Geometric Models for Critical Systems in $2+\varepsilon$ Dimensions

D. Friedan

Lawrence Berkeley Laboratory

and

Department of Physics

University of California,

Berkeley, California 94720

Expanded version of a talk given at the Nuffield Workshop on Quantum Gravity, DAMTP, Cambridge University, August 13, 1979.

Abstract

Generalizations of the non-linear sigma model are described in which the fields take values in a compact manifold M, the couplings are the Riemannian metrics on M and the action is $S(\phi) = \frac{1}{2T}\int d^d x \; g_{ij}(\phi(x))\partial_\mu \phi^i(x)\partial_\mu \phi^j(x)$. Renormalizability in $2+\epsilon$ dimensions is proved. The renormalization group acts on the infinite dimensional space of couplings. Its fixed points are the metrics satisfying an extension of Einstein's equation. At the fixed points all but a finite dimensional subspace of couplings become irrelevant. Possibilities for exotic critical behaviour are considered.

Some forms of critical behaviour in extended physical systems are due to large-scale fluctuations among a variety of easily accessible equilibrium states. When the typical extent of a fluctuation becomes large compared to microscopic and thermal lengths the long distance static properties of the fluctuations can be reproduced in a classical Euclidean statistical field theory. Because susceptibilities diverge in the critical regime, the values of the field can be restricted to lie among the extreme equilibrium states available to the system. The effective action of the field theoretic model will encourage order and will be local if long range physical interactions can be ignored. The standard example is the ideal ferromagnet near the Curie temperature. Its pure equilibrium states are represented by directions of magnetization. The corresponding field theory is the non-linear sigma model, or, with lattice cutoff, the classical Heisenberg model.

Invariance of the long distance physics under the renormalization group ensures that there be relatively few distinct models, all associated with unstable asymptotic behaviour of the group orbits.[1] Here I consider the possibilities when the pure equilibria form a compact differentiable manifold and when all mixtures of pure states are distinct. The corresponding models are generalisations of the Heisenberg model in which the field ϕ takes values $\phi(x)$ not in the 2-sphere but in the m-dimensional manifold M.

Polyakov[2] found the renormalization group transformation for the Heisenberg model simplified drastically at low effective temperature in dimensions near two and that perturbative methods could be used to find fixed points and critical scaling behaviour. Brèzin and Zinn-Justin[3] sharpened this result by showing that the low temperature expansion was in fact a renormalizable perturbation series in $2+\varepsilon$ dimensions, so that critical indices could be calculated systematically as asymptotic expansions in ε. The results in dimension two are exact and in $2+\varepsilon$ give, it is hoped, at least qualitative information about the possibilities in dimension three. The same strategy can be made to work in the general case.

1. Perturbation theory and its renormalization

The partition function for the fluctuations in the presence of external fields is

$$Z(h) = \int \mathcal{D}(\phi)\, e^{-S(\phi) + H(\phi)}$$

where $\quad \mathcal{D}(\phi) = \prod_x dvol(\phi(x))$,

$dvol(\cdot)$ is a volumn element on M,

$S(\cdot)$ is the effective action,

and $H(\cdot)$ is the effective source.

The effective action has a continuum limit when the significant forces are short-range:

$$S(\phi) = \int d^dx \left(\tfrac{1}{2T}\right) g_{ij}(\phi(x))\partial_\mu\phi^i(x)\partial_\mu\phi^j(x) \quad + \quad \text{terms with more derivatives}$$

where the coupling $\frac{1}{T} g_{ij}$ is some symmetric, non-negative tensor field on M. (Greek indices are used for external space, Latin for internal space.) The source takes the form

$$H(\phi) = \int d^dx\; h(x, \phi(x))$$

where, for each x, h(x,$\cdot$) is a function on M. With an assumption of non-degeneracy, the coupling becomes a (positive-definite) Riemannian metric on M. The volumn element $dvol(\cdot)$ can be taken to be the metric volumn. Naive power counting indicates that the terms in the effective action with more than two derivatives are irrelevant to the low temperature expansion when d is asymptotically close to 2. $\mathcal{Z}(h)$ is generating functional for the correllation functions

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle = \frac{1}{\mathcal{Z}(0)} \int \mathcal{D}(\phi)\, e^{-S(\phi)} \phi(x_1) \cdots \phi(x_k)$$

which should be thought of as a probability measure on M^K. In particular, the order parameter $\langle \phi(x) \rangle$ is a probability measure on M.

In the low temperature limit the constant fields dominate the functional integral. The expansion in T is obtained by integrating perturbatively the small fluctuations about the constants. Around a particular constant field ϕ this is done by choosing a system of coordinates, to be thought of as an identification of a neighbourhood of ϕ in M with a neighbourhood of the origin in the tangent space to ϕ, $T_\phi M$. The small fluctuations become linear fields $\sigma^i(x)$; the action is written as a power series in σ and Feynman rules are derived. Unfortunately the part of the action quadratic in σ is the action of a massless free scalar field, whose propagator has low momentum pathology in two dimensions. To provide infra-red regulation the system is placed in a box and periodic boundary conditions imposed. Compactness of M ensures existence of the infinite volumn limit of $\mathcal{Z}(h)$;[5] the boundary conditions are chosen in order that the infinite volumn limit commute with the low temperature limit in two dimensions.[6]

Since space is now compact (a torus) the functional integral must include small fluctuations about each of the constant fields: one perturbative integration for each point ϕ in M. To avoid double counting a gauge condition (like $\int d^d x\, \sigma^i(x) = 0$) and a finite dimensional Faddeev-Popov determinant are required in each perturbative integration. The functional integral becomes

$$Z(h) = \int_M d\,vol(\phi) \left(\prod_x \int_{T_\phi M} d^n \sigma(x) \right) \int_{T_\phi^* M} d^n \gamma \int_{\Lambda^* T_\phi M \otimes \Lambda^* T_\phi^* M} d^n c \wedge d^n c^*$$

$$\times \exp \left\{ - \tilde{S}_\phi(\sigma) + \tilde{H}_\phi(\sigma) + i \gamma_i \tilde{P}_\phi(\sigma)^i + c_i \tilde{F}_\phi(\sigma)^i_j c^{*j} \right\}$$

where $\tilde{P}_\phi(\sigma)^i$. is the gauge function at ϕ (expanded as a power series in σ); γ_i is a multiplier enforcing the gauge condition; $\det(\tilde{F}_\phi(\sigma))$ is the Faddeev-Popov determinant (F also expanded in powers of σ); c_i, c^{*j} are anti-commuting ghosts used to calculate the determinant; and $\tilde{S}_\phi$, $\tilde{H}_\phi$ are the action and source term written in coordinates and expanded in σ:

$$\tilde{S}_\phi(\sigma) = \frac{1}{2T} \int d^d x \left[\tilde{g}_{ij}(0) + (\partial_{k_1} \tilde{g}_{\phi\, ij})(0)\, \sigma^{k_1}(x) + \tfrac{1}{2} (\partial_{k_1} \partial_{k_2} \tilde{g}_{\phi\, ij})(0)\, \sigma^{k_1}(x) \sigma^{k_2}(x) + \cdots \right] \partial_\mu \sigma^i(x)\, \partial_\mu \sigma^j(x)$$

$$\tilde{H}_\phi(\sigma) = \int d^d x \left[\tilde{h}_\phi(x,0) + (\partial_{k_1} \tilde{h}_\phi)(x,0)\, \sigma^{k_1}(x) + \tfrac{1}{2} (\partial_{k_1} \partial_{k_2} \tilde{h}_\phi)(x,0)\, \sigma^{k_1}(x) \sigma^{k_2}(x) + \cdots \right]$$

$\tilde{g}_\phi(\sigma)$ and $\tilde{h}_\phi(x,\sigma)$ being the metric and source function pulled back to the coordinate system σ^i in the tangent space at ϕ . The change from non-linear to linear functional measure on the σ field gives rise to cutoff dependent terms usually written $\sim \delta^{j}(0)$. Dimensional regularization conveniently allows these to be ignored, because they vanish in the continuum limit for sufficiently low dimension.

By naive power counting the perturbative action (in curly brackets) for each ϕ is renormalizable, since all possible vertices which might arise as primitive divergences are already present in the Taylor series making up the action. But the original Taylor series for the various ϕ are not independent, being the expressions of a single metric and source on M in a particular collection of coordinate systems. Unless the bare vertices are similarly derived from bare metric, source and coordinates, the renormalization procedure will abandon the original space of couplings for a much larger class of theories having no infinite volumn limit in two dimensions. Proof of renormalizability requires a demonstration that compatibility of metric, source and coordinates persists after renormalization, order by order in perturbation theory. Since the asymptotic expansion of $\mathcal{Z}(h)$ in powers of T does not depend on the coordinate systems used in its calculation, it can be made finite in the continuum limit by using from the beginning a bare metric and source depending only on the regularization scheme and on the renormalized metric and source.

The proof of renormalizability uses a certain amount of abstract geometric machinery and will be given elsewhere. The crucial idea is that the bundle of perturbation series contains redundant information: the contents of one can be derived from any of its near neighbours by variation of the gauge condition. A differential operator $\mathcal{R}$, generalizing the BRS transformation,[7] is constructed to connect infinitesimally close perturbation theories and to encode their redundant information in its annihilation of the perturbative action as a function of ϕ, σ, γ, c and c*. Legendre transformation of the perturbative action for each ϕ separately converts the differential identity to a quadratic functional equation on the resulting effective actions, again connecting neighbouring perturbation theories. This is used to prove inductively that the bare action as function of ϕ, σ, etc. is annihilated by a bare version of ρ, implying that the compatibility conditions are indeed renormalized.

The same geometric apparatus is used to derive recursion relations for the partial derivatives of the metric (and $\tilde{F}_{\phi}{}^{i}{}_{j}$) and the source, allowing their systematic calculation in terms of curvature and covariant derivatives.[8] Renormalization should preserve internal symmetry. This is guaranteed by a choice of coordinate system which is natural with respect to the action of the diffeomorphism group, Diff (M), on metrics. The obvious choice is geodesic or normal coordinates, in which the Taylor series begin

$$\tilde{g}_{\phi\, ij}(\sigma) = g_{ij}(\phi) + \tfrac{1}{3}\sigma^{k_1}\sigma^{k_2} R_{ik_1k_2j}(\phi) + \tfrac{1}{6}\sigma^{k_1}\sigma^{k_2}\sigma^{k_3}\nabla_{k_1}R_{ik_2k_3j}(\phi)$$
$$+ \sigma^{k_1}\sigma^{k_2}\sigma^{k_3}\sigma^{k_4}\Big[\tfrac{1}{20}\nabla_{k_1}\nabla_{k_2}R_{ik_3k_4j}(\phi) + \tfrac{1}{45}\big(R_{ik_1k_2\ell}(\phi)R^{\ell}{}_{k_3k_4j}(\phi) + i\leftrightarrow j\big)\Big] + \cdots$$

$$\tilde{h}_{\phi}(x,\sigma) = h(x,\phi) + \sigma^{k_1}\nabla_{k_1}h(x,\phi) + \tfrac{1}{2}\sigma^{k_1}\sigma^{k_2}\nabla_{k_1}\nabla_{k_2}h(x,\phi) + \cdots$$

$$\tilde{F}_{\phi}(\sigma)^{i}{}_{j} = \delta^{i}_{j} + \tfrac{1}{3}\sigma^{k_1}\sigma^{k_2}R^{i}{}_{k_1k_2j}(\phi) + \tfrac{1}{12}\sigma^{k_1}\sigma^{k_2}\sigma^{k_3}\nabla_{k_1}R^{i}{}_{k_2k_3j}$$
$$+ \tfrac{1}{30}\sigma^{k_1}\sigma^{k_2}\sigma^{k_3}\sigma^{k_4}\Big(\tfrac{1}{2}\nabla_{k_1}\nabla_{k_2}R^{i}{}_{k_3k_4j} - \tfrac{2}{3}R^{i}{}_{k_1k_2\ell}R^{\ell}{}_{k_3k_4j}\Big) + \cdots$$

(where $\tilde{P}_\phi(\sigma)^i = \int d^dx\, \sigma^i(x)$).

Primitive divergences are cancelled by replacing these at order K with regulator dependent power series $\tilde{g}^{(K)}_{\phi\, ij}(\sigma)$, $\tilde{h}^{(K)}_\phi(\sigma)$, and $\tilde{F}^{(K)}_\phi(\sigma)^i_j$. The bare metric and source are then given by

$$g^{bare}_{ij}(\phi) = \tilde{g}^{(\infty)}_{\phi\, i'j'}(0) \quad \tilde{F}_\phi{}^{i'}_i(0)\, \tilde{F}_\phi{}^{j'}_j(0)$$

$$h^{bare}(x,\phi) = \tilde{h}^{(\infty)}_\phi(0).$$

The divergences from source vertices are soft so the external field is renormalized linearly.

To calculate renormalization coefficients it is only necessary to consider those irreducible diagrams with primitive divergences and either: (1) two external legs, no ghost or source vertices; or (2) one source vertex, no external legs, no ghost vertices. The renormalization of the ghost vertices can be derived from that of the sources.

The renormalization group equation satisfied by the partition function, derived in standard fashion, is

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \int d^dx\, \tilde{\gamma}(g)\, h(x,\cdot)\frac{\partial}{\partial h(x,\cdot)}\right] Z(h) = 0$$

with

$$\beta(g/T)_{ij} = -\frac{\varepsilon}{T} g_{ij} + R_{ij} + \frac{T}{2} R_{i k_1 k_2 k_3} R_{j k_1 k_2 k_3} + O(T^2)$$

$$\gamma(g/T) = -\frac{T}{2}\nabla_k\nabla_k + O(T^3)$$

$$\tilde{\gamma} = -d + \gamma$$

all to two loops.[9] The β-function is a vector field on the coupling space, assigning to each metric g a small perturbation, i.e. a symmetric tensor field $\beta(g)_{ij}$. Similarly, $\gamma(g)h$ is a vector field on the space of sources, linear in h; $\gamma(g)$ is a linear operator on functions on M.

The free energy as a function of the order parameter $\Phi(x)$ satisfies

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \int d^dx\ \gamma(g)^*\ \Phi(x)\cdot\frac{\partial}{\partial\Phi(x)}\right]\Gamma(\Phi) = 0$$

where $\gamma(g)^*$ acting on measures on M is dual to $\gamma(g)$.

Effective coupling, source and order parameter are found by flowing along the rg trajectories

$$\mu\frac{\partial}{\partial\mu}\, g_{eff}(\mu) = -\beta(g_{eff}(\mu))$$

$$\mu\frac{\partial}{\partial\mu}\, h_{eff}(\mu) = -\tilde{\gamma}(g_{eff}(\mu))\, h_{eff}(\mu)$$

$$\mu\frac{\partial}{\partial\mu}\, \Phi_{eff}(\mu) = \gamma(g_{eff}(\mu))\, \Phi_{eff}(\mu).$$

The coefficients of the rg equations are in every order natural local invariants of the metric (combinations of curvature and covariant derivatives) of appropriate homogeneity degree. This is guaranteed by the use of natural methods in the calculation and by the nature of the Taylor series of the metric. If a metric has a symmetry then all its local invariants will also, so the β-function will too. The same is true of local symmetries, at least in perturbation theory. This means that the rg trajectories stay within the subspace of couplings of any fixed symmetry. If a symmetry group is assumed to act transitively on M, so M is a homogeneous space, then the space of invariant metrics is finite dimensional. Thus renormalizability in the standard sense is proved for the familiar non-linear sigma models.

2. The β-function

Consider zeros of the β-function which depend smoothly ~~and~~ on ε and have a zero temperature limit as ε vanishes.

For $\varepsilon < 0$ these are the solutions of Einstein's equation $0 = -\frac{\varepsilon}{T} g_{ij} + R_{ij}$. All the known examples are either locally symmetric or Kahler.[10] Of the locally symmetric spaces those with no two dimensional factors are infra-red stable fixed points,[11] so are not critical points. The Kahler examples also show no instability, at least through two loops, but they do show marginality. That is, rather than an isolated zero there is a finite dimensional space of zeros. It is not clear whether the marginality persists to higher order or even whether any of the space of zeros remains when higher order corrections to the β-function are taken into account. The two dimensional examples are the surfaces of constant negative curvature. They have no instability, but they have marginality which persists to all orders.

Each of these Einstein metrics give also Gaussian fixed points at zero temperature which are once unstable, in the direction of increasing temperature. Physical intuition about dimensions slightly below two is difficult to come by, but the absence of non-Gaussian critical fixed points among two large classes of fixed points suggests the impossibility of such a creature. In two dimensions the fixed point at $T = 0(\varepsilon)$ merges with the $T = 0$ fixed point and might be interpreted as describing the power law correlations in the long distance limit of a theory whose critical region is out of reach of perturbation theory. The presence of a finite correlation length in the high temperature expansion[12] indicates some sort of phase transition at finite temperature. The locally symmetric and some of the Kahler examples have non-trivial fundamental groups, so might go critical when bound vortices dissociate, as in the XY-model,[13] but without a marginal temperature variable. Other of the Kahler examples are simply connected so would require another mechanism to produce critical behaviour.

When $\varepsilon > 0$ two types of zero are possible: (1) the metric satisfies the Einstein condition $0 = -\frac{\varepsilon}{T} g_{ij} + R_{ij}$ and the zero is at $T = 0(\varepsilon)$; or (2) the metric is Ricci-flat, $R_{ij} = 0$, and the zero is at $T = 0(\varepsilon^{\frac{1}{2}})$. Both are unstable in the temperature variable.

Examples of compact Einstein manifolds with positive scalar curvature (type (1) above) are scarce. The obvious ones are the homogeneous spaces with symmetry group so large that only one invariant symmetric form is possible, up to constant multiples.[14] The corresponding fixed points are essentially already known.[2,3] Assumption of homogeneity gives an algebraic Einstein equation, some solutions of which are known not to be forced by symmetry.[15] These are applicable to frustrated spin systems for which the possible couplings are the left-invariant metrics on a Lie group.[16]

Only one non-homogeneous Einstein metric with positive scalar curvature is known.[17] Existence of the corresponding critical model will not be established until it is shown that linearization of the one loop β-function at the zero is non-degenerate, making it impossible for higher order corrections to remove the zero. The model would be interesting because it would show "spontaneous symmetry breakdown" without the symmetry. Once one of these metrics is known there seems no reason to suppose there are not others.

About the possibility of additional instability there is little of generality to be said. A certain amount of variation of sectional curvature is required, so the manifolds of constant positive curvature, the spheres, are unstable only in the temperature. ~~They~~ There are homogeneous spaces with more instability.

Zeros near the Ricci-flat metrics (type (2) above) are found by consulting the two loop term in β. The integral over M of its contraction is balanced against the zero loop term to fix the temperature. This works unless the metric is flat, in which case its perturbative renormalization is trivial. With the temperature held fixed the metric is perturbed slightly to give enough of a Ricci tensor to cancel the remainder of the two loop term in β . As long as the linearization of the one loop term is non-degenerate this will succeed. But if there are moduli for the Ricci-flat metric then the two loop term will have to satisfy a finite number of linear conditions for the zero to be present. (Its projection onto the moduli space must vanish.) Non-flat Ricci-flat metrics are known to exist.[10] All have moduli; all satisfy the linear conditions so the zeros exist, with marginality, at two-loops. There is no instability beyond that of temperature. It is difficult to find guarantees that any of these zeros will survive higher order corrections. The three loop term might clarify the matter.

Normally 'fixed point of the renormalization group' and 'zero of the β -function' are synonymous, but for the geometric models this is not true. Additional structure is provided by the diffeomorphisms of M acting as covariance group for the renormalization procedure. The simplest manifestation is the dependence of the correlation length not on the coupling itself but only on its equivalence class under the action of Diff(M). The fixed points are defined as the couplings at which the renormalization group makes no change in the correlation length. This happens when flow along the β -function gives the effect of transformation by a diffeomorphism, that is, when $\beta(g)_{ij} = \nabla_i X_j + \nabla_j X_i$ for some vector field X on M. The basic fixed point equations are now (#) $-\frac{\varepsilon}{T} g_{ij} + R_{ij} = \nabla_i X_j + \nabla_j X_i$

with $\varepsilon < 0$, $\varepsilon = 0$ or $\varepsilon > 0$. Temperature instability is present if and only if $\varepsilon \geq 0$.

I have no examples with X non-trivial but I can suggest likely places to look. From the Bianchi identity X must satisfy

$(\flat)$ $-\nabla_k \nabla_k X_i + R_{ik} X_k = 0$ and for non-triviality it must not be a Killing vector. It follows that the scalar curvature must not be constant, so the metric is not homogeneous. If the metric is Kahler then $(\flat)$ is just the statement that X is the real part of a holomorphic vector field, so the candidates for X are known from the complex structure prior to the choice of Kahler metric. This suggests looking for solutions of (#) among the Kahler metrics on complex manifolds with holomorphic vector fields. The two-sphere is one such, but if an SO(2) symmetry is assumed in order to make the problem tractable there is no solution. Any Kahler Einstein manifold with holomorphic vector fields, or equivalently Killing fields, has infinitesmal perturbations solving the linearization of (#). These are responsible for exceptional 1-loop marginality. Unfortunately for the prospects of finding a "quasi-Einstein" fixed point, the marginality turns to stability at two loops in the available examples. Non-homogeneous zeros of the one loop β -function, for instance the Page metric, might be turned into non-zero fixed points by higher order corrections. This is another reason for interest in them.

Covariance under Diff(M) is responsible for extraordinary universal properties of non-zero fixed points (if they exist). The usual local universal properties of a fixed point are its invariants under reparametrizations of couplings and fields. At a non-degenerate zero of the renormalization group generator the invariants are the eigenvalues of its linearization. But in the geometric models reparametrization must commute with the action of Diff(M). At fixed points which are zeros of β this makes no difference, except that only physically distinct eigenvectors of the linearization matter. At a non-zero fixed point the situation would be completely different. In addition to the standard eigenvalues (which could become complex here) the vector field X would itself be universal up to equivalence under Diff(M). Under special circumstances even more subtle invariants would exist. The richness that is possible in the topological structure of the universal vector field associated with a quasi-Einstein metric would be reflected in the structure of the corresponding critical model.

All of the fixed points of the geometric β -function have one loop linearization given by an elliptic pseudo-differential operator bounded above. The dimensions of the eigenspaces with zero or positive eigenvalues must therefore be finite. These are the directions of marginality (or, potentially, instability at higher order) and instability. It is generally expected that only a finite number of parameters should be needed to characterize a given critical system (or quantum field theory). Locally these expectations are met here.

The fixed points for $\epsilon > 0$ all describe critical phenomena associated with continuous phase transitions. At low temperature the models are accurately described by mean field theory (in the measure valued order parameter). In the absence of external field there is coexistence of a collection of pure, ordered equilibrium states indexed by the points in M.[18] The possibility of varying the ordered state continuously implies massless modes (a geometric Goldstone phenomenon). As zero temperature is raised the point measures diffuse outward, gradually losing memory of their origins until at the critical temperature they become identical. The system them has a unique, disordered equilibrium state. About the high temperature phase only critical scaling behaviour can be learned from the perturbative renormalization group. When $\varepsilon = 0$ these fixed points all define asymptotically free theories and describe their short distance scaling behaviour.

The possibility of more complicated asymptotic behaviour of the renormalization group trajectories than that determined by the fixed points cannot be ruled out. The β-function is not in general the gradient of a Diff(M) invariant potential (its behaviour near the Fubini-Study metric on $CP^{m/2}$ rules this out). It is such a gradient in the special cases with m = 2 with the notable exception of the 2-sphere.

The β-function restricted to a finite dimensional space of homogeneous metrics has a gradient-like flow. Complex crossover behaviour between competing fixed points remains possible, however, and is accessible to study because the β-function is now algebraic. In simple examples a fixed point can occur on the boundary of the space of metrics, indicating premature disordering along a submanifold of M and identical to a fixed point of the model based on the submanifold alone. Questions are raised about the meaning of stability when the β-function can connect metrics on different manifolds.[19]

Finally it should be noted that the key question on global properties of the β-function is not whether it itself is a gradient but whether its projection to the space of equivalence classes of metrics is one. It seems not to be, but I have no proof.

My partial analysis of the β-function has benefited from advice and information from several members of Berkeley geometry community: N. Ercolani, S. Kobayashi, P. Li, J. E. Marsden, I. M. Singer and J. A. Wolf. I learned of the relevance of geometric ideas to perturbative quantum field theory from Singer's 1977-78 lectures on gauge theory.

I thank the Nuffield Foundation, the Department of Applied Mathematics and Theoretical Physics, the Institute of Astronomy, Stephen Hawking and Martin Rees for their hospitality.

This work was supported in part by the U.S. Department of Energy.

References

1. K. G. Wilson and J. B. Kogut, Physics Reports, 12C, 2 (1974).

2. A. M. Polyakov, Physics Letters, 59B, no. 1, 79-81 (1975). See also A. A. Migdal, Sov. Phys. JETP, 42, 4, 745-6 (1975).

3. E. Brezin and J. Zinn-Justin, Phys. Rev. Letters, 36, 13 691-4 (1976).
and Phys. Rev. B, 14, 7, 3110-9 (1976).
and J. C. Le Guillon, Phys. Rev. D, 14, 10, 2651-21 (1976).
See also W. A. Bardeen, B. W. Lee, R. F. Shrock, Phys. Rev. D 14, 4, 985-1005 (1976).
and W. A. Bardeen, K. Shizuya, Phys. Rev. D, 18, 6, 1969-82 (1978).

4. The geometric character of this action seems to have been noted first in K. Meetz, J. Math. Phys., 10, 4, 589-93 (1969) and C. J. Isham, Il Nuovo Cimento, LXI A, 2, 188-202 (1969). Both assumed homogeneity. Besides myself its recent reinventors are A. M. Perelemov, Comm. Math. Phys. 63, 237-42 (1978) (assuming Kahler), C. W. Misner, Phys. Rev. D, 18, 12 4510-24 (1978), and M. Stone, Nucl. Phys. B 152, 1, 97-108 (1979).

5. (Reference unavailable,

6. See S. Elitzur, IAS Preprint (on S^m models).

7. C. Becchi, A. Ronet, R. Stora in Renormalization Theory, eds. G. Velo, A. S. Wightman, D. Reidel (1976).

8. The Tayler series of the metric in normal coordinates was given recursively by E. Cartan, "Lecons sur la Geometrie Differentielle" (1924).

9. This calculation of the two loop renormalization group equations reproduces the results of Brezin and Zinn-Justin, ref. 3 on the s^m models and was confirmed by the calculation

of S. Hikami, Kyoto preprint (1979) on the $CP^{m/2}$ models.

10. S. T. Yau, Proc. Natl. Acad. Sci. USA, 74, 5, 1798-9 (1977) and Comm. Pure Appl. Math. 31, 339-411 (1978).

11. A. Weil, Ann. Math. 75, 578-602 (1962).

12. See Stone, ref. 4.

13. J. M. Kosterlitz and D. J. Thouless, J. Phys. C6, 1181 (1973).

14. J. A.Wolf, "The Geometry and Structure of Isotropy Irreducible Homogeneous Spaces" (location unavailable).

15. G. R. Gensen, Indiana Math. J., 20, 12 (1971).

16. G. Toulouse in Physics Reports 49, 2 (1979).

17. D. N. Page, Phys. Lett. 79B, 235 (1978).

18. In a sense, this serves to define the absence of external field.

19. Cf. S. W. Hawking, Nucl. Phys. B. 144, 349-62 (1978).