
\begin{align*}
\left\langle F_{+}^{\dagger}\left(\bar{\xi}_{1}\right) F_{-}\left(\xi_{2}\right)\right\rangle & =0  \tag{B.31}\\
\left\langle F_{-}^{\dagger}\left(\bar{\xi}_{1}\right) F_{+}\left(\xi_{2}\right)\right\rangle & =0  \tag{B.32}\\
\left\langle F_{+}^{\dagger}\left(\bar{\xi}_{1}\right) F_{+}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[G\left(\overline{P_{-} \xi_{1}}, \xi_{2}\right)+\frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.33}\\
& =B_{n}\left[G\left(\bar{\xi}_{1}, P_{+} \xi_{2}\right)-\frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.34}\\
\left\langle F_{-}^{\dagger}\left(\bar{\xi}_{1}\right) F_{-}\left(\xi_{2}\right)\right\rangle & =B_{n}\left[G\left(\overline{P_{+} \xi_{1}}, \xi_{2}\right)-\frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right]  \tag{B.35}\\
& =B_{n}\left[G\left(\bar{\xi}_{1}, P_{-} \xi_{2}\right)+\frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \xi_{2}\right\rangle\right] \tag{B.36}
\end{align*}
$$

The S-D equations follow immediately,

$$
\begin{align*}
& \left\langle F_{+}^{\dagger}\left(\bar{\xi}_{1}\right) F_{+}\left(\partial \xi_{2}\right)\right\rangle=B_{n} \frac{i}{4} I\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle \quad=-B_{n} \frac{i}{4} I_{M}\left\langle\partial \bar{\xi}_{1}, \xi_{2}\right\rangle  \tag{B.37}\\
& \left\langle F_{+}^{\dagger}\left(\overline{\xi \xi}_{1}\right) F_{+}\left(\xi_{2}\right)\right\rangle=-B_{n} \frac{i}{4} I_{M}\left\langle\partial \bar{\xi}_{1}, \xi_{2}\right\rangle=B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle  \tag{B.38}\\
& \left\langle F_{-}^{\dagger}\left(\bar{\xi}_{1}\right) F_{-}\left(\partial \xi_{2}\right)\right\rangle=-B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle=B_{n} \frac{i}{4} I_{M}\left\langle\partial \bar{\xi}_{1}, \xi_{2}\right\rangle  \tag{B.39}\\
& \left\langle F_{-}^{\dagger}\left(\bar{\partial}_{1}\right) F_{-}\left(\xi_{2}\right)\right\rangle=B_{n} \frac{i}{4} I_{M}\left\langle\partial \bar{\xi}_{1}, \xi_{2}\right\rangle=-B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle . \tag{B.40}
\end{align*}
$$

These S-D equations are compatible with $d A_{ \pm}=F_{ \pm}$. Integrate them to get

$$
\begin{align*}
\left\langle A_{+}^{\dagger}\left(\bar{\xi}_{0}\right) F_{+}\left(\partial \xi_{2}\right)\right\rangle & =-B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{0}, \xi_{2}\right\rangle  \tag{B.41}\\
\left\langle F_{+}^{\dagger}\left(\overline{\partial \xi}_{2}\right) A_{+}\left(\xi_{0}\right)\right\rangle & =B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{2}, \xi_{0}\right\rangle  \tag{B.42}\\
\left\langle A_{-}^{\dagger}\left(\bar{\xi}_{0}\right) F_{-}\left(\partial \xi_{2}\right)\right\rangle & =B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{0}, \xi_{2}\right\rangle  \tag{B.43}\\
\left\langle F_{-}^{\dagger}\left(\overline{\partial \xi}_{2}\right) A_{-}\left(\xi_{0}\right)\right\rangle & =-B_{n} \frac{i}{4} I_{M}\left\langle\bar{\xi}_{2}, \xi_{0}\right\rangle \tag{B.44}
\end{align*}
$$

The normalization constants $B_{n}$ are fixed by matching to a standard convention in $d=2$ dimensions, as described in section B. 6 below,

$$
\begin{equation*}
B_{n}=8 \pi \tag{B.45}
\end{equation*}
$$

## B. 5 Summary

The Schwinger-Dyson equations for the chiral fields are

$$
\begin{align*}
& \left\langle F_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{1}\right) F_{\beta}\left(\partial \xi_{2}\right)\right\rangle=2 \pi i \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\bar{\xi}_{1}, \partial \xi_{2}\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\partial \bar{\xi}_{1}, \xi_{2}\right\rangle  \tag{B.46}\\
& \left\langle A_{\bar{\alpha}}^{\dagger}\left(\bar{\xi}_{0}\right) F_{\beta}\left(\partial \xi_{2}\right)\right\rangle=-2 \pi i \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\bar{\xi}_{0}, \xi_{2}\right\rangle \tag{B.47}
\end{align*}
$$

and the complex conjugate equations

$$
\begin{align*}
& \left\langle F_{\bar{\beta}}^{\dagger}\left(\overline{\partial \xi}_{2}\right) F_{\alpha}\left(\xi_{1}\right)\right\rangle=-2 \pi i \gamma_{\bar{\beta} \alpha} I_{M}\left\langle\partial \bar{\xi}_{2}, \xi_{1}\right\rangle=2 \pi i \gamma_{\bar{\beta} \alpha} I_{M}\left\langle\bar{\xi}_{2}, \partial \xi_{1}\right\rangle  \tag{B.48}\\
& \left\langle F_{\bar{\beta}}^{\dagger}\left(\overline{\partial \xi}_{2}\right) A_{\alpha}\left(\xi_{0}\right)\right\rangle=2 \pi i \gamma_{\bar{\beta} \alpha} I_{M}\left\langle\bar{\xi}_{2}, \xi_{0}\right\rangle \tag{B.49}
\end{align*}
$$

where

$$
\begin{equation*}
\gamma_{\mp+}=1, \quad \gamma_{\mp-}=0, \quad \gamma_{=+}=0, \quad \gamma=-=-1 . \tag{B.50}
\end{equation*}
$$

In terms of the fields $F, F^{*}, A, A^{*}$,

$$
\begin{equation*}
F=F_{+}+F_{-}, \quad F^{*}=\frac{1}{\epsilon_{n} \delta_{n}}\left(F_{+}-F_{-}\right), \quad A=A_{+}+A_{-}, \quad A^{*}=\frac{1}{\epsilon_{n} \delta_{n}}\left(A_{+}-A_{-}\right) \tag{B.51}
\end{equation*}
$$

the nontrivial S-D equations are

$$
\begin{align*}
\left\langle F^{* \dagger}\left(\bar{\xi}_{1}\right) F\left(\partial \xi_{2}\right)\right\rangle & =4 \pi i \delta_{n} I_{M}\left(\bar{\xi}_{1}, \partial \xi_{2}\right)=4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\overline{\partial \xi}_{1}, \xi_{2}\right)  \tag{B.52}\\
\left\langle F^{\dagger}\left(\bar{\xi}_{1}\right) F^{*}\left(\partial \xi_{2}\right)\right\rangle & =4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\bar{\xi}_{1}, \partial \xi_{2}\right)=4 \pi i \delta_{n} I_{M}\left(\overline{\partial \xi_{1}}, \xi_{2}\right)  \tag{B.53}\\
\left\langle A^{* \dagger}\left(\bar{\xi}_{0}\right) F\left(\partial \xi_{2}\right)\right\rangle & =4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\bar{\xi}_{0}, \xi_{2}\right)  \tag{B.54}\\
\left\langle A^{\dagger}\left(\bar{\xi}_{0}\right) F^{*}\left(\partial \xi_{2}\right)\right\rangle & =4 \pi i \delta_{n} I_{M}\left(\bar{\xi}_{0}, \xi_{2}\right) \tag{B.55}
\end{align*}
$$

and their complex conjugates (recalling that $F^{\dagger}=\bar{F}, F^{* \dagger}=-\bar{F}^{*}$ ),

$$
\begin{align*}
\left\langle F^{\dagger}\left(\overline{\partial \xi}_{2}\right) F^{*}\left(\xi_{1}\right)\right\rangle & =4 \pi i \delta_{n} I_{M}\left(\xi_{1}, \overline{\partial \xi}_{2}\right)=4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\partial \xi_{1}, \bar{\xi}_{2}\right)  \tag{B.56}\\
\left\langle F^{* \dagger}\left(\overline{\partial \xi}_{2}\right) F\left(\xi_{1}\right)\right\rangle & =4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\xi_{1}, \overline{\partial \xi}_{2}\right)=4 \pi i \delta_{n} I_{M}\left(\xi_{1}, \overline{\partial \xi_{2}}\right)  \tag{B.57}\\
\left\langle F^{\dagger}\left(\overline{\partial \xi}_{2}\right) A^{*}\left(\xi_{0}\right)\right\rangle & =4 \pi i \delta_{n}(-1)^{n} I_{M}\left(\xi_{0}, \bar{\xi}_{2}\right)  \tag{B.58}\\
\left\langle F^{* \dagger}\left(\overline{\partial g}_{2}\right) A\left(\xi_{0}\right)\right\rangle & =4 \pi i \delta_{n} I_{M}\left(\xi_{0}, \bar{\xi}_{2}\right) \tag{B.59}
\end{align*}
$$

## B. $6 d=2$

In $d=2$ dimensions, on $\mathbb{R}^{2}$, using coordinates $z=x^{1}+i x^{2}, w=y^{1}+i y^{2}$, a standard convention is

$$
\begin{align*}
& F_{+}=j(z) d z, \quad F_{+}^{\dagger}=j^{\dagger}(z) d z, \quad F_{-}=\bar{\jmath}(\bar{z}) d \bar{z}, \quad F_{-}^{\dagger}=\bar{\jmath}^{\dagger}(\bar{z}) d \bar{z}  \tag{B.60}\\
&\left\langle j^{\dagger}(z) j(w)\right\rangle=\frac{-2}{(z-w)^{2}},\left\langle\bar{\jmath}^{\dagger}(\bar{z}) \jmath(\bar{w})\right\rangle=\frac{-2}{(\bar{z}-\bar{w})^{2}} \tag{B.61}
\end{align*}
$$

The identity

$$
\begin{equation*}
\partial_{\bar{z}}\left(\frac{1}{z-w}\right)=\pi \delta^{2}(z-w) \tag{B.62}
\end{equation*}
$$

is

$$
\begin{equation*}
\frac{1}{z-w}=\int \frac{d^{2} p}{(2 \pi)^{2}} e^{i p(x-y)} \frac{(-4 \pi i) p_{z}}{p^{2}} \tag{B.63}
\end{equation*}
$$

where

$$
\begin{equation*}
p(x-y)=p_{z}(z-w)+p_{\bar{z}}(\bar{z}-\bar{w}), \quad p^{2}=4 p_{z} p_{\bar{z}} \tag{B.64}
\end{equation*}
$$

So

$$
\begin{equation*}
\frac{-2}{(z-w)^{2}}=2 \partial_{z}\left(\frac{1}{z-w}\right)=\int \frac{d^{2} p}{(2 \pi)^{2}} e^{i p(x-y)} \frac{8 \pi p_{z} p_{z}}{p^{2}} . \tag{B.65}
\end{equation*}
$$

Comparing to equation ( $\overline{\mathrm{B} .13}$ ) gives

$$
\begin{equation*}
\left\langle j^{\dagger}(z) d z j(w) d w\right\rangle=8 \pi G(x, y)_{\mu ; \nu} d x^{\mu} d x^{\nu} \tag{B.66}
\end{equation*}
$$

Comparing to equation (B.34) - away from coincident points so the contact terms can be ignored - gives the normalization

$$
\begin{equation*}
B_{n}=8 \pi \tag{B.67}
\end{equation*}
$$

## C Vertex operators and the Dirac quantization condition

## C. 1 The vertex operators

The vertex operators for the complex scalar on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ are, in terms of $\phi$ and $\phi^{*}$,

$$
\begin{gather*}
V_{p, p^{*}}(\eta)=e^{i p \cdot \phi(\eta)+i p^{*} \cdot \phi^{*}(\eta)}, \quad \eta \in \mathcal{D}_{0}^{\text {int }}\left(\mathcal{E}_{\partial \xi}^{\mathbb{C}}\right)  \tag{C.1}\\
p \cdot \phi(\eta)=\frac{1}{2}\left[\bar{p} \phi(\eta)+p \phi^{\dagger}(\bar{\eta})\right], \quad p^{*} \cdot \phi^{*}(\eta)=\frac{1}{2}\left[\bar{p}^{*} \phi^{*}(\eta)+p^{*} \phi^{* \dagger}(\bar{\eta})\right] . \tag{C.2}
\end{gather*}
$$

The reality condition is $p=\bar{p}, p^{*}=\bar{p}^{*}$. In terms of the chiral fields, the vertex operators are

$$
\begin{gather*}
V_{p^{+}, p^{-}}(\eta)=e^{i p^{+} \cdot \phi_{+}(\eta)+i p^{-} \cdot \phi_{-}(\eta)}  \tag{C.3}\\
p^{+} \cdot \phi_{+}(\eta)=\frac{1}{2}\left[\overline{p^{+}} \phi_{+}(\eta)+p^{+} \phi_{+}^{\dagger}(\bar{\eta})\right], \quad p^{-} \cdot \phi_{-}(\eta)=\frac{1}{2}\left[\overline{p^{-}} \phi_{-}(\eta)+p^{-} \phi_{-}^{\dagger}(\bar{\eta})\right], \tag{C.4}
\end{gather*}
$$

$$
\begin{equation*}
p^{+}=p+\epsilon_{n} p^{*}, \quad p^{-}=p-\epsilon_{n} p^{*} \tag{C.5}
\end{equation*}
$$

The vertex operators satisfy the operator product equations

$$
\begin{align*}
& V_{p^{+}, p^{-}}\left(\eta_{0}\right) j_{\beta}\left(\partial \eta_{2}\right)=-\pi i \bar{p}^{\bar{\alpha}} \gamma_{\bar{\alpha} \beta} I_{M}\left\langle\bar{\eta}_{0}, \eta_{2}\right\rangle V_{p^{+}, p^{-}}\left(\eta_{0}\right)  \tag{C.6}\\
& j_{\bar{\beta}}^{\dagger}\left(\overline{\partial \eta_{2}}\right) V_{p^{+}, p^{-}}\left(\eta_{0}\right)=\pi i p^{\alpha} \gamma_{\bar{\beta} \alpha} I_{M}\left\langle\bar{\eta}_{2}, \eta_{0}\right\rangle V_{p^{+}, p^{-}}\left(\eta_{0}\right) \tag{C.7}
\end{align*}
$$

expressing the generalized $U(1)$ charges [8] of the extended objects.

## C. 2 The Dirac quantization condition

The Dirac quantization condition is derived from the requirement that the correlation functions of the vertex operators be single-valued on $\mathcal{E}_{\partial \xi}$.

Let $\xi_{0}$ be a point in $\mathcal{E}_{\partial \xi}$ and let $\eta_{0}=\delta_{\xi_{0}}$ be the 0 -current in $\mathcal{E}_{\partial \xi}$ representing $\xi_{0}$. Let $\eta_{2}$ represent a 2 -disk in $\mathcal{E}_{\partial \xi}$ such that the $(n-1)$-current $\xi_{0}=\Pi_{*} \eta_{0}$ and the ( $n+1$ )-current $\Pi_{*} \eta_{2}$ have intersection number 1 in $M$,

$$
\begin{equation*}
\Pi^{*} I_{M}\left(\eta_{0}, \eta_{2}\right)=1 \tag{C.8}
\end{equation*}
$$

Then the skew-hermitian $M$-intersection form has values

$$
\begin{equation*}
\Pi^{*} I_{M}\left\langle\bar{\eta}_{0}, \eta_{2}\right\rangle=\epsilon_{n}, \quad \Pi^{*} I_{M}\left\langle\bar{\eta}_{2}, \eta_{0}\right\rangle=-\bar{\epsilon}_{n} \tag{C.9}
\end{equation*}
$$

Consider the product of vertex operators

$$
\begin{equation*}
V_{p^{+}, p^{-}}\left(\delta_{\xi_{0}}\right) V_{q^{+}, q^{-}}\left(\delta_{\xi_{1}}\right) \tag{C.10}
\end{equation*}
$$

as $\xi_{1}$ moves around the boundary $\partial \eta_{2}$ of the disk in $\mathcal{E}_{\partial \xi}$ represented by $\eta_{2}$. The monodromy will be

$$
\begin{equation*}
e^{-\left\langle p^{+} \cdot \phi_{+}\left(\eta_{0}\right) q^{+} \cdot j_{+}\left(\partial \eta_{2}\right)\right\rangle-\left\langle p^{-} \cdot \phi_{-}\left(\eta_{0}\right) q^{-} \cdot j_{-}\left(\partial \eta_{2}\right)\right\rangle} . \tag{C.11}
\end{equation*}
$$

The S-D equations combined with (C.9) give

$$
\begin{align*}
& \left\langle p^{+} \cdot \phi_{+}\left(\eta_{0}\right) q^{+} \cdot j_{+}\left(\partial \eta_{2}\right)\right\rangle=-\frac{1}{2} \pi i\left(\bar{\epsilon}_{n} \overline{p^{+}} q^{+}+\epsilon_{n} p^{+} \overline{q^{+}}\right)  \tag{C.12}\\
& \left\langle p^{-} \cdot \phi_{-}\left(\eta_{0}\right) q^{-} \cdot j_{-}\left(\partial \eta_{2}\right)\right\rangle=\frac{1}{2} \pi i\left(\bar{\epsilon}_{n} \overline{p^{-}} q^{-}+\epsilon_{n} p^{-} \overline{q^{-}}\right) \tag{C.13}
\end{align*}
$$

so the Dirac quantization condition - the condition that the correlation function be single-valued, that the monodromy equal 1 - is

$$
\begin{equation*}
\frac{1}{4}\left[\overline{p^{+}} q^{+} \bar{\epsilon}_{n}+p^{+} \overline{q^{+}} \epsilon_{n}-\overline{p^{-}} q^{-} \bar{\epsilon}_{n}-p^{-} \overline{q^{-}} \epsilon_{n}\right] \in \mathbb{Z} \tag{C.14}
\end{equation*}
$$

which can be written

$$
\begin{equation*}
\frac{1}{2}\left(\epsilon_{n} p^{+}\right) \cdot q^{+}-\frac{1}{2}\left(\epsilon_{n} p^{-}\right) \cdot q^{-} \in \mathbb{Z} \tag{C.15}
\end{equation*}
$$

where the euclidean inner product on complex charges is

$$
\begin{equation*}
p \cdot q=\frac{1}{2}(\bar{p} q+p \bar{q}) . \tag{C.16}
\end{equation*}
$$

Substituting the electric and magnetic charges, the Dirac quantization condition becomes

$$
\begin{equation*}
p \cdot q^{*}+(-1)^{n-1} p^{*} \cdot q \in \mathbb{Z} \tag{C.17}
\end{equation*}
$$

The Dirac quantization condition in the real case, $p, p^{*}, q, q^{*} \in \mathbb{R}$, is

$$
\begin{equation*}
p q^{*}+(-1)^{n-1} p^{*} q \in \mathbb{Z} \tag{C.18}
\end{equation*}
$$

The charges lie in real lattices

$$
\begin{equation*}
p=\frac{m}{R}, \quad q=\frac{n}{R}, \quad p^{*}=\frac{m^{*}}{R^{*}}, \quad q^{*}=\frac{n^{*}}{R^{*}} \tag{C.19}
\end{equation*}
$$

so the Dirac quantization condition is $R R^{*}=1$. In the complex case, the charges $p, q$ lie in a two-dimensional lattice $L \subset \mathbb{C}$ and the $p^{*}, q^{*}$ lie in a two-dimensional lattice $L^{*} \subset \mathbb{C}$. The Dirac quantization condition (C.17) is the condition that $L$ and $L^{*}$ are dual lattices - the euclidean inner-product between an element of $L$ and an element of $L^{*}$ is always an integer.

## C. $3 d=2$

We can check the Dirac quantization condition in $d=2$ using the explicit formulas of section B.6. The scalar fields are given by

$$
\begin{equation*}
\partial \phi_{+}(z)=j(z),, \quad \bar{\partial} \phi_{-}(\bar{z})=j(\bar{z}) . \tag{C.20}
\end{equation*}
$$

Their two-point functions are

$$
\begin{equation*}
\left\langle\phi_{+}^{\dagger}(z) \phi_{+}(w)\right\rangle=-2 \ln (z-w), \quad\left\langle\phi_{-}^{\dagger}(\bar{z}) \phi_{-}(\bar{w})\right\rangle=-2 \ln (\bar{z}-\bar{w}) \tag{C.21}
\end{equation*}
$$

The vertex operators

$$
\begin{equation*}
V_{p^{+}, p^{-}}(z, \bar{z})=e^{i p^{+} \cdot \phi_{+}(z)+i p^{-} \cdot \phi_{-}(\bar{z})} \tag{C.22}
\end{equation*}
$$

have two-point functions

$$
\begin{equation*}
\left\langle V_{p^{+}, p^{-}}(z, \bar{z}) V_{q^{+}, q^{-}}(w, \bar{w})\right\rangle=(z-w)^{p^{+} \cdot q^{+}}(\bar{z}-\bar{w})^{p^{-} \cdot q^{-}} \tag{C.23}
\end{equation*}
$$

which are single-valued iff

$$
\begin{equation*}
p^{+} \cdot q^{+}-p^{-} \cdot q^{-} \in \mathbb{Z} \tag{C.24}
\end{equation*}
$$

The reality condition for $d=2$ is $p^{ \pm}=\overline{p^{ \pm}}$, giving

$$
\begin{equation*}
p^{+} \cdot q^{+}=p^{+} q^{+}, \quad p^{-} \cdot q^{-}=p^{-} q^{-} \tag{C.25}
\end{equation*}
$$

The usual sign choices in $d=2$ are $\epsilon_{1}=1, \delta_{1}=1$, for which

$$
\begin{equation*}
p^{ \pm}=\frac{1}{2}\left(p \pm p^{*}\right) \tag{C.26}
\end{equation*}
$$

so the Dirac quantization condition is

$$
\begin{equation*}
p q^{*}+p^{*} q \in \mathbb{Z} \tag{C.27}
\end{equation*}
$$

which, for $p=\frac{m}{R}, p^{*}=\frac{m^{*}}{R^{*}}, q=\frac{n}{R}, p^{*}=\frac{n^{*}}{R^{*}}$, gives $R R^{*}=1$.

## D Complex conjugation and reality conditions

The difference between $n$ even and $n$ odd shows up in the class of 2 d quantum field theories that live on the quasi-holomorphic curves. The S-D equations on the quasi-holomorphic curves, written in terms of $J$ and the skew-hermitian $M$-intersection form, are equations on complex fields $j$ and $\phi$. For $n$ odd, both $J$ and $I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ are real and $\mathcal{E}_{\partial \xi}$ can remain a real space. The quasi-holomorphic curve $C$ is a real map from $\Sigma$ as a real two-dimensional manifold to $\mathcal{E}_{\partial \xi}$. The reality condition $F=\bar{F}$ on the $n$-form field becomes the reality condition $j=\bar{j}, \phi=\bar{\phi}$ on the 2 d fields. The 2 d quantum field theory is the gaussian model of a single real 1-form $j$. For $n$ even, both $J$ and $I_{M}\left\langle\bar{\eta}_{1}, \eta_{2}\right\rangle$ are imaginary and $\mathcal{E}_{\partial \xi}$ must be complexified to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The quasi-holomorphic curve $C$ will be a map from $\Sigma$ to $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$. The fields $j$ and $\phi$ on the q-h curve remain complex. The 2 d quantum field theory is a two-component gaussian model of a complex 1-form $j$. The reality condition $F=\bar{F}$ on the $n$-form field becomes invariance under a $\mathbb{Z}_{2}$ symmetry of the 2 d quantum field theory on the quasi-holomorphic curve, that combines complex conjugation on the fields and orientation reversal on the Riemann surface. Any construction on 2 d field theories will preserve this $\mathbb{Z}_{2}$ symmetry, so, for $n$ even, the general class of 2 d quantum field theories to be considered are the 2 d theories with this symmetry.

## D. 1 Complex conjugation on the fields

Because the almost complex structure is given by $J=\epsilon_{n} *$ with $\epsilon_{n}^{2}=(-1)^{n-1}$, and $P_{ \pm}=\frac{1}{2}\left(1 \pm i^{-1} J\right)$, complex conjugation acts differently for $n$ odd and $n$ even:

$$
\bar{J}=\left\{\begin{array}{ll}
J, & n \text { odd, }  \tag{D.1}\\
-J, & n \text { even, }
\end{array} \quad \bar{P}_{ \pm}= \begin{cases}P_{\mp}, & n \text { odd } \\
P_{ \pm}, & n \text { even } .\end{cases}\right.
$$

So, for $F$ a complex $n$-form field on $M$, using equations (B.11-B.12) for the adjoint fields,

$$
\begin{array}{cll}
\text { for } n \text { odd : } & F_{ \pm}^{\dagger}(\bar{\xi})=\bar{F}_{\mp}(\bar{\xi})=\bar{F}_{ \pm}(\bar{\xi}), & A_{ \pm}^{\dagger}(\bar{\xi})=\bar{A}_{\mp}(\bar{\xi})=\bar{A}_{ \pm}(\bar{\xi}) \\
\text { for } n \text { even }: & F_{ \pm}^{\dagger}(\bar{\xi})=\bar{F}_{\mp}(\bar{\xi})=\bar{F}_{\mp}(\bar{\xi}), & A_{ \pm}^{\dagger}(\bar{\xi})=\bar{A}_{\mp}(\bar{\xi})=\bar{A}_{\mp}(\bar{\xi}) . \tag{D.3}
\end{array}
$$

## D. 2 Reality conditions on the fields

The condition that $F$ is a real field, $F=\bar{F}$, is

$$
\begin{array}{lll}
\text { for } n \text { odd : } & F_{ \pm}^{\dagger}=F_{ \pm}, & A_{ \pm}^{\dagger}=A_{ \pm} \\
\text {for } n \text { even }: & F_{ \pm}^{\dagger}=F_{\mp}, & A_{ \pm}^{\dagger}=A_{\mp} \tag{D.5}
\end{array}
$$

As a check, re-write these relations in terms of the the usual fields $F$ and $F^{*}=i^{-1} * F$,

$$
\begin{align*}
F & =F_{+}+F_{-}, & A & =A_{+}+A_{-},  \tag{D.6}\\
\epsilon_{n} F^{*} & =F_{+}-F_{-}, & \epsilon_{n} A^{*} & =A_{+}-A_{-},
\end{align*}
$$

getting, for real $F$,

$$
\begin{equation*}
F^{\dagger}=F, \quad F^{* \dagger}=F^{*}, \quad A^{\dagger}=A, \quad A^{* \dagger}=A^{*} \tag{D.7}
\end{equation*}
$$

which is indeed self-adjointness of the magnetic and electric fields and the gauge potentials.
The reality condition on the 1 -form field $j$ and the 0 -form field $\phi$ on $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$ is the transcription,

$$
\begin{array}{cll}
\text { for } n \text { odd : } & j_{ \pm}^{\dagger}=j_{ \pm}, & \phi_{ \pm}^{\dagger}=\phi_{ \pm} \\
\text {for } n \text { even : } & j_{ \pm}^{\dagger}=j_{\mp}, & \phi_{ \pm}^{\dagger}=\phi_{\mp} . \tag{D.9}
\end{array}
$$

The 1-form and 0 -form fields pulled back along a quasi-holomorphic curve $C$ to the Riemann surface $\Sigma$,

$$
\begin{array}{ll}
j_{+}(z)=j_{+}\left(C_{*} \delta_{z}\right), & \phi_{+}(z)=\phi_{+}\left(C_{*} \delta_{z}\right), \\
j_{-}(\bar{z})=j_{-}\left(C_{*} \delta_{z}\right), & \phi_{-}(\bar{z})=\phi_{-}\left(C_{*} \delta_{z}\right), \tag{D.11}
\end{array}
$$

will satisfy the same reality condition (D.8 D.9) .
For $n$ odd, the reality condition (D.8) is the reality condition satisfied by a real 1-form on the Riemann surface. So the 2 d conformal field theory on a q -h curve is the theory of a free real 1-form.

For $n$ even, the reality condition (D.9) is not the reality condition of a real 1-form field on the Riemann surface $\Sigma$. The 2 d conformal field theory on the q -h curve is the free complex 1 -form. The reality condition on $M$ becomes a symmetry condition, $j_{ \pm}^{\dagger}=j_{\mp}$, $\phi_{ \pm}^{\dagger}=\phi_{\mp}$. This is an anti-linear symmetry that reverses the 2 d orientation. The fields in the real theory on the space-time $M$ correspond to the invariant subalgebra of fields in the complex 1-form theory on the Riemann surface $\Sigma$.

## D. 3 Reality conditions on the vertex operators

For the vertex operators

$$
\begin{gather*}
V_{p^{+}, p^{-}}(\eta)=e^{i p^{+} \cdot \phi_{+}(\eta)+i p^{-} \cdot \phi_{-}(\eta)}  \tag{D.12}\\
p^{+} \cdot \phi_{+}(\eta)=\frac{1}{2}\left[\overline{p^{+}} \phi_{+}(\eta)+p^{+} \phi_{+}^{\dagger}(\bar{\eta})\right], \quad p^{-} \cdot \phi_{-}(\eta)=\frac{1}{2}\left[\overline{p^{-}} \phi_{-}(\eta)+p^{-} \phi_{-}^{\dagger}(\bar{\eta})\right]  \tag{D.13}\\
p^{+}=p+\epsilon_{n} p^{*}, \quad p^{-}=p-\epsilon_{n} p^{*} \tag{D.14}
\end{gather*}
$$

The reality condition $p=\bar{p}, p^{*}=\bar{p}^{*}$ becomes

$$
\begin{align*}
\text { for } n \text { odd : } & p^{ \pm}=\overline{p^{ \pm}},  \tag{D.15}\\
\text {for } n \text { even : } & p^{ \pm}=\overline{p^{\mp}} . \tag{D.16}
\end{align*}
$$

For $n$ odd, the vertex operators of the real $n$-form on space-time are exactly the vertex operators of the real 1-form on the Riemann surface. For $n$ even, the space-time vertex
operators of the real $n$-form theory correspond to the subalgebra of the vertex operators of the 2 d theory,

$$
\begin{equation*}
\left\{V_{p^{+}, p^{-}}: p^{-}=\overline{p^{+}}\right\} . \tag{D.17}
\end{equation*}
$$

The Dirac quantization condition on $M$ is

$$
\begin{equation*}
\frac{1}{2}\left[\left(\epsilon_{n} p^{+}\right) \cdot q^{+}-\left(\epsilon_{n} p^{-}\right) \cdot q^{-}\right] \in \mathbb{Z} \tag{D.18}
\end{equation*}
$$

The Dirac quantization condition on $\Sigma$ is

$$
\begin{equation*}
\frac{1}{2}\left(p^{+} \cdot q^{+}-p^{-} \cdot q^{-}\right) \in \mathbb{Z} \tag{D.19}
\end{equation*}
$$

For $n$ odd, these are the same, so the real theory on $M$ corresponds to the real theory on $\Sigma$. For $n$ even, the Dirac quantization conditions (D.18) and (D.19) are different. Writing $p_{\Sigma}, p_{\Sigma}^{*}$ for the magnetic and electric charges on $\Sigma$, and writing $p_{M}, p_{M}^{*}$ for the charges on $M$ (formerly $p, p^{*}$ ),

$$
\begin{equation*}
p_{\Sigma}=\frac{1}{2}\left(p^{+}+p^{-}\right)=p_{M}, \quad \epsilon_{1} p_{\Sigma}^{*}=\frac{1}{2}\left(p^{+}-p^{-}\right)=\epsilon_{n} p_{M}^{*} . \tag{D.20}
\end{equation*}
$$

For $n$ even, real magnetic charges on $M$ become imaginary magnetic charges in the complex 1-form theory on $\Sigma$. The real charges $p_{M}, p_{M}^{*}$ on $M$ lie in dual lattices $L \subset \mathbb{R}$ and $L^{*} \subset \mathbb{R}$. The corresponding charges in the complex 1 -form theory on $\Sigma$ lie in $L \subset \mathbb{C}$ and $i L^{*} \subset \mathbb{C}$. So the full lattice of electric charges $p_{\Sigma}$ on $\Sigma$ must be $L \oplus i L \subset \mathbb{C}$, and the full lattice of magnetic charges $p_{\Sigma}^{*}$ must be $L^{*} \oplus i L^{*} \subset \mathbb{C}$,

$$
\begin{equation*}
p_{\Sigma}=\frac{m_{1}+i m_{2}}{R}, \quad p_{\Sigma}^{*}=\frac{m_{1}^{*}+i m_{2}^{*}}{R^{*}} \tag{D.21}
\end{equation*}
$$

Complex conjugation on $M$ is

$$
\begin{equation*}
p_{\Sigma} \mapsto \bar{p}_{\Sigma}, \quad p_{\Sigma}^{*} \mapsto-\overline{p_{\Sigma}^{*}} . \tag{D.22}
\end{equation*}
$$

All of the vertex operators with charges (D.21) are well-defined on $\Sigma$, but only those invariant under complex conjugation on $M$ are well-defined on $\mathcal{E}_{\partial \xi}$ and on $M$. The q-h curve lies not in $\mathcal{E}_{\partial \xi}$ but in $\mathcal{E}_{\partial \xi}^{\mathbb{C}}$, so vertex operators can have well-defined correlation functions on the q-h curve, yet not be well-defined on $\mathcal{E}_{\partial \xi}$.

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