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Precise lower bound on Monster brane boundary entropy

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Abstract: In this paper we develop further the linear functional method of deriving lower bounds on the boundary entropy of conformal boundary conditions in 1+1 dimensional conformal field theories (CFTs). We show here how to use detailed knowledge of the bulk CFT spectrum. Applying the method to the Monster CFT with $c = \bar{c} = 24$ we derive a lower bound $s > -3.02\times10^{-19}$ on the boundary entropy $s = \ln g$, and find compelling evidence that the optimal bound is $s \geq 0$. We show that all g=1 branes must have the same low-lying boundary spectrum, which matches the spectrum of the known g=1 branes, suggesting that the known examples comprise all possible g=1 branes, and also suggesting that the bound $s \geq 0$ holds not just for critical boundary conditions but for all boundary conditions in the Monster CFT. The same analysis applied to a second bulk CFT — a certain $c = 2$ Gaussian model — yields a less strict bound, suggesting that the precise linear functional bound on s for the Monster CFT is exceptional.

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1 Introduction

Two-dimensional conformal field theories (CFTs) with boundaries have many applications to condensed matter systems. They describe critical boundaries, defects, and junctions in 1+1-dimensional quantum critical systems. In string theory, CFTs with boundary describe the worldsheets of open strings.

Consider a 1-dimensional system of length L with the same boundary condition at each end. The logarithm of partition function $Z(\beta) = \operatorname{Tr} e^{-\beta H}$ in the limit $L/\beta \to \infty$ behaves as

$$\ln Z(\beta) = c\frac{\pi}{6}\frac{L}{\beta} + 2\ln g\,. \tag{1.1}$$

Here β is the inverse temperature and c is the conformal central charge of the bulk CFT. For a critical boundary condition g is a number characterising the boundary condition—the *universal noninteger ground-state degeneracy* [3]. (In equation (1.1) there is a factor 2 multiplying $\ln g$ because there is a contribution $\ln g$ from each of the two boundaries.) The

g-theorem conjectured in [3] states that for any renormalization group (RG) flow between critical boundary conditions for a given fixed bulk CFT, the number g is always smaller at the IR (final) fixed point than at the UV (initial) fixed point.

For an arbitrary — not necessarily critical — boundary condition on a CFT the boundary entropy $s(\beta)$ is defined in the same fashion, by subtracting the universal bulk entropy from the total entropy $S(\beta)$ in the limit $L/\beta \to \infty$,

$$S(\beta) = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta) = c\frac{\pi}{3}\frac{L}{\beta} + 2s(\beta)\,. \tag{1.2}$$

For a critical boundary condition, the boundary entropy is $s = \ln g$, independent of temperature. It was shown in [7] that the boundary entropy $s(\beta)$ for a general boundary condition always decreases with decreasing temperature (contingent on certain regularity assumptions on the ultraviolet properties of the boundary condition). Equivalently, $s(\beta)$ decreases under the renormalization group flow. This proved the g-theorem as a corollary.

In order to control the IR behavior of the renormalization group — the low temperature behavior of the boundary system — it is not enough to have a quantity $s(\beta)$ that decreases under the RG flow. A lower bound on $s(\beta)$ is needed. Without a lower bound, the RG flow might go on forever towards $s(\infty) = -\infty$. A lower bound on $s(\beta)$ would imply that every boundary condition becomes critical at zero temperature. No way has yet been found to put a lower bound on $s(\beta)$.

A more modest goal is to establish a lower bound on $s = \ln g$ for all the *conformal* boundary conditions for each given bulk CFT. That would at least exclude the possibility of flows that end at critical boundary conditions with arbitrarily low values of s. Once we have a lower bound on the critical values of s, it becomes interesting to look for critical boundary conditions that saturate the bound. If such minimal conformal boundary conditions exist and if any of them has a relevant perturbation, the corresponding outgoing RG trajectory would have to go on forever, to $s(\infty) = -\infty$. On the other hand, if such boundary conditions exist and have no relevant perturbations, it would suggest that the lower bound on s applies to all boundary conditions, not just the critical ones.

In [8] the present authors demonstrated the existence of a lower bound on the boundary entropy $s = \ln g$ of all conformal boundary conditions for any given unitary bulk CFT subject to the condition that the lowest scaling dimension Δ_1 of the spin-0 bulk fields satisfies $\Delta_1 > (c-1)/12$. Only CFTs with $c \geq 1$ were considered because the conformal boundary conditions for the $c < 1$ unitary CFTs are already completely classified. No attempt was made in [8] to obtain an optimal lower bound. The goal was only to show existence of a bound.

Once existence of a lower bound is known, the goal becomes to find the best possible lower bound for any given bulk CFT. Nothing can be assumed about the boundary condition beyond what is implied by conformal invariance and the general principles of boundary quantum field theory. On the other hand, all available knowledge of the bulk CFT can be used. The bound obtained in [8] used only the values of c and Δ_1. A sharper lower bound can be obtained for any given bulk CFT by exploiting more detailed information about the bulk CFT. Here, we carry out this program for two specific bulk CFTs.

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Our main example is the Monster CFT. The Monster CFT is the direct product of the right-moving $c = 24$ chiral Monster CFT with the parity conjugate left-moving $\bar{c} = 24$ chiral Monster CFT. The chiral Monster CFT was constructed in [9]. Its internal symmetry group is the Monster — the largest finite simple group. Its fields are all right-moving (holomorphic in 2-dimensional euclidean spacetime). The scaling dimensions of the fields of the chiral theory are $2, 3, 4, \ldots$, so the Monster CFT has spin-0 (scalar) fields of dimensions $\Delta_k = 4, 6, 8, \ldots$. All the spin-0 local couplings are irrelevant, which makes the Monster CFT of interest for constructing critical quantum circuits [10, 11].

A variety of conformal boundary conditions (branes) for the Monster CFT are known [12]. The lowest value of the boundary entropy among these branes is s=0 (g=1). The known g=1 branes are the boundary conditions that respect the full chiral algebra. The incoming chiral fields are transformed by an element of the Monster symmetry group and reflected into outgoing chiral fields. They all have the same boundary spectrum $h_j = 2, 3, 4, \cdots$, with the same multiplicities.

The linear functional (LF) method used in [8] can exploit the complete knowledge of the primary spin-0 bulk scaling dimensions Δ_k. The LF method provides a series of larger and larger numerical computations, each of which gives a rigorous lower bound on g. We find that this series of numerically derived lower bounds on g converges spectacularly closely to 1. We are lead to conjecture that $s \geq 0$ ($g \geq 1$) is the exact lower bound for the Monster CFT.

Moreover, the LF method can pin down the low-lying spectrum of boundary scaling dimensions and their multiplicities, for any g=1 boundary condition, i.e., any boundary condition that saturates the lower bound. We show that there are no relevant boundary perturbations — that the lowest nonzero boundary scaling dimension is greater than 1. This suggests that the bound $s \geq 0$ holds for all boundary conditions and that the generic boundary RG flows ends at a g=1 conformal boundary condition. We find strong evidence that the low-lying boundary spectrum for g=1 branes — including multiplicities — is uniquely determined, matching the boundary spectrum of the known g=1 branes. This suggests that the known examples comprise all possible g=1 branes.

The second bulk CFT we study is a certain $c = 2$ Gaussian model — a nonlinear model with a particular two-torus as target manifold. Again, we know the complete spectrum of bulk scaling dimensions and can use the linear functional method with that knowledge to get a succession of numerical lower bounds on g that converge rapidly to a limit. In this case, no known brane saturates the lower bound. Moreover, we show that no such minimal conformal boundary condition can exist, because the LF method fixes the multiplicity of the lowest lying boundary dimension to lie in a range of real numbers that does not include an integer. We conclude that the success of the linear functional method for the Monster CFT is exceptional.

The method is described in sections 2 and 3. Section 4 presents the results for the Monster CFT. Section 5 presents the results for the $c = 2$ Gaussian model. In section 6 we discuss possible improvements on the linear functional method that might give strict lower bounds on g and consistent boundary spectra for minimal boundary conditions for the general bulk CFT.

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2 The linear functional method

The linear functional method of producing bounds on quantities in conformal field theories was originated in [1, 2] (see appendix A in particular) where it was used to constrain operator dimensions and state degeneracies in a two-dimensional CFT. Similar methods are used in CFTs in higher dimensions, starting from [5], and the method was later applied in two dimensions in particular in [4]. Here we briefly summarize the use of the method in [8] to get bounds on the boundary entropy.

2.1 The modular duality equation

Consider a given bulk CFT with bulk central charge $c > 1$. A conformal boundary condition is described by a certain bulk state $|B\rangle$. The partition function of the boundary system can be calculated in a second way, as a matrix element in the bulk state. The equivalence of the two calculations gives the modular duality equation [14]

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H_{\mathrm{bdry}}} = \langle B| e^{-(2\pi/\beta) H_{\mathrm{bulk}}} |B\rangle\,. \tag{2.1}$$

Here H_{bulk} is the hamiltonian for the CFT on a circular space of length 2π, without boundary. H_{bdry} is the hamiltonian for the CFT on a line interval of length $L = 1$ with the same boundary condition at each end. The two sides of the duality equation are the two operator interpretations of the partition function of the euclidean CFT on a 2-dimensional annulus. On the lhs, the annulus is the product of the spatial interval of length $L = 1$ with the circle of periodic euclidean time of length β. On the rhs, the annulus is the conformally equivalent product of the euclidean time interval of length $2\pi/\beta$ with spatial circle of length 2π.

Expanding both sides of equation (2.1) in Virasoro characters and eliminating the common factor of $1/\eta(i\beta)$ in all the characters (see [8] for details, with slightly different notation), we obtain

$$f_0 - f_1 + \sum_j N(h_j) f_{h_j} = g^2(\tilde{f}_0 - \tilde{f}_1) + \sum_k b^2(\Delta_k) \tilde{f}_{\frac{1}{2}\Delta_k} \tag{2.2}$$

where

$$f_h = \beta^{\frac{1}{4}} q^{h-\gamma}\,, \quad q = e^{-2\pi\beta}\,, \quad \gamma = \frac{c-1}{24}\,, \quad \tilde{f}_{\tilde{h}} = \tilde{\beta}^{\frac{1}{4}} \tilde{q}^{\tilde{h}-\gamma}\,, \quad \tilde{\beta} = \beta^{-1}\,, \quad \tilde{q} = e^{-2\pi\tilde{\beta}}\,, \tag{2.3}$$

and

- The Δ_k are the distinct scaling dimensions of the primary spin-0 bulk fields besides the identity, ordered so that $0 < \Delta_1 < \Delta_2 < \cdots$.
- The h_j are the distinct scaling dimensions of the primary boundary fields besides the identity, ordered so that $0 < h_1 < h_2 < \cdots$.
- $N(h_j)$, an integer ≥ 1, is the multiplicity of h_j.
- $g = \langle B|0\rangle$ is the overlap between the boundary state and the bulk ground state $|0\rangle$.

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- $b^2(\Delta_k) = \sum_\alpha \langle B|\alpha\rangle^2$ where the sum is over all primary spin-0 bulk fields of dimension Δ_k, and $|\alpha\rangle$ is the bulk state corresponding to the primary field ϕ_α.

The numbers h_j, $N(h_j)$, g^2, and $b^2(\Delta_k)$ are properties of the boundary condition, so we assume nothing about them besides the basic constraints from unitarity

- $h_j > 0$,
- $N(h_j)$ integer ≥ 1,
- $g^2 \geq 0$,
- $b^2(\Delta_k) \geq 0$.

On the other hand, we do use the knowledge we have of the Δ_k, which are properties of the bulk CFT.

2.2 Bounds from linear functionals

A linear functional ρ acting on functions $F(\beta)$ is a distribution

$$\rho(F) = \int_0^\infty d\beta\, \hat{\rho}(\beta) F(\beta)\,. \tag{2.4}$$

Applying a linear functional ρ to both sides of (2.2) we obtain

$$\rho(f_0 - f_1) + \sum_j N(h_j)\rho(f_{h_j}) = g^2 \rho(\tilde{f}_0 - \tilde{f}_1) + \sum_k b^2(\Delta_k)\rho(\tilde{f}_{\frac{1}{2}\Delta_k}) \tag{2.5}$$

If we can choose ρ so that

$$\rho(f_h) \geq 0 \qquad \forall h > 0\,, \tag{2.6}$$

$$\rho\big(-\tilde{f}_{\frac{1}{2}\Delta_k}\big) \geq 0 \qquad \forall \Delta_k \tag{2.7}$$

we get an inequality

$$g^2 \rho(\tilde{f}_0 - \tilde{f}_1) \geq \rho(f_0 - f_1)\,. \tag{2.8}$$

It was shown in [8] that condition (2.6) implies $\rho(\tilde{f}_0 - \tilde{f}_1) > 0$, so the inequality is a lower bound on g^2. Equations (2.5)–(2.8) are indifferent to positive rescalings of ρ, so we might as well impose the normalization condition

$$\rho(\tilde{f}_0 - \tilde{f}_1) = 1\,. \tag{2.9}$$

Equation (2.5) becomes

$$g^2 = \rho(f_0 - f_1) + \sum_j N(h_j)\rho(f_{h_j}) + \sum_k b^2(\Delta_k)\rho(-\tilde{f}_{\frac{1}{2}\Delta_k}) \tag{2.10}$$

and the lower bound is

$$g^2 \geq \rho(f_0 - f_1)\,. \tag{2.11}$$

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Maximizing over all distributions ρ subject to the positivity conditions (2.6) and (2.7) and the normalization condition (2.9), we obtain the optimal linear functional bound

$$g^2 \geq g_B^2 = \max_\rho \rho(f_0 - f_1)\,. \tag{2.12}$$

The lower bound g_B^2 depends on the bulk central charge c and on the entire bulk spin-0 spectrum Δ_k. In [8], the goal was to show the existence of a lower bound on g for as general a class of bulk CFTs as possible, so condition (2.7) was replaced by the stronger condition

$$\rho\big(-\tilde{f}_{\frac{1}{2}\Delta}\big) \geq 0\,, \quad \forall \Delta \geq \Delta_1\,, \tag{2.13}$$

which gives a lower bound that depends only on c and Δ_1. It was shown that conditions (2.6) and (2.13) can be satisfied together if and only if $\Delta_1 > 2\gamma = (c-1)/12$.

2.3 Practical calculations

In practice, the method is to maximize over larger and larger finite dimensional subspaces of linear functionals of the form

$$\rho(F) = \mathcal{D}F(\beta)\,, \tag{2.14}$$

where $\mathcal{D}$ is a polynomial differential operator in β of order $2n-1$, and $\mathcal{D}F$ is evaluated at some fixed value of β. The order of the differential operator must be odd because of the positivity conditions. For each n, we maximize over the $2n$-dimensional space of differential operators of order $2n-1$.

It is impractical to enforce the positivity condition (2.7) for the infinite collection of Δ_k. Instead, we enforce the stronger condition

$$\rho(-\tilde{f}_{\frac{1}{2}\Delta}) \geq 0 \quad \text{for } \Delta = \Delta_1, \Delta_2, \ldots, \Delta_{N-1}, \text{ and for } \Delta \geq \Delta_N\,. \tag{2.15}$$

For each value of n and N, we get a lower bound on g. As we increase n or N, the lower bound gets larger. The limit $N \to \infty$ will realize the positivity condition (2.7). The limit $n \to \infty$ will exhaust the space of linear functionals because any linear functional on real analytic functions can be approximated by a differential operator $\mathcal{D}$ acting at a single point β. The combined limit $N, n \to \infty$ gives the optimal LF bound.

In practice, we solve the maximization problem numerically for various values of the parameters n, N, limited by computational resources. We use a more or less arbitrary value of β. The numerical solution of each maximization problem is of course not an exact solution. The numerical solution does provide a concrete linear functional $\rho_{n,N}$. We verify that $\rho_{n,N}$ satisfies the positivity conditions. Then we calculate a rigorous lower bound on g^2 using equation (2.8). Thus each numerical maximization provides a rigorous lower bound on g^2.

2.4 Integrality constraints

Note that the linear functional method makes no use of the fact that the boundary multiplicities $N(h_j)$ must be integers ≥ 0. The linear functional bound is a necessary condition for the existence of a solution to equation (2.2) with real $N(h_j) > 0$, which of course is also a necessary condition for a solution with integer $N(h_j) > 0$. We comment in the final section on the possibility of finding better lower bounds on g that take account of the integrality constraints.

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2.5 The existence of solutions to the modular duality equation

The optimal linear functional bound is a necessary *and* sufficient condition for existence of a solution to equation (2.2) with $N(h_j)$ real, by a small variation of an argument used in [19]. Let $\mathcal{F}$ be the space of functions of $\beta > 0$ (suitably defined). For any real non-negative measure $N(h)$ on $h > 0$, and any collection of numbers $b^2(\Delta_k) \geq 0$ define $f[N, b^2] \in \mathcal{F}$ by

$$f[N, b^2] = \int_0^\infty dh\, N(h) f_h + \sum_k b^2(\Delta_k) \left(-\tilde{f}_{\frac{1}{2}\Delta}\right) . \tag{2.16}$$

The functions $f[N, b^2]$ form a convex cone C in $\mathcal{F}$

$$C = \left\{ f[N, b^2] \right\}. \tag{2.17}$$

Define a vector $v \neq 0$ in $\mathcal{F}$

$$v = g^2(\tilde{f}_0 - \tilde{f}_1) - (f_0 - f_1) . \tag{2.18}$$

There exists a real solution of equation (2.2) iff $v \in C$. The *Generalized Farkas Lemma* [20] says

> $v \in C$ iff there is no hyperplane separating v from C or, equivalently, iff $\rho(v) > 0$ for all linear functionals ρ satisfying $\rho(C) \geq 0$.

The condition $\rho(C) \geq 0$ is exactly conditions (2.6) and (2.7). The condition $\rho(v) > 0$ is exactly equation (2.11). Therefore there exists a real solution of equation (2.2) iff g^2 satisfies the optimal linear function bound.

3 Map to an SDP problem

We next recast the maximization problem as a semidefinite programming (SDP) problem, following [5, 6]. An SDP problem is an optimization over a set of positive-semidefinite matrices — the SDP variables. The problem is to minimize an *objective function* $\mathcal{O}$ which is linear in the SDP variables, subject to a collection of equality constraints also linear in the SDP variables. Effective codes are available for solving SDP problems numerically.

The general differential operator $\mathcal{D}$ of equation (2.14) can be written $\mathcal{D} = D(-4\beta\partial_\beta)$ for $D(z)$ a polynomial of degree $2n-1$. Recall that $\mathcal{D}$ is acting at a specific fixed value of β. Maximizing over differential operators $\mathcal{D}$ is equivalent to maximizing over polynomials $D(z)$. Now define two polynomials $p(x)$ and $\tilde{p}(x)$, each of degree $2n-1$, by

$$p(x) = x^{-\frac{1}{4}} e^{x/4} D(-4x\partial_x) \left(x^{\frac{1}{4}} e^{-x/4}\right) , \quad \tilde{p}(x) = -x^{-\frac{1}{4}} e^{x/4} D(4x\partial_x) \left(x^{\frac{1}{4}} e^{-x/4}\right) . \tag{3.1}$$

We will see shortly that the map from polynomials $D(z)$ to polynomials $p(x)$ is invertible, as is the map from $D(z)$ to $\tilde{p}(x)$. Thus we can maximize over polynomials $p(x)$, or over polynomials $\tilde{p}(x)$. Actually, we will maximize over pairs of polynomials $p(x)$, $\tilde{p}(x)$ subject to the constraint that they come from the same differential operator $\mathcal{D}$.

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The definitions of $p(x)$ and $\tilde{p}(x)$ were designed so that

$$\mathcal{D}f_h = P(h)f_h\,, \qquad \text{where } P(h) = p(x(h))\,, \quad x(h) = 8\pi\beta(h-\gamma)\,, \tag{3.2}$$

$$\mathcal{D}\tilde{f}_{\tilde{h}} = -\tilde{P}(\tilde{h})f_{\tilde{h}}\,, \qquad \text{where } \tilde{P}(\tilde{h}) = \tilde{p}(\tilde{x}(\tilde{h}))\,, \quad \tilde{x}(\tilde{h}) = 8\pi\tilde{\beta}(\tilde{h}-\gamma)\,, \tag{3.3}$$

so the positivity conditions (2.6) and (2.15) on the differential operator $\mathcal{D}$ are equivalent to positivity conditions on the polynomials $p(x)$ and $\tilde{p}(x)$,

$$p(x) \geq 0 \quad \text{for } x \geq x(0) \tag{3.4}$$

$$\tilde{p}(\tilde{x}) \geq 0 \quad \text{for } \tilde{x} = \tilde{x}_1, \ldots, \tilde{x}_{N-1} \text{ and } \tilde{x} \geq \tilde{x}_N\,, \quad \text{where } \tilde{x}_k = \tilde{x}\left(\tfrac{1}{2}\Delta_k\right)\,. \tag{3.5}$$

Equation (2.5) — which is $\mathcal{D}$ applied to both sides of the modular duality equation (2.2) — now reads

$$P(0)f_0 - P(1)f_1 + \sum_j N(h_j)P(h_j)f_{h_j} = g^2[\tilde{P}(1)\tilde{f}_1 - \tilde{P}(0)\tilde{f}_0] - \sum_k b^2(\Delta_k)\tilde{P}\left(\frac{1}{2}\Delta_k\right)\tilde{f}_{\frac{1}{2}\Delta_k}\,. \tag{3.6}$$

The normalization condition (2.9) becomes

$$-\tilde{P}(0)\tilde{f}_0 + \tilde{P}(1)\tilde{f}_1 = 1\,, \tag{3.7}$$

giving

$$g^2 = g_B^2[p,\tilde{p}] + \sum_j N(h_j)P(h_j)f_{h_j} + \sum_k b^2(\Delta_k)\tilde{P}\left(\frac{1}{2}\Delta_k\right)\tilde{f}_{\frac{1}{2}\Delta_k} \tag{3.8}$$

where

$$g_B^2[p,\tilde{p}] = P(0)f_0 - P(1)f_1 \tag{3.9}$$

is the lower bound to be maximized over pairs of polynomials $p(x)$, $\tilde{p}(x)$ to get the optimal bound for each n, N,

$$g_{n,N}^2 = \max_{p,\tilde{p}}\left[P(0)f_0 - P(1)f_1\right]. \tag{3.10}$$

Again following [5], we write the general solution of the continuum positivity constraints on the polynomials $p(x)$ and $\tilde{p}(\tilde{x})$ in terms of positive semidefinite $n\times n$ matrices Y_α [6],

$$p(x) = \sum_{k=0}^{2n-1} p_k x^k = \mathbf{x}^t Y_1 \mathbf{x} + (x - x(0))\mathbf{x}^t Y_2 \mathbf{x} \tag{3.11}$$

$$\tilde{p}(x) = \sum_{k=0}^{2n-1} \tilde{p}_k x^k = \mathbf{x}^t Y_3 \mathbf{x} + (x - \tilde{x}_N)\mathbf{x}^t Y_4 \mathbf{x}\,.$$

where $\mathbf{x}$ is the n-vector with components $(1, x, x^2, \ldots, x^{n-1})$. Note that the polynomial coefficients p_k and $\tilde{p}_k$ are linear functions of the matrix elements of the Y_α. The remaining positivity constraints are

$$\tilde{p}(\tilde{x}_k) \geq 0\,, \quad k = 1, \ldots, N-1\,. \tag{3.12}$$

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These are solved by introducing $N-1$ auxiliary 1×1 positive semidefinite matrices y_k subject to the $N-1$ equality constraints

$$y_k = \tilde{p}(\tilde{x}_k)\,, \quad k = 1, \ldots, N-1\,. \tag{3.13}$$

Finally, we need to impose the condition that the polynomials $p(x)$ and $\tilde{p}(\tilde{x})$ come from the same differential operator $\mathcal{D} = D(-4\beta\partial/\partial\beta)$. The differential operator $\mathcal{D} = D(-4\beta\partial_\beta)$,

$$D(z) = \sum_{l=0}^{2n-1} d_l z^l\,, \tag{3.14}$$

is determined by the coefficients of either of the two polynomials by the equations

$$d_l = \sum_{k\geq l} g_{lk} p_k = -\sum_{k\geq l} (-1)^l g_{lk} \tilde{p}_k \tag{3.15}$$

where the numbers g_{lk} — which depend only on n —- are calculated in appendix A. Therefore the condition that the two polynomials come from the same differential operator is expressed by $2n$ equality constraints

$$\sum_{k\geq l} [g_{lk} p_k + (-1)^l g_{lk} \tilde{p}_k] = 0\,, \quad l = 0, 1, \ldots, 2n-1 \tag{3.16}$$

which are linear constraints on the matrix elements of the semidefinite matrices.

The maximization problem is now re-formulated as an SDP problem:

- The SDP variables are the semidefinite matrices Y_α, $\alpha = 1, \ldots, 4$ and y_k, $k = 1, \ldots, N-1$. The Y_α are $n\times n$ matrices. The y_k are 1×1 matrices.
- There are $2n$ equality constraints given by equation (3.16) and $N-1$ equality constraints given by equation (3.13).
- The objective function to be maximized is $\mathcal{O} = g_B^2[p, \tilde{p}]$ given by equation (3.9).

The equality constraints and the objective function are all linear functions of the matrix elements of the semidefinite matrices.

Following the lead of [6], we used the arbitrary precision SDP solver SPDA-GMP [15], which calculates using the GMP arbitrary precision arithmetic libraries. We found it necessary to calculate using extended precision floating point arithmetic in order to obtain stable numerical solutions to the SDP problems. In practice, we found it useful and feasible to solve our SDP problems with 400 decimal digits of precision.

We prepare the SPDA-GMP problem specifications in the Sage symbolic mathematics program [16]. The input for each run consists of

- the central charge c and the list of the low-lying Δ_k, $k = 1, \ldots, N$ in the bulk CFT,
- the integer n specifying the rank of each of the semidefinite matrices Y_α and the order $2n-1$ of the differential operator.

We scan increasing values of n and N to the limits of our computational resources.

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3.1 Verification of numerical solutions

For each choice of n and N, the SDP solver returns a set of semidefinite matrices Y_α that solves the optimization problem approximately. The solver is a black box to us, so we cannot take the solution at face value. We verify that the SDP solution actually provides a rigorous lower bound on g^2.

From the solution matrices Y_α provided by the solver, we calculate the polynomials $p(x)$ and $\tilde{p}(x)$ by equation (3.11). From each of the two polynomials, we calculate the coefficients of the corresponding differential operator $\mathcal{D}$. This gives us two slightly different differential operators, because the solver does not impose the equality constraints exactly. Then we reverse the calculation for each of the two differential operators. From the differential operator we calculate the corresponding polynomials $p(x)$ and $\tilde{p}(x)$ and check that they satisfy the positivity constraints. We check $\tilde{p}(x_k) \geq 0$ for $k = 1, \ldots, N$ by direct calculation. We check positivity in the half-line, $p(x) \geq 0$ for $x \geq x_1$ and $\tilde{p}(x) \geq 0$ for $x \geq \tilde{x}_N$, in two ways. First, we find all real roots of $p(x)$ numerically (in Sage) to check that all are less than than x_1. We do the analogous check for $\tilde{p}(x)$. Second, we check that the absolute minimum of $p(x)$ for $x \geq x_1$ is nonnegative by finding all real roots of $p'(x)$ with $x \geq x_1$ and then finding the minimum value of $p(x)$ at those roots of $p'(x)$. We do the analogous check for $\tilde{p}(x)$.

Sometimes the positivity checks fail, presumably because the SDP solver enforces the positivity constraints with too much tolerance. When the positivity constraints are satisfied for at least one of the two differential operators reconstructed from the Y_α, the resulting lower bound on g^2 is rigorous, since it derives from a specific linear functional given as a specific differential operator acting at a specific value of β. The calculated lower bound is not the best possible bound for the given values of n and N, but it is a rigorous bound. Strictly speaking, we should control the rounding errors by using rigorous interval arithmetic in the calculations to check the validity of the solutions. We do not go to such lengths. Instead, we do the numerical calculations with a floating point precision of 400 decimal digits, which is far more than enough to allow us to disregard rounding errors. We have checked that Sage calculates the roots of polynomials accurately to within a few digits of the floating point precision, and that the positivity checks are passed by tolerances which are hundreds of orders of magnitude larger than our floating point precision.

4 Numerical results for the Monster CFT

The $c = 24$ chiral Monster CFT [9] is the algebra of right-moving (holomorphic) fields $\mathcal{H}_{\mathrm{M}}$ constructed as the chiral $\mathbb{Z}_2$ orbifold of the holomorphic vertex operator algebra associated to the 24-dimensional self-dual Leech lattice (see e.g. [13] or [12] section 2 for details of the construction). The Monster group — the largest finite simple group — is the internal symmetry group of the chiral Monster CFT. Each element γ of the Monster group acts on the fields of the chiral Monster CFT by $\phi(z) \mapsto \phi^\gamma(z)$. All we use from this construction is the spectrum $h_k = 2, 3, 4, \ldots$ of distinct non-zero primary conformal weights.

The Monster CFT is made by tensoring together the right-moving chiral Monster CFT with its left-moving conjugate $\mathcal{H}_{\mathrm{bulk}} = \mathcal{H}_{\mathrm{M}} \otimes \bar{\mathcal{H}}_{\mathrm{M}}$. Each primary field of the Monster CFT

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has the form $\phi(z,\bar{z}) = \phi_R(z)\bar{\phi}_L(\bar{z})$, with scaling dimension $\Delta = h + \bar{h}$ and spin $h - \bar{h}$, where h and $\bar{h}$ are the conformal weights of the chiral primaries ϕ_R and ϕ_L. So the distinct scaling dimensions of the spin-0 primary fields are

$$\Delta_k = 2k + 2\,, \quad k \geq 1\,. \tag{4.1}$$

We took $\beta = 0.93$ for all our Monster calculations. This is approximately the value of β which maximizes the $n = 1$, $N = 1$ lower bound on g.

The Monster branes [12] with the smallest known value g=1 are in one-to-one correspondence with the elements γ of the Monster group. The brane is given by a bulk state $\langle\gamma|$ on the unit circle $|z| = 1$ satisfying

$$\langle\gamma|\,\phi(z)(dz)^h = \langle\gamma|\,\bar{\phi}^{\gamma}(1/\bar{z})(d(1/\bar{z}))^h \tag{4.2}$$

for all the primary chiral fields $\phi(z)$. For each of these branes, the partition function of the unit interval is the same

$$Z = J(i\beta) = j(i\beta) - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + O(q^4) \tag{4.3}$$

where $j(\tau)$ is the j-invariant. Since $j(\tau) = j(-1/\tau)$, each of these branes has g=1. To find the boundary primary dimensions h_j and their multiplicities $N(h_j)$, we expand the partition function in the $c = 24$ Virasoro characters

$$Z = \frac{(1-q)q^{-1} + \sum_j N(h_j)q^{h_j - 1}}{\prod_{n=1}(1-q^n)} \tag{4.4}$$

to get

$$\sum_j N(h_j)q^{h_j-1} = 196883q + 21296876q^2 + 842609326q^3 + O(q^4)\,. \tag{4.5}$$

So the spectrum is $h_j = 2, 3, 4, \ldots$ with multiplicities $N(2) = 196883$, $N(3) = 21296876$, $N(4) = 842609326$, ...

4.1 Lower bounds on g

Let us write the rigorous lower bounds in the form

$$g^2 \geq g^2_{n,N} = 1 - \epsilon_{n,N}\,. \tag{4.6}$$

The following table gives the values of $\epsilon_{n,N}$ we found for SDP solutions that passed the positivity tests:

		N		
		15	31	41
n	15	1.93×10^{-4}		
	24			1.08×10^{-9}
	30		1.12×10^{-10}	6.23×10^{-11}
	36		7.25×10^{-13}	
	42		6.03×10^{-19}	

(4.7)

Our best lower bound is

$$g^2 \geq g^2_{42,31} = 1 - 6.03\times10^{-19} \tag{4.8}$$

which gives a lower bound on the boundary entropy

$$s \geq -3.02\times10^{-19}\,. \tag{4.9}$$

This is a rigorous bound, since it is derived from a specific differential operator that satisfies the positivity constraints.

It seems reasonably clear that the lower bounds shown in table 4.7 are converging to the optimal bound $g^2 \geq 1$, $s \geq 0$. Given that the smallest value of g for the known conformal boundary conditions [12] is $g = 1$, our numerically derived bounds give very strong indication that $g \geq 1$ is the exact lower bound for all possible Monster branes.

4.2 The boundary spectrum for minimal branes

Let us call a conformal boundary condition *minimal* if it saturates the optimal linear functional lower bound, $g = g_B$. Equation (3.8) for an optimal linear functional is

$$g^2 = g_B^2 + \sum_j N(h_j) P_{opt}(h_j) f_{h_j} + \sum_k b^2(\Delta_k) \tilde{P}_{opt}\left(\frac{1}{2}\Delta_k\right) \tilde{f}_{\frac{1}{2}\Delta_k} \tag{4.10}$$

where P_{opt} and $\tilde{P}_{opt}$ are nonnegative. Thus $g = g_B$ requires the h_j to lie at zeros of the function $P_{opt}(h) f_h$. (It also follows that the $b^2(\Delta_k)$ can be non-vanishing only when $\Delta_k/2$ is at a zero of $\tilde{P}_{opt}(\tilde{h})\tilde{f}_{\tilde{h}}$, but we do not pursue this point.)

For each of the solutions returned by the SDP solver, we calculated the local minima of the function $P(h)f_h$ for $h \geq 0$. As n and N increase, successively more of the local minima approach the values $2, 3, 4, \ldots$ and the values of $P(h)f_h$ at those local minima approach zero. For our best solution, with $n = 42$, $N = 31$, the first 8 local minima are presented in the table below.

h	$P(h)$	$P(h)f_h$
$2 - 1.014107\times10^{-17}$	2.547717×10^{-23}	5.685816×10^{-26}
$3 + 3.532221\times10^{-16}$	8.126899×10^{-23}	5.258071×10^{-28}
$4 - 3.099596\times10^{-15}$	7.080475×10^{-22}	1.328079×10^{-29}
$5 + 2.776280\times10^{-14}$	1.063502×10^{-20}	5.783089×10^{-31}
$6 + 1.070319\times10^{-14}$	2.274723×10^{-19}	3.585999×10^{-32}
$7 - 1.168940\times10^{-12}$	6.281304×10^{-18}	2.870723×10^{-33}
$8 - 2.778270\times10^{-11}$	2.108470×10^{-16}	2.793631×10^{-34}
$9 - 3.057268\times10^{-10}$	8.266698×10^{-15}	3.175364×10^{-35}

The evidence seems reasonably strong that $P_{opt}(h)f_h$ will have zeros for $h > 0$ exactly at $h = 2, 3, 4, \cdots$, so any g=1 brane must have boundary spectrum $h_j = 2, 3, 4, \cdots$.

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4.3 Stability of g=1 branes

Our numerical results prove that any g=1 brane must be stable, i.e. that the lowest boundary scaling dimension of a g=1 brane satisfies $h_1 > 1$.

For the $n = 42$, $N = 31$ solution, equation (3.8) becomes

$$(g^2 - 1) + 6.03\times10^{-19} = \sum_j N(h_j)P(h_j)f_{h_j} + \sum_k b^2(\Delta_k)\tilde{P}\left(\frac{1}{2}\Delta_k\right)\tilde{f}_{\frac{1}{2}\Delta_k} \tag{4.11}$$

When $g = 1$, this implies

$$6.03\times10^{-19} > P(h_1)f_{h_1} \tag{4.12}$$

since $N(h_1) \geq 1$. We check that $P(h)f_h > 6.03\times10^{-19}$ for $0 < h \leq 2 - 2.1\times10^{-7}$, so we conclude that

$$h_1 > 2 - 2.1\times10^{-7}\,. \tag{4.13}$$

Therefore any g=1 brane must be stable.

4.4 Boundary multiplicities

A small modification of the SDP problem gives upper and lower bounds on the boundary multiplicities $N(h_j)$ for any g=1 brane with boundary spectrum $h_j = 2, 3, 4, \ldots$. The multiplicities must be integers, so sufficiently tight bounds fix them precisely. We find $N(2) = 196883$, $N(3) = 21296876$, $N(4) = 842609326$ in exact agreement with the boundary multiplicities of the known g=1 branes as given by equation (4.5).

When $g = 1$, equation (3.6) becomes

$$\sum_j N(h_j)P(h_j)f_{h_j} + \sum_k b^2(\Delta_k)\tilde{P}\left(\frac{1}{2}\Delta_k\right)\tilde{f}_{\frac{1}{2}\Delta_k} = -[\tilde{P}(0)\tilde{f}_0 - \tilde{P}(1)\tilde{f}_1 + P(0)f_0 - P(1)f_1] \tag{4.14}$$

which gives an inequality

$$N(h_1)P(h_1)f_{h_1} \leq -[\tilde{P}(0)\tilde{f}_0 - \tilde{P}(1)\tilde{f}_1 + P(0)f_0 - P(1)f_1] \tag{4.15}$$

if we enforce the positivity conditions

$$P(h_j) \geq 0 \qquad \text{for } j = 2, 3, \ldots \tag{4.16}$$

$$\tilde{P}\left(\frac{1}{2}\Delta_k\right) \geq 0 \qquad \text{for } k = 1, 2, 3, \ldots\,. \tag{4.17}$$

Using the normalization condition $P(h_1)f_{h_1} = 1$ we get an upper bound

$$N(h_1) \leq -[\tilde{P}(0)\tilde{f}_0 - \tilde{P}(1)\tilde{f}_1 + P(0)f_0 - P(1)f_1]\,, \tag{4.18}$$

while using the normalization $P(h_1)f_{h_1} = -1$ gives a lower bound

$$\tilde{P}(0)\tilde{f}_0 - \tilde{P}(1)\tilde{f}_1 + P(0)f_0 - P(1)f_1 \leq N(h_1)\,. \tag{4.19}$$

For both normalizations, we want to maximize the objective function

$$\mathcal{O} = \tilde{P}(0)\tilde{f}_0 - \tilde{P}(1)\tilde{f}_1 + P(0)f_0 - P(1)f_1 \tag{4.20}$$

to get the optimal bounds on $N(h_1)$.

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For computability, as before, we replace the infinite series of positivity conditions with the stronger conditions

$$P(h) \geq 0 \quad \text{for } h = h_j, \qquad j = 2, 3, \ldots, N-1 \quad \text{and} \quad h \geq h_N \tag{4.21}$$

$$\tilde{P}(\tilde{h}) \geq 0 \quad \text{for } \tilde{h} = \frac{1}{2}\Delta_k, \qquad k = 1, 2, \ldots, N-1 \quad \text{and} \quad \tilde{h} \geq \frac{1}{2}\Delta_N\,. \tag{4.22}$$

Some numerical results are given in the table below.

n	N	bounds on $\delta = N(2) - 196883$
8	5	$-0.79 < \delta < 0.74$
10	10	$-2.7\times10^{-5} < \delta < 4.8\times10^{-4}$
12	8	$-3.6\times10^{-6} < \delta < 8.3\times10^{-5}$
12	10	$-2.9\times10^{-7} < \delta < 6.2\times10^{-6}$
15	20	$-1.3\times10^{-10} < \delta < 5.5\times10^{-12}$

(4.23)

The $n = 8$, $N = 5$ bounds are enough to fix $N(2) = 196883$, since the multiplicities $N(h_j)$ must be integers. The additional results illustrate convergence to a sharp optimal bound. These calculations were done at $\beta = 1.0$.

Now we can substitute $N(2) = 196883$ into equation (4.14) to get bounds on $N(3)$.

n	N	bounds on $\delta = N(3) - 21296876$
8	5	$-38 < \delta < 61$
10	5	$-2.6 < \delta < 4.4$
10	8	$-8.5\times10^{-3} < \delta < 8.1\times10^{-3}$

(4.24)

So we have $N(3) = 21296876$ and can calculate bounds on $N(4)$.

n	N	bounds on $\delta = N(4) - 842609326$
10	10	$-8.3\times10^{-2} < \delta < 2.2\times10^{-2}$
10	12	$-8.3\times10^{-2} < \delta < 3.6\times10^{-3}$
15	15	$-1.2\times10^{-9} < \delta < 7.0\times10^{-9}$

(4.25)

So $N(4) = 842609326$.

At this point we extrapolate to the conclusion that any g=1 brane must have the same spectrum and multiplicities as the known g=1 branes.

5 $c = 2$ Gaussian model

Our second example is a certain $c = 2$ Gaussian model — a nonlinear model whose target space is a 2-torus whose radii are both equal to $R = \sqrt{2}R_{\text{sd}}$ where R_{sd} is the self-dual radius. There is no B-field in this example. All known conformal boundary conditions for this CFT have $g \geq 0.5$ [17].

We show in appendix B that the spin-0 scaling dimensions of the Virasoro primary fields are

$$\{\Delta_k\} = \left\{ \frac{m}{4} \;:\, m > 0, m \equiv 0, 1, 2, 4, 5 \,(\text{mod}\, 8) \right\} \tag{5.1}$$

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Using this list of scaling dimensions, we calculated lower bounds on g^2 as before, only with a different value of c and a different list of Δ_k. For $n = 36$, $N = 40$ (using $\beta = 1.0$) we obtained the lower bound

$$g^2 > 0.1009\,. \tag{5.2}$$

The bound did not improve appreciably when we increased n from 24 to 30 and then to 36. This linear functional bound is well below the smallest known value $g^2 = 0.25$.

We next explored the possibility of a minimal brane, that saturates the linear functional bound. For a minimal brane, the boundary scaling dimensions h_j must lie among the zeros of the function $P_{opt}(h)f_h$, as in section 4.2. For the $n = 36$, $N = 40$ solution, the first ten local minima of the function $P(h)f_h$ are shown in the following table.

h	$P(h)f_h$
2.527099	1.801512×10^{-64}
4.281833	9.260807×10^{-65}
5.802231	5.951241×10^{-65}
7.160648	3.699803×10^{-65}
8.443321	2.115923×10^{-65}
9.768486	1.235336×10^{-65}
11.03488	8.684430×10^{-66}
12.33631	5.141452×10^{-66}
13.67932	3.325586×10^{-66}
15.06931	2.052798×10^{-66}

(5.3)

The low-lying h_j should be from this list.

With these values for the low-lying h_j, we determined the boundary multiplicities as in section 4.4. Using $n = 10$, $\beta = 1.0$, and taking account of the first $N = 24$ of the bulk Δ_k, we obtained the bounds

$$6.30974 < N(h_1) < 6.30978\,. \tag{5.4}$$

But $N(h_1)$ is an integer. Therefore the linear functional bound on g cannot be saturated. The true lower bound on g must be higher than the linear functional bound.

6 Conclusions

We have proved by numerical computation (1) a lower bound on the boundary entropy s of a Monster brane, (2) the stability of branes saturating the bound, and (3) the uniqueness of the low-lying boundary spectrum of such extremal branes. Our numerical results give strong evidence for the exact s=0 (g=1) lower bound on the boundary entropy of Monster branes and for the uniqueness of the boundary spectrum of such extremal branes. The lower bound on s and the stability of the extremal branes suggests that $s \geq 0$ for all boundary conditions, conformal or not.

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The $c = 2$ example shows that this situation is exceptional, that in general the optimal linear function bound may not be the true lower bound. It would be interesting to have some clues as to when the LF method provides the true bound. In such situations we expect that the method of sections 4.2, 4.4 can be used to constrain the spectrum of the extremal boundary conditions. One speculation is the LF bound is the true bound when the bulk CFT is itself an extremal solution to the bulk modular invariance equations. It would be interesting to check this by numerical calculations.

It might be possible to improve the LF method so as to produce true lower bounds on s for CFTs such as the $c = 2$ example or for CFTs with $\Delta_1 \leq (c-1)/12$ where the present LF method gives no bound at all. One could try to incorporate the constraint that the boundary multiplicities $N(h_j)$ are nonnegative integers. This does not seem possible with the SDP technique, but one could instead generate from the modular transform equation a linear programming problem as was done for the three and four-dimensional CFT bootstrap equations [18, 19]. In our case since the modular duality equation (2.2) contains both positive integer variables $N(h_i)$ and real positive variables $b^2(\Delta_k)$ we get a mixed integer linear programming (MIP) problem. Software packages are available for solving such problems. A disadvantage of the MIP technique is that unlike the SDP method it does not produce rigorous bounds. First one has to make assumptions about the spectrum of dimensions (in our case about h_j) e.g. putting them on a grid [18, 19]. Second, one has to have faith in the MIP solver when it says that there exists no solution to the MIP problem. There is no way to verify the non-existence. On a practical level, the extended numerical precision that we have needed with the SDP technique is not currently available in MIP software packages.

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A Coefficients g_{lk}

The polynomials

$$D(z) = \sum_{l=0}^{2n-1} d_l z^l\,, \qquad p(x) = \sum_{k=0}^{2n-1} p_k x^k\,, \qquad \tilde{p}(x) = \sum_{k=0}^{2n-1} \tilde{p}_k x^k \tag{A.1}$$

are related by equation (3.1)

$$p(x) = x^{-\frac{1}{4}} e^{x/4} D(-4x\partial_x)\left(x^{\frac{1}{4}} e^{-x/4}\right)\,, \quad \tilde{p}(x) = -x^{-\frac{1}{4}} e^{x/4} D(4x\partial_x)\left(x^{\frac{1}{4}} e^{-x/4}\right) \tag{A.2}$$

Define polynomials

$$g_k(z) = \sum_l g_{lk} z^l \tag{A.3}$$

by

$$x^k = x^{-\frac{1}{4}} e^{x/4} g_k(-4x\partial_x) \left(x^{\frac{1}{4}} e^{-x/4} \right) \tag{A.4}$$

so

$$D(z) = \sum_k p_k g_k(z)\,, \qquad D(-z) = -\sum_k \tilde{p}_k g_k(z)\,. \tag{A.5}$$

Now calculate

$$\begin{aligned} g_{k+1}(-4x\partial_x) \left(x^{\frac{1}{4}} e^{-x/4} \right) &= x^{k+1} x^{\frac{1}{4}} e^{-x/4} \\ &= x^k x^{\frac{1}{4}} \left(-4x\partial_x\right) e^{-x/4} \\ &= \left(-4x\partial_x + 4k + 1\right) x^k x^{\frac{1}{4}} e^{-x/4} \end{aligned} \tag{A.6}$$

so

$$g_0(z) = 1\,, \qquad g_{k+1}(z) = (z + 4k + 1)\, g_k(z) \tag{A.7}$$

so

$$g_{00} = 1\,, \quad g_{0,k+1} = (4k+1) g_{0,k}\,, \quad g_{l,k+1} = g_{l-1,k} + (4k+1) g_{l,k}\,, \quad l = 1, 2, \ldots, k+1\,. \tag{A.8}$$

Equation (A.5) now gives equation (3.15)

$$d_l = \sum_{k \geq l} g_{lk} p_k = -\sum_{k \geq l} (-1)^l g_{lk} \tilde{p}_k\,. \tag{A.9}$$

B Scaling dimensions in the $c = 2$ Gaussian model

We need a list of the scaling dimensions Δ_k of the spin-0 Virasoro primary fields. The vertex operators — the primary fields for the U(1)×U(1) current algebra — have conformal weights

$$h = p^2 = p_1^2 + p_2^2\,, \qquad \bar{h} = \bar{p}^2 = \bar{p}_1^2 + \bar{p}_2^2 \tag{B.1}$$

$$p_\mu = \frac{1}{2}(m_\mu R + n_\mu R^{-1})\,, \qquad \bar{p}_\mu = \frac{1}{2}(m_\mu R - n_\mu R^{-1})\,, \quad m_\mu, n_\mu \in \mathbb{Z}\,, \quad \mu = 1, 2\,. \tag{B.2}$$

Let $N_{h,\bar{h}}$ be the multiplicity of the Virasoro representation with weights $h, \bar{h}$. The partition function $\mathrm{tr}\big(q^{L_0} \bar{q}^{\bar{L}_0}\big)$ (stripped of the factor $q^{-c/24} \bar{q}^{-c/24}$)

$$\sum_{p,\bar{p}} \frac{q^{p^2} \bar{q}^{\bar{p}^2}}{\prod_n |1 - q^n|^4} = \frac{|1-q|^2 + \sum N_{h,0} q^h (1-\bar{q}) + \sum N_{0,\bar{h}} (1-q) q^{\bar{h}} + \sum_{h,\bar{h} \neq 0} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}}}{\prod_n |1 - q^n|^2}\,. \tag{B.3}$$

can be expanded in the characters of the U(1)×U(1) current algebra (on the left) or in the characters of the two Virasoro algebras (on the right). Multiply by the denominator on the right and rearrange to get

$$\sum_{h,\bar{h} \neq 0} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}} = \sum_{p,\bar{p}} \frac{q^{p^2} \bar{q}^{\bar{p}^2}}{\prod_n |1 - q^n|^2} - |1-q|^2 - \sum N_{h,0} q^h (1-\bar{q}) - \sum N_{0,\bar{h}} (1-q) q^{\bar{h}} \tag{B.4}$$

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Write $P_{s=0}$ for the projection on the spin-0 part of a sum over powers of q and $\bar{q}$ — the terms with the same power of q and $\bar{q}$, and apply it to both sides of the above identity.

$$\sum_{h=\bar{h}\neq 0} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}} = P_{s=0} \sum_{p,\bar{p}} \frac{q^{p^2}\bar{q}^{\bar{p}^2}}{|\prod_n (1-q^n)|^2} - 1 - q\bar{q} + N_{1,0} q\bar{q} + N_{0,1} q\bar{q} \tag{B.5}$$

Look at the q and $\bar{q}$ terms in (B.4). There are no $p, \bar{p}$ with $p^2 = 1$, $\bar{p} = 0$ or $p^2 = 0$, $\bar{p} = 1$, so

$$\begin{aligned} 0 &= q + q - N_{1,0} q \\ 0 &= \bar{q} + \bar{q} - N_{0,1} \bar{q} \end{aligned} \tag{B.6}$$

so $N_{1,0} = N_{0,1} = 2$. Equation (B.5) becomes

$$\sum_{h=\bar{h}\neq 0} N_{h,\bar{h}} q^h \bar{q}^{\bar{h}} = P_{s=0} \sum_{p,\bar{p}} \frac{q^{p^2}\bar{q}^{\bar{p}^2}}{|\prod_n (1-q^n)|^2} - 1 + 3q\bar{q}\,. \tag{B.7}$$

By inspection, $N_{h,h} = N(\Delta) \neq 0$ exactly for all $\Delta = 2h$ of the form

$$\Delta = p^2 + \bar{p}^2 + |p^2 - \bar{p}^2| + 2r\,, \qquad r = 0, 1, \ldots \tag{B.8}$$

which is

$$\Delta = m_1^2 + m_2^2 + \frac{1}{4}(n_1^2 + n_2^2) + |m_1 n_1 + m_2 n_2| + 2r\,, \qquad r = 0, 1, \ldots \tag{B.9}$$

or

$$4\Delta = 4(m_1^2 + m_2^2 + |m_1 n_1 + m_2 n_2|) + n_1^2 + n_2^2 + 8r\,, \qquad r = 0, 1, \ldots \tag{B.10}$$

Consider the cases

$$(m_1, m_2, n_1, n_2) = (0,0,0,0),\ (0,0,1,0),\ (0,0,1,1),\ (1,0,0,0),\ (1,0,0,1) \tag{B.11}$$

to get

$$N(\Delta) \neq 0 \text{ for } 4\Delta \equiv 0, 1, 2, 4, 5 \,(\mathrm{mod}\, 8) \tag{B.12}$$

Finally, we show that $N(\Delta) = 0$ for $4\Delta \equiv 3, 6, 7 \,(\mathrm{mod}\, 8)$ which is to say for (1) $4\Delta \equiv -1 \,(\mathrm{mod}\, 4)$, and (2) $4\Delta \equiv 6 \,(\mathrm{mod}\, 8)$.

For (1), note that $4\Delta \equiv n_1^2 + n_2^2 \,(\mathrm{mod}\, 4)$. If n_1 and n_2 are both even, then $4\Delta \equiv 0 \,(\mathrm{mod}\, 4)$. If both are odd, $4\Delta \equiv 2 \,(\mathrm{mod}\, 4)$. If one is even and the other is odd, $4\Delta \equiv 1 \,(\mathrm{mod}\, 4)$. So $4\Delta \not\equiv -1 \,(\mathrm{mod}\, 4)$.

For (2), suppose that $4\Delta \equiv 6 \,(\mathrm{mod}\, 8)$ then $4\Delta \equiv 2 \,(\mathrm{mod}\, 4)$ so n_1 and n_2 must both be odd, say $n_1 = 2k_1 + 1$, $n_2 = 2k_2 + 1$. Then

$$4\Delta = 4(m_1^2 + m_2^2 + |2m_1 k_1 + m_1 + 2m_2 k_2 + m_2|) + 4k_1(k_1+1) + 4k_2(k_2+1) + 2 + 8r \tag{B.13}$$

so $m_1^2 + m_2^2 + |2m_1 k_1 + m_1 + 2m_2 k_2 + m_2|$ must be odd, so $m_1^2 + m_2^2 + m_1 + m_2$ must be odd, which is impossible. So $4\Delta \not\equiv 6 \,(\mathrm{mod}\, 8)$.

Therefore

$$\{\Delta_k\} = \left\{ \frac{m}{4} : m > 0, m \equiv 0, 1, 2, 4, 5 \,(\mathrm{mod} 8) \right\}. \tag{B.14}$$

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