

# The link between integrability, level crossings, and exact solution in quantum models

**Emil Yuzbashyan**

***Rutgers University***

**Collaborators: Haile K. Owusu and Kshitij Wagh**

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Consider  $H(u)$  – general  $N \times N$  Hermitian matrix  
 $u$  – real parameter (coupling constant)

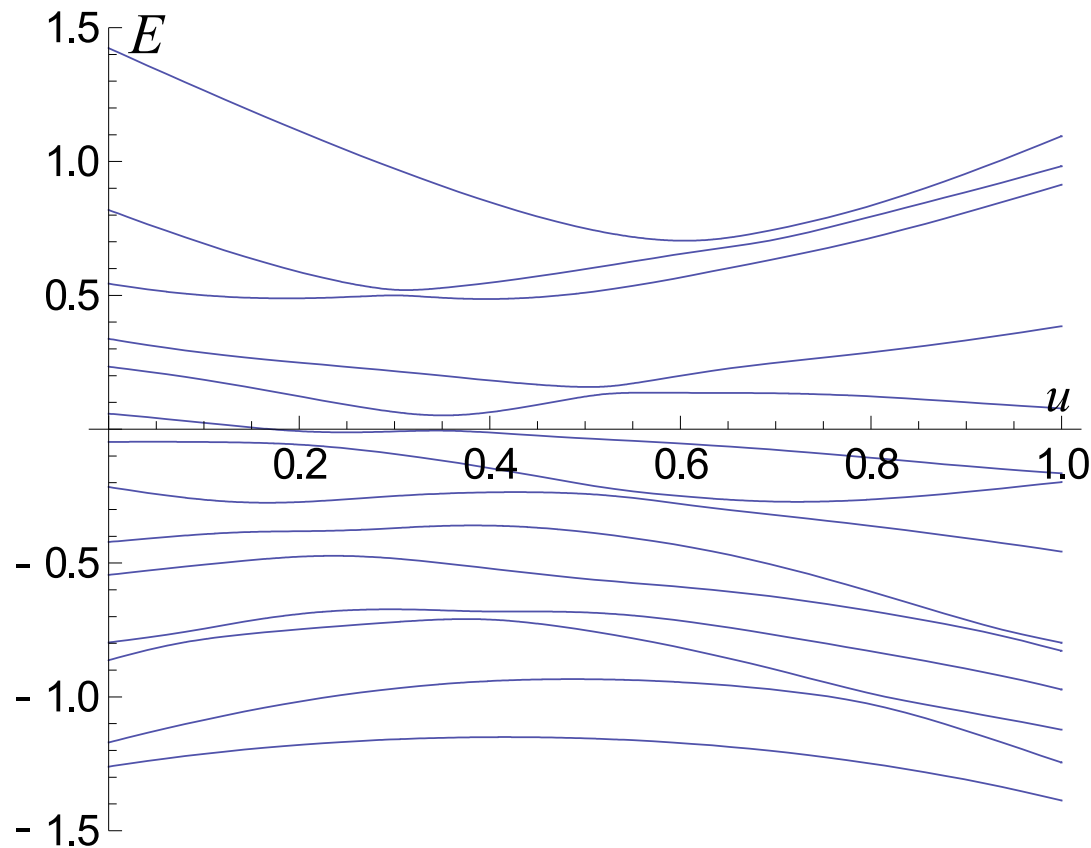
**Q:** What can we say about evolution of eigenvalues with  $u$ ?

For example, let  $H(u)$  be a block of (many-body, lattice) Hamiltonian characterized by a complete set of quantum numbers, i.e.

$$[S, H(u)] = 0 \implies S = aI$$

(Any  $u$ -independent symmetry can be removed by going to smaller blocks until all symmetry is exhausted.)

For a **typical**  $H(u)$  energy levels never cross!!!



$H(u) = T + uV$ , where independent matrix elements of  $T$  and  $V$  are uniformly distributed random numbers

# Noncrossing rule (theorem)

For a **typical**  $H(u)$  energy levels with same quantum numbers cannot cross

F. Hund, *Z. Phys.* v.40, p.742, 1927

“Proof” J. v. Neumann & E. Wigner, *Z. Phys.* v.30, p.467, 1929

Textbooks (e.g. Landau & Lifshitz, *Quantum Mechanics*) usually use the version due to E. Teller, *J. Phys. Chem.* 41 109, 1937

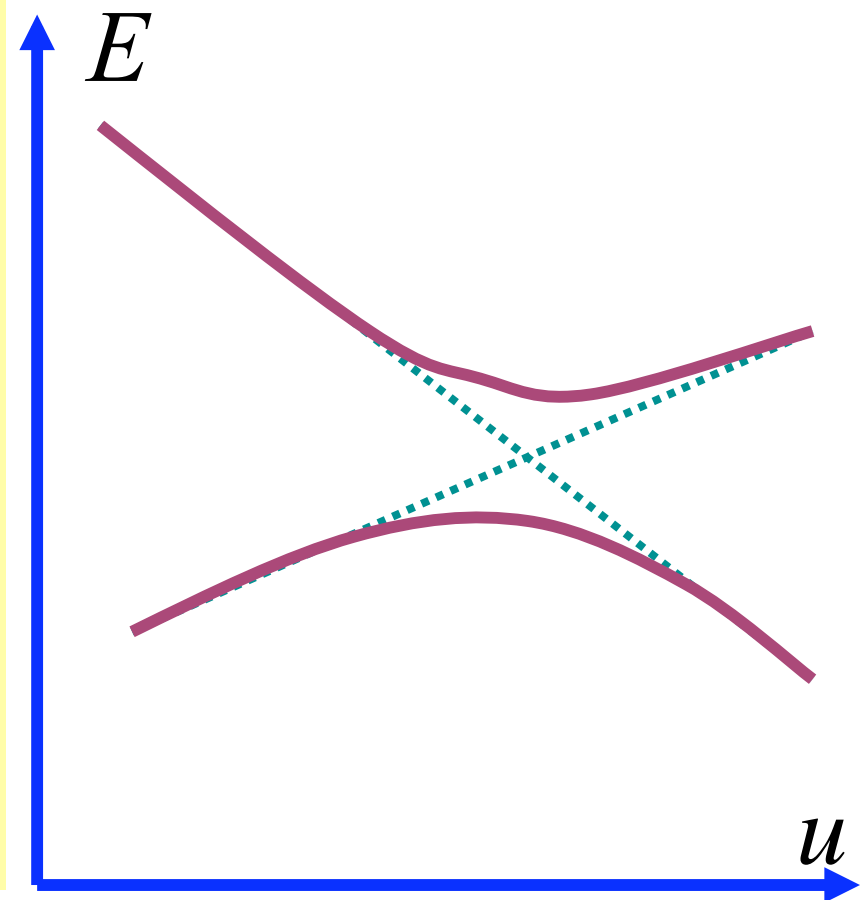
Need two or more real parameters to get finite number of crossings per unit volume in parameter space

Numerous restatements/ refinements in various contexts, e.g. in Random Matrix Theory (level repulsion)

**V. I. Arnold**, *Mathematical Methods of Classical Mechanics*  
(Springer-Verlag: New York), Appendix 10, 1989

Quadratic forms (real symmetric case,  $H_{ij}(u)=H_{ji}(u)$ )

Under a change of parameter in the **typical** one-parameter family the eigenvalues can approach closely, but when they are sufficiently close, it is as if they begin to repel one another. The eigenvalues again diverge, disappointing the person who hoped, by changing the parameter to achieve a multiple spectrum.



“Typical” suggests that there are atypical  $N \times N$  Hermitian matrices (of some measure zero) that violate the noncrossing rule

What are these atypical Hamiltonians?

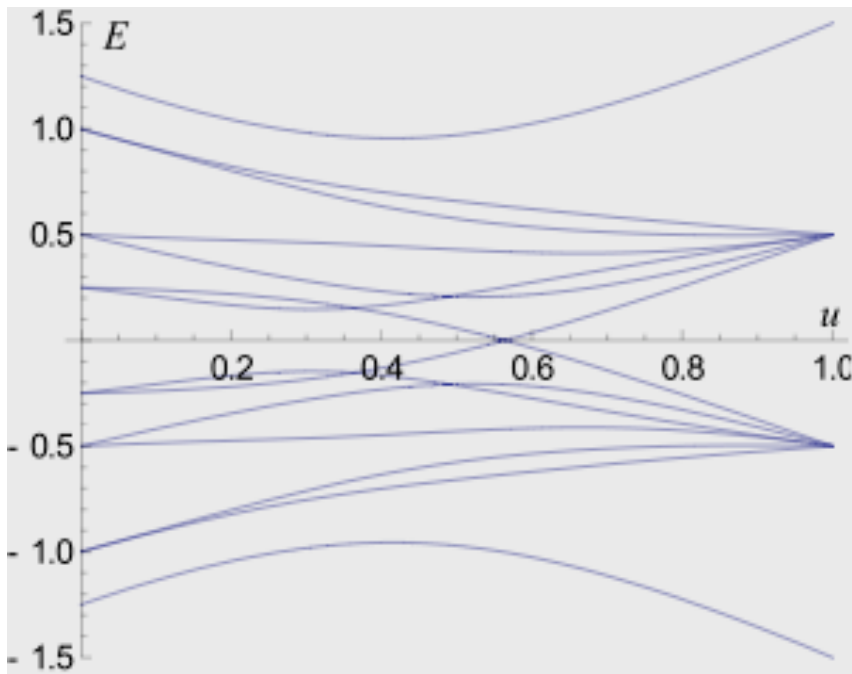
Common belief: Atypical  $H(u)$  is usually a quantum integrable Hamiltonian

At least proliferation of such crossings  $\longrightarrow$  integrable  $H(u)$

Q: 1) Is this true? 2) What is an integrable  $N \times N$  parameter dependent Hermitian matrix? Does this make sense?

# Violation of the noncrossing rule in 1d Hubbard model

**O. J. Heilmann and E. H. Lieb,** *Ann. N. Acad. Sci. v.172, p. 583 (1971)*



Energies for a **14 x 14** block of 1d Hubbard on six sites characterized by a complete set of quantum numbers

$$u = U / (U - 4T)$$

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c_{js}^\dagger c_{j+1s} + c_{j+1s}^\dagger c_{js}) + U \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

Tight-binding + onsite interactions, electrons on a chain

$$\hat{H} = \hat{T} + u\hat{V}, \quad u \in [0, 1]$$

“The noncrossing rule is apparently violated in the case of the 1d Hubbard Hamiltonian for benzene molecule [six sites]...”

Other integrable Hamiltonians known to violate the noncrossing rule: 1d anisotropic Heisenberg, reduced Bardeen-Cooper-Schrieffer model etc. All integrable models with finite state space(?)

**Q:** Why integrable Hamiltonians  $H(u)$  show crossings of levels with same quantum #s? What is a finite-dimensional ( $N \times N$ ) integrable quantum Hamiltonian?

In quantum mechanics degeneracies (crossings) are usually “explained” by symmetry (e.g. Landau & Lifshitz)

But here there is no usual (parameter-independent) symmetry:

$$[S, H(u)] = 0 \implies S = aI$$

Integrable quantum models have nontrivial parameter-dependent conservation laws (dynamical symmetries, conserved currents)

$$[H(u), \tilde{H}(u)] = 0 \quad \text{for all } u$$

Hamiltonian ( $N \times N$   
Hermitian matrix)

Dynamical symmetry (another  $N \times N$   
Hermitian matrix)

In fact, quantum integrability is often understood as the existence of sufficiently many (?) of such conserved currents. (Usually, goes hand in hand with exact solution via Bethe Ansatz.)

Q: Can dynamical symmetries explain violations of the noncrossing rule?

# Conserved currents for 1d Hubbard model

**B. S. Shastry**, *Phys. Rev. Lett.* **56**, 1529(1986); *Phys. Rev. Lett.* **56**, 2453 (1986); *J. Stat. Phys.* **50**, 57 (1988)

$$H(u) \equiv H_1(u) = \sum_{j=1}^N \sum_{s=\uparrow\downarrow} (c_{js}^\dagger c_{j+1s} + c_{j+1s}^\dagger c_{js}) + u \sum_{j=1}^N \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

$$\begin{aligned} \tilde{H}(u) \equiv H_2(u) = & i \sum_{j=1}^N \sum_{s=\uparrow\downarrow} (c_{j+2s}^\dagger c_{js} - c_{js}^\dagger c_{j+2s}) - \\ & iu \sum_{j=1}^N \sum_{s=\uparrow\downarrow} (c_{j+1s}^\dagger c_{js} - c_{js}^\dagger c_{j+1s}) (\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1) \end{aligned}$$

**Both the Hamiltonian and first conserved current are linear in  $u$**

$H_2(u), H_3(u), H_4(u), \dots$  - in principle, infinitely many **dynamical symmetries**, but not all of them are nontrivial for finite  $N$

**Q: Can dynamical symmetries explain violations of noncrossing rule in integrable Hamiltonians?**

$$[H(u), \tilde{H}(u)] = 0 \implies \text{level crossings?}$$

**Problems:**

**1) What constitutes a nontrivial integral of motion?**

$$\tilde{H}(u) = H^2(u), H^3(u), \dots, \text{projectors } P_n(u) = |n(u)\rangle\langle n(u)|$$

**always commute with  $H(u)$  – restrictions on the form of integrals are necessary...**

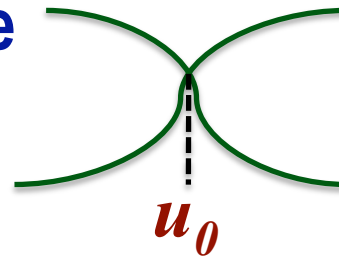
**2) How many independent integrals one needs to call an  $N \times N$  Hermitian matrix integrable?**

**In other words, quantum integrability is not well-defined at least for finite  $N$ ...**

## Problems: (cont'd)

3) Usual argument that levels of different symmetry can cross doesn't work for dynamical symmetries

4) Given a crossing can always cook-up a “dynamical symmetry”, i.e. restrictions on the form of integrals are necessary

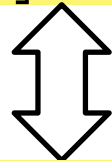


$$\tilde{H}(u) = A\delta_{uu_0}$$

Hamiltonian & conserved currents for physical integrable models are often linear in the coupling  $u$ , e.g. BCS and anisotropic Heisenberg models, Hubbard Hamiltonian etc. Let us consider this case from now on.

**Let**  $H(u) = T + uV, \quad \tilde{H}(u) = \tilde{T} + u\tilde{V}$

$$[H(u), \tilde{H}(u)] = 0 \quad \text{for all } u$$



$$[T, \tilde{T}] = [V, \tilde{V}] = 0, \quad [T, \tilde{V}] = [\tilde{T}, V] \quad (1)$$

**nontrivial (independent) current:**  $\tilde{H}(u) \neq cH(u) + (a + bu)I$

**Now commutation condition (1) severely constrains matrices  $T$  &  $V$**

**E.g. in  $3 \times 3$  case commutation condition (1) is equivalent to a single algebraic constraint on matrix elements  $T_{ij}$  &  $V_{ij}$**

$$f(T_{ij}, V_{ij}) = 0$$


Crossing condition in  $3 \times 3$  case:

$$H(u) = T + uV$$

$$\exists u_0 : \text{Discriminant}_\lambda |H(u_0) - \lambda I| = 0$$

is also a single algebraic constraint on matrix elements  $T_{ij}$  &  $V_{ij}$   $g(T_{ij}, V_{ij}) = 0$

Moreover,  $g = f$ , i.e. crossing condition = commutation condition in  $3 \times 3$  case!!!  $[H(u), \tilde{H}(u)] = 0 \implies$  crossings in  $3 \times 3$  case!

 No longer true for  $4 \times 4$  matrices  $H(u) = T + uV$ , i.e. the existence of a single nontrivial integral of motion (linear in  $u$ ) does not guarantee crossings (explicit example on file)

**Q:** How many linear in  $u$  integrals of motion are necessary to call an  $N \times N$  Hamiltonian  $H(u) = T + uV$  integrable?

# Classical integrability

A classical Hamiltonian  $H(p_i, q_i)$  with  $n$  degrees of freedom ( $n$  coordinates) is said to be integrable if it has the maximum possible number ( $n$ ) of independent Poisson-commuting integrals of motion



Liouville-Arnold theorem: the dynamics of  $H(p_i, q_i)$  is exactly solvable by quadratures

**Q:** Can we adopt a similar notion of integrability for  $N \times N$  Hermitian matrices? Is there a quantum extension of the Liouville-Arnold theorem? Can we prove that these integrable quantum Hamiltonians violate the noncrossing rule?

**Yes we can!** (B. H. Obama, 2008, unpublished?)

## Maximally commuting Hamiltonians (matrices)

Let  $H(u)=T+uV$  be an  $N \times N$  Hermitian matrix and let us require that the existence of the maximum possible number (which turns out to be  $N-1$ ) of linearly independent conservation laws  $H_i(u)=T_i+uV_i$   $H_1(u) \equiv H(u)$

$$\boxed{[H_i(u), H_j(u)] = 0, \text{ for all } u \text{ and } i, j = 1, \dots, N} \quad (1)$$

Using this commutation property **alone** one can (for any  $N$ ):

1) Explicitly parameterize a general maximal Hamiltonian, i.e. solve nonlinear commutation relations (1) (impossible for submaximal systems)

**Maximal Hamiltonians**  $H_i(u) = T_i + uV_i$  (having maximum number of independent commuting partners)  $H_1(u) \equiv H(u)$

$$[H_i(u), H_j(u)] = 0, \text{ for all } u \text{ and } i, j = 1, \dots, N \quad (1)$$

Using this commutation property **alone** it turns out to be possible (for any  $N$ ):

- 1) Explicitly parameterize a general maximal Hamiltonian
- 2) Prove a theorem: Any maximally commuting Hamiltonian has at least one level crossing for  $N > 2$
- 3) Obtain an exact solution for the spectrum of each  $H_i(u)$  (of Bethe Ansatz type) – quantum version of the Liouville-Arnold theorem

Can also show that less than maximum number of commuting partners does not guarantee level crossings

**Parameterization** of maximal  $N \times N$  Hamiltonians  $H_i(u) = T_i + uV_i$  (having maximum number of independent commuting partners)

**Maximal Hamiltonians form an  $(N+1)$ -dimensional vector space**

**general element (vector):**  $H(u) = \sum_{i=1}^N d_i H^i(u) + aI, d_i, a \in R$

**In a convenient basis can resolve commutation relations**  $[H_i(u), H_j(u)] = 0$

$$\begin{aligned} [H^i(u)]_{ij} &= \frac{\gamma_i \gamma_j}{\varepsilon_i - \varepsilon_j}, \quad i \neq j, \\ [H^i(u)]_{jj} &= -\frac{\gamma_i^2}{\varepsilon_i - \varepsilon_j}, \quad i \neq j, \\ [H^i(u)]_{ii} &= u - \sum_{j \neq i} \frac{\gamma_j^2}{\varepsilon_i - \varepsilon_j}. \end{aligned}$$

$\gamma_i, \varepsilon_i - 2N$   
real parameters

# Maximally commuting Hamiltonians map to Gaudin magnets!!!

$$\hat{H}^i(u) = -u \hat{s}_i^z + \sum_{k=1}^N \frac{\hat{\vec{s}}_i \cdot \hat{\vec{s}}_k}{\varepsilon_i - \varepsilon_k}, \quad i = 1, \dots, N$$

$\hat{\vec{s}}_i$  - quantum spins of length  $s_i$ ,  $u$  - magnetic field,  $\varepsilon_i$  - real parameters

The  $z$  component of the total spin  $\hat{J}^z = \sum_{i=1}^N \hat{s}_i^z$  is conserved

In the sector  $J_z = \max - 1$  (next to highest weight) matrix elements of Gaudin magnets coincide with those of maximally commuting Hamiltonians

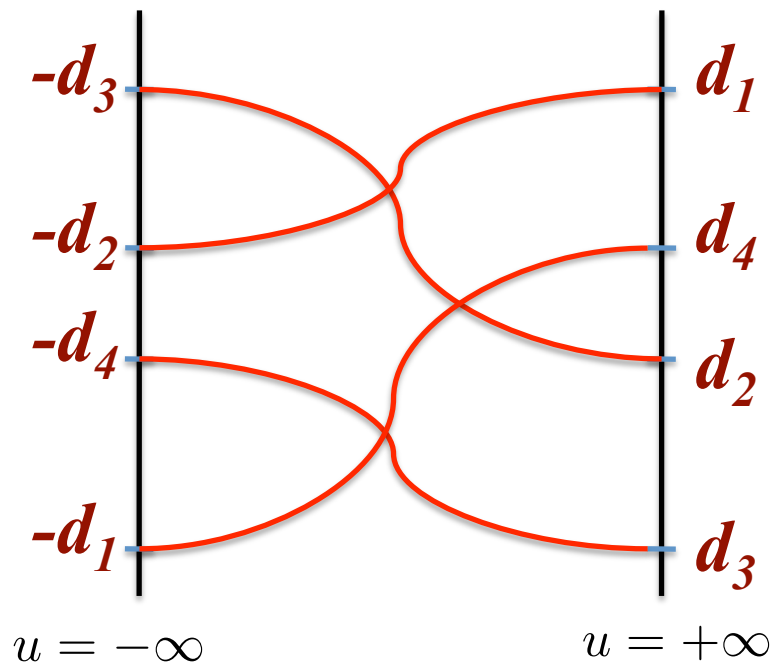
Exact solution for the spectrum (of Bethe Ansatz type), M. Gaudin (1972 – 1976) “*Diagonalisation d’une classe d’hamiltoniens de spin*”

# What about level crossings?

$H(u) = T + uV$  – general maximal Hamiltonian (maximal number of Hermitian commuting partners linear in  $u$ )

As  $u \rightarrow \pm\infty$  eigenvalues  $E_m(u) \rightarrow \pm|u|d_m$ , where  $d_m$  are the eigenvalues of  $V$

E.g. consider an  $N=4$  example and let  $d_1 > d_4 > d_2 > d_3$



Exact solution allows to track the evolution of eigenvalues with  $u$ !

Three crossings for this ordering

# What about level crossings?

$H(u)=T+uV$  – general maximal Hamiltonian (maximal number of Hermitian commuting partners linear in  $u$ )

Using exact solution for the spectrum of  $H(u)$  can demonstrate:

**Theorem:** All maximal  $N \times N$  Hamiltonians have at least one level crossing for  $N > 2$  and

$$\# \text{ of crossings} = \frac{(N-1)(N-2)}{2} - 2K$$

maximum possible # of crossings

nonnegative integer

For example, for  $N=6$  allowed number of crossings = 2, 4, 6, 8, 10

# Summary

1. Established the relationship between integrability (existence of nontrivial commuting partners) on one hand and level crossings & exact solution on the other for  $N \times N$  quantum Hamiltonians  $H(u) = T + uV$
2. Existence of maximum number of independent commuting partners linear in  $u$  guarantees i) exact solution for the spectrum (quantum extension of Liouville-Arnold theorem) ii) level crossings in violation of the noncrossing rule
3. Explicitly parameterized the most general maximally commuting Hamiltonians. Exact solution via mapping to Gaudin magnets
4. Submaximal Hamiltonians do not have to have level crossing, i.e. maximal Hamiltonians are the only ones that for sure violate the noncrossing rule