Modulated superradiance and synchronized chaos in clouds of atoms in a bad cavity

Emil Yuzbashyan

Collaborators: Aniket Patra (Rutgers) and Boris Altshuler (Columbia)
Ultra-cold atoms coupled to a bad cavity: superradiant laser

- Ultra-stable, sub-milihertz linewidth optical laser (quality factor $\sim 10^{18}$)
- Ultra-high precision atomic clock
- Extreme bad cavity limit (cavity photon decay rate – largest rate)
- $N \sim 10^6$ of $^{87}$Rb atoms

Bohnet et. al., EPJ Web of Conferences, v. 57, 2013

Atom + Cavity Hamiltonian in the rotating frame of the cavity field:

$$
\hat{H} = \Delta \hat{S}_z + \frac{\Omega_d}{2} \left( \hat{a}^\dagger \hat{S}_- + \hat{S}_+ \hat{a} \right)
$$

Collective spin: $\hat{S}_z = \frac{1}{2} \sum_{j=1}^{N} \hat{\sigma}_j^z$, $\hat{S}_\pm = \frac{1}{2} \sum_{j=1}^{N} \hat{\sigma}_j^\pm$

Cavity mode: $\hat{a}, \hat{a}^\dagger$

Detuning from the cavity mode: $\Delta$

---

**Figure 2.** (a) Ultranarrow superradiant laser (b) Raman superradiant laser (c) Experimental optical cavity

Optical Cavity

Mirrors

MOT

Location

Ultra-stable, sub-milihertz linewidth optical laser (quality factor $\sim 10^{18}$)

Ultra-high precision atomic clock

Extreme bad cavity limit (cavity photon decay rate – largest rate)

$N \sim 10^6$ of $^{87}$Rb atoms

**Bohnet et. al., EPJ Web of Conferences, v. 57, 2013**

Collective enhancement with direct analogy to a classical phased-array antenna [31].

For the atoms to collectively establish a strong superradiant coupling to the cavity mode, providing a presence of the cavity. However, the collective coupling particle weak coupling limit.

Our studies from $\hat{a}$ such that a single atom scarcely experiences the decay rate $S_2$ that can be controlled with the $|g\rangle$, $|e\rangle$

J $\sim 1~\text{mHz}$

Ultra-narrow superradiant laser

Raman superradiant laser

Sample technology uninteresting as a stable optical phase reference. However, the tunability of intensity of the dressing laser. We typically operate with $\tilde{W}$ for details) has an $\epsilon_i$ e $\epsilon_i$

Simplified energy level diagrams of (a) a proposed superradiant light source using an ultra-narrow atomic clock.

This is applied non-resonantly along the cavity axis. The repumping light (green) is applied perpendicular to the cavity optical transition in alkaline-earth atoms (Sr, Yb, Ca, etc.) and (b) a superradiant Raman laser. Proposed superradiant Raman laser uses an $\epsilon_i$ $\epsilon_i$ e $\epsilon_i$ $\epsilon_i$

The laser operates in the single and is equivalent to the Purcell factor [30]. Here the emitted light (blue) is detected in heterodyne with the dressing laser, removing the dressing laser's phase noise for the decay rate $S_3$.

The emitted light is detected in heterodyne with the dressing laser, removing the dressing laser's phase noise for the decay rate $S_3$.

The spontaneous synchronization of the individual atomic dipoles leads us to represent the collective enhancement with direct analogy to a classical phased-array antenna [31].

For the atoms to collectively establish a strong superradiant coupling to the cavity mode, providing a presence of the cavity. However, the collective coupling particle weak coupling limit.

Our studies from $\hat{a}$ such that a single atom scarcely experiences the decay rate $S_2$ that can be controlled with the $|g\rangle$, $|e\rangle$

J $\sim 1~\text{mHz}$

Ultra-narrow superradiant laser

Raman superradiant laser

Sample technology uninteresting as a stable optical phase reference. However, the tunability of intensity of the dressing laser. We typically operate with $\tilde{W}$ for details) has an $\epsilon_i$ e $\epsilon_i$ $\epsilon_i$ $\epsilon_i$

Simplified energy level diagrams of (a) a proposed superradiant light source using an ultra-narrow atomic clock.

This is applied non-resonantly along the cavity axis. The repumping light (green) is applied perpendicular to the cavity optical transition in alkaline-earth atoms (Sr, Yb, Ca, etc.) and (b) a superradiant Raman laser. Proposed superradiant Raman laser uses an $\epsilon_i$ $\epsilon_i$ e $\epsilon_i$ $\epsilon_i$
Two (or more) clouds in the cavity: synchronization of atomic clocks

Xu et. al., PRL 113, 154101, 2014

Christiaan Huygens
“an odd kind of sympathy” in maritime clocks
Letters to de Sluse, 1665

Oliveira & Melo, Scientific Reports 5:11548, 2015
Two (or more) clouds in the cavity: synchronization of atomic clocks

Two collective spins:

\[
\hat{S}_{z}^{A,B} = \frac{1}{2} \sum_{j=1}^{N} \hat{\sigma}_{jz}^{A,B}, \quad \hat{S}_{\pm}^{A,B} = \sum_{j=1}^{N} \hat{\sigma}_{j\pm}^{A,B}
\]

\[
\hat{H} = \frac{\delta}{2} \left( \hat{S}_{z}^{A} - \hat{S}_{z}^{B} \right) + \frac{\Omega}{2} \left( \hat{a}^{\dagger} \hat{S}_{-}^{A} + \hat{S}_{+}^{A} \hat{a} + \hat{a}^{\dagger} \hat{S}_{-}^{B} + \hat{S}_{+}^{B} \hat{a} \right)
\]

(Can always set \(\omega_{A} + \omega_{B} = 0 \Rightarrow \omega_{A,B} = \pm \delta/2\))
Rich collective dynamics much beyond synchronization of atomic clocks

- Two types of limit cycles
- Chaos via quasiperiodicity
- Synchronized chaos
- Regions of coexistence

Nonequilibrium phase diagram for two ensembles of atoms in a bad cavity (mean-field)
$W$ – pumping, $\delta$ – detuning from the cavity mode
\[
\dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}_z^A - \hat{S}_z^B, \rho \right] - \frac{i\Omega}{2} \left[ \hat{a}^\dagger \hat{J}_- + \hat{J}_+ \hat{a}, \rho \right] + \kappa \mathcal{L}[\hat{a}]\rho + W \sum_{\tau=A,B} \sum_{j=1}^{N} \mathcal{L}[\hat{\sigma}_{j+}^\tau] \rho
\]

\[
\mathbf{J} = \mathbf{S}^A + \mathbf{S}^B
\]

Lindblad super-operators: \[
\mathcal{L}[\hat{O}]\rho = \frac{1}{2} \left( 2\hat{O}\rho\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\rho - \rho\hat{O}^\dagger\hat{O} \right)
\]
\[
\dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}_z^A - \hat{S}_z^B, \rho \right] - \frac{i\Omega}{2} \left[ \hat{a}^\dagger \hat{J}_- + \hat{J}_+ \hat{a}, \rho \right] + \kappa \mathcal{L}[\hat{a}] \rho + W \sum_{\tau=A,B} \sum_{j=1}^N \mathcal{L}[\hat{\sigma}_j^\tau] \rho
\]

\[
\mathbf{J} = \mathbf{S}^A + \mathbf{S}^B
\]

Lindblad super-operators: \[
\mathcal{L}[\hat{O}] \rho = \frac{1}{2} \left( 2\hat{O} \rho \hat{O}^\dagger - \hat{O}^\dagger \hat{O} \rho - \rho \hat{O}^\dagger \hat{O} \right)
\]

Adiabatic elimination: (bad cavity limit) \[
\frac{d\langle \hat{a} \rangle}{dt} = -\frac{\kappa}{2} \langle \hat{a} \rangle - i \frac{\Omega}{2} \langle \hat{J}_- \rangle \implies \hat{a} \approx -i \frac{\Omega}{\kappa} \hat{J}_-, \quad \tau \gg \kappa^{-1}
\]

Master equation

\[ \dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}^A_z - \hat{S}^B_z, \rho \right] - \frac{i\Omega}{2} \left[ \hat{a}^\dagger \hat{J}_- + \hat{J}_+ \hat{a}, \rho \right] + \kappa \mathcal{L}[\hat{a}]\rho + W \sum_{\tau=A,B} \sum_{j=1}^N \mathcal{L}[\hat{\sigma}_{j+}^\tau] \rho \]

Cavity intensity decay rate  

Pump rate

\[ \mathbf{J} = \mathbf{S}^A + \mathbf{S}^B \]

Lindblad super-operators:  

\[ \mathcal{L}[\hat{O}]\rho = \frac{1}{2} \left( 2\hat{O}\rho\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\rho - \rho\hat{O}^\dagger\hat{O} \right) \]

Adiabatic elimination:  

(bad cavity limit)  

\[ \frac{d\langle \hat{a} \rangle}{dt} = -\frac{\kappa}{2} \langle \hat{a} \rangle - i \frac{\Omega}{2} \langle \hat{J}_- \rangle \implies \hat{a} \approx -i \frac{\Omega}{\kappa} \hat{J}_- , \quad \tau \gg \kappa^{-1} \]


Effective atomic master equation:

\[ \dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}^A_z - \hat{S}^B_z, \rho \right] + \Gamma_c \mathcal{L}[\hat{J}_-]\rho + W \sum_{\tau=A,B} \sum_{j=1}^N \mathcal{L}[\hat{\sigma}_{j+}^\tau] \rho \]

Collective decay rate,  

\[ \Gamma_c = \frac{\kappa^2}{\kappa} \]
System size expansion

\[ \dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}_z^A - \hat{S}_z^B, \rho \right] + \Gamma_c \mathcal{L}[\hat{J}_-] \rho + W \sum_{\tau=A,B} \sum_{j=1}^{N} \mathcal{L}[\hat{\sigma}_j^\tau] \rho \]

Set \( N\Gamma_c = 1 \)

\[ \dot{\rho} = -\frac{i\delta}{2} \left[ \hat{S}_z^A - \hat{S}_z^B, \rho \right] + \Gamma_c \mathcal{L}[\hat{J}_-] \rho + W \sum_{\tau=A,B} \sum_{j=1}^{N} \mathcal{L}[\hat{\sigma}_{j+}^\tau] \rho \]

Set \( N\Gamma_c = 1 \)


- Define the characteristic function and the associated quasiprobability distribution

\[ \chi_N(\xi_A, \xi_A^*, \eta_A, \xi_B, \xi_B^*, \eta_B) \equiv \text{tr} \left[ \rho e^{i\xi_A^* \hat{S}_+} e^{\eta_A \hat{S}_z} e^{i\xi_A \hat{S}_-} e^{\xi_B^* \hat{S}_+} e^{\eta_B \hat{S}_z} e^{i\xi_B \hat{S}_-} \right] \]

\[ \equiv \int d^2v_A \int dm_A \int d^2v_B \int dm_B P(v_A^*, v_A, m_A, v_B^*, v_B, m_B) \times \]

\[ \times e^{i\xi_A^* v_A^*} e^{i\xi_A v_A} e^{\eta_A m_A} e^{i\xi_B v_B^*} e^{i\xi_B v_B} e^{\eta_B m_B} \]

Glauber–Sudarshan P representation, PRL 10, 277; Phys. Rev. 131 (1963)

- Expand in \( \frac{1}{\sqrt{N}} \) (system size expansion)
0th order – semiclassical equations of motion

Ensemble A:

\[
\dot{s}_x^A = -\frac{\delta}{2} s_y^A - \frac{W}{2} s_x^A + \frac{1}{2} s_z^A J_x \\
\dot{s}_y^A = \frac{\delta}{2} s_x^A - \frac{W}{2} s_y^A + \frac{1}{2} s_z^A J_y \\
\dot{s}_z^A = W(1 - s_z^A) - \frac{1}{2} s_x^A J_x - \frac{1}{2} s_y^A J_y
\]

Ensemble B:

\[
\dot{s}_x^B = \frac{\delta}{2} s_y^B - \frac{W}{2} s_x^B + \frac{1}{2} s_z^B J_x \\
\dot{s}_y^B = -\frac{\delta}{2} s_x^B - \frac{W}{2} s_y^B + \frac{1}{2} s_z^B J_y \\
\dot{s}_z^B = W(1 - s_z^B) - \frac{1}{2} s_x^B J_x - \frac{1}{2} s_y^B J_y
\]

\[s = \frac{2}{N} \langle \hat{S} \rangle, \quad J = s^A + s^B, \quad N\Gamma_c = 1\]
\[ \text{Ensemble A:} \]
\[
\begin{align*}
\dot{s}_x^A &= -\frac{\delta}{2} s_y^A - \frac{W}{2} s_x^A + \frac{1}{2} s_z^A J_x \\
\dot{s}_y^A &= \frac{\delta}{2} s_x^A - \frac{W}{2} s_y^A + \frac{1}{2} s_z^A J_y \\
\dot{s}_z^A &= W(1 - s_z^A) - \frac{1}{2} s_x^A J_x - \frac{1}{2} s_y^A J_y
\end{align*}
\]

\[ \text{Ensemble B:} \]
\[
\begin{align*}
\dot{s}_x^B &= \frac{\delta}{2} s_y^B - \frac{W}{2} s_x^B + \frac{1}{2} s_z^B J_x \\
\dot{s}_y^B &= -\frac{\delta}{2} s_x^B - \frac{W}{2} s_y^B + \frac{1}{2} s_z^B J_y \\
\dot{s}_z^B &= W(1 - s_z^B) - \frac{1}{2} s_x^B J_x - \frac{1}{2} s_y^B J_y
\end{align*}
\]

\[ \dot{s} = -s \times H \]
\[ H = \pm \frac{\delta}{2} \hat{z} \]

\[ s = \frac{2}{N} \langle \hat{S} \rangle, \quad J = s^A + s^B, \quad N \Gamma_c = 1 \]
0th order – semiclassical equations of motion

Ensemble A:

\[
\begin{align*}
\frac{\dot{s}_x^A}{2} &= -\frac{\delta}{2} s_y^A - \frac{W}{2} s_x^A + \frac{1}{2} s_z^A J_x \\
\frac{\dot{s}_y^A}{2} &= \frac{\delta}{2} s_x^A - \frac{W}{2} s_y^A + \frac{1}{2} s_z^A J_y \\
\frac{\dot{s}_z^A}{2} &= W(1 - s_z^A) - \frac{1}{2} s_x^A J_x - \frac{1}{2} s_y^A J_y
\end{align*}
\]

Ensemble B:

\[
\begin{align*}
\frac{\dot{s}_x^B}{2} &= -\frac{\delta}{2} s_y^B - \frac{W}{2} s_x^B + \frac{1}{2} s_z^B J_x \\
\frac{\dot{s}_y^B}{2} &= \frac{\delta}{2} s_x^B - \frac{W}{2} s_y^B + \frac{1}{2} s_z^B J_y \\
\frac{\dot{s}_z^B}{2} &= W(1 - s_z^B) - \frac{1}{2} s_x^B J_x - \frac{1}{2} s_y^B J_y
\end{align*}
\]

Landau-Lifshitz damping

\[
\frac{1}{2} s \times (J \times \hat{z})
\]

Single ensemble, \( W = 0 \) – LL equation

\[
\dot{s} = -s \times H - \lambda s \times (s \times H)
\]

\[
H = \pm \frac{\delta}{2} \hat{z}
\]

Two coupled LL eqs + pumping

\[
s = \frac{2}{N} \langle \hat{S} \rangle, \quad J = s^A + s^B, \quad N \Gamma_c = 1
\]
Next order in system size expansion (leading quantum correction) – Fokker-Planck equation

\[ \frac{\partial}{\partial t} \bar{P}(\nu_A, \nu_A^*, \mu_A, \nu_B, \nu_B^*, \mu_B, t) = \left\{ \left( \frac{i \delta}{2} + \frac{W}{2} - NT_c \bar{S}_z^A(t) \right) \frac{\partial}{\partial \nu_A} \nu_A + \left( -\frac{i \delta}{2} + \frac{W}{2} - NT_c \bar{S}_z^A(t) \right) \frac{\partial}{\partial \nu_A^*} \nu_A^* + \right. \\
\left. + W \frac{\partial}{\partial \mu} \mu_A \right\} + \left\{ \delta \rightarrow -\delta, \ A \rightarrow B \right\} \bar{P} + \left\{ \right. \\
\left. \left. + NT_c \left( \bar{S}_z^A(t) + \bar{S}_z^A(t) \bar{S}_z^B(t) \right) \frac{\partial^2}{\partial \nu_A^*} \right. \\
\left. \left. + \frac{W}{16} - \frac{W}{4} \bar{S}_z^A(t) - \frac{NT_c}{16} \left( 2\bar{S}_+^A(t) \bar{S}_-^A(t) + \bar{S}_+^A(t) \bar{S}_-^B(t) + \bar{S}_-^A(t) \bar{S}_+^B(t) \right) \right. \\
\left. \left. \frac{\partial^2}{\partial \mu_A^2} + W \frac{\partial^2}{\partial \nu_A \partial \nu_A^*} - \frac{W \bar{S}_-^A(t)}{2} \frac{\partial^2}{\partial \nu_A \partial \mu_A} - \right. \\
\left. \left. - \frac{W \bar{S}_+^A(t)}{2} \frac{\partial^2}{\partial \nu_A \partial \mu_A} \right\} + \left\{ A \leftrightarrow B \right\} \right\} \bar{P}, \right. \\
\bar{P}(\nu_A, \nu_A^*, \mu_A, \nu_B, \nu_B^*, \mu_B, t) = N^3 P(\nu_A, \nu_A^*, m_A, \nu_B, \nu_B^*, m_B, t) \\
\bar{S}_{\pm, z}^A, B(t) = \frac{1}{N} S_{\pm, z}^A, B(t) \equiv \frac{1}{N} \langle \hat{S}_{\pm, z}^A, B(t) \rangle \]
Ensemble A:

\[
\begin{align*}
\dot{s}_x^A &= -\frac{\delta}{2} s_y^A - \frac{W}{2} s_x^A + \frac{1}{2} s_z^A J_x \\
\dot{s}_y^A &= \frac{\delta}{2} s_x^A - \frac{W}{2} s_y^A + \frac{1}{2} s_z^A J_y \\
\dot{s}_z^A &= W (1 - s_z^A) - \frac{1}{2} s_x^A J_x - \frac{1}{2} s_y^A J_y
\end{align*}
\]

Ensemble B:

\[
\begin{align*}
\dot{s}_x^B &= \frac{\delta}{2} s_y^B - \frac{W}{2} s_x^B + \frac{1}{2} s_z^B J_x \\
\dot{s}_y^B &= -\frac{\delta}{2} s_x^B - \frac{W}{2} s_y^B + \frac{1}{2} s_z^B J_y \\
\dot{s}_z^B &= W (1 - s_z^B) - \frac{1}{2} s_x^B J_x - \frac{1}{2} s_y^B J_y
\end{align*}
\]

Symmetries

Eqs of motion invariant with respect to:

Rotations around z-axis (choice of axes in xy-plane is arbitrary)

\[\mathbb{Z}_2\] symmetry (similar to particle-hole)

\[
\begin{align*}
s_x^A &\rightarrow s_x^B \\
s_y^A &\rightarrow -s_y^B \\
s_z^A &\rightarrow s_z^B
\end{align*}
\]
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: fixed points

Trivial steady state:

$$s^{A,B}_{x,y} = 0, \quad s^{A,B}_z = 1$$

**Ensemble A:**

$$\dot{s}^A_x = -\frac{\delta}{2} s^A_y - \frac{W}{2} s^A_x + \frac{1}{2} s^A_z J_x$$

$$\dot{s}^A_y = \frac{\delta}{2} s^A_x - \frac{W}{2} s^A_y + \frac{1}{2} s^A_z J_y$$

$$\dot{s}^A_z = W (1 - s^A_z) - \frac{1}{2} s^A_x J_x - \frac{1}{2} s^A_y J_y$$

**Ensemble B:**

$$\dot{s}^B_x = \frac{\delta}{2} s^B_y - \frac{W}{2} s^B_x + \frac{1}{2} s^B_z J_x$$

$$\dot{s}^B_y = -\frac{\delta}{2} s^B_x - \frac{W}{2} s^B_y + \frac{1}{2} s^B_z J_y$$

$$\dot{s}^B_z = W (1 - s^B_z) - \frac{1}{2} s^B_x J_x - \frac{1}{2} s^B_y J_y$$

Eq's of motion invariant with respect to:

Rotations around z-axis (choice of axes in xy-plane is arbitrary)

$\mathbb{Z}_2$ symmetry (similar to particle-hole)

$$s^A_x \rightarrow s^B_x$$

$$s^A_y \rightarrow -s^B_y$$

$$s^A_z \rightarrow s^B_z$$
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: fixed points

Trivial steady state:

\[ s_{x,y}^{A,B} = 0, \quad s_z^{A,B} = 1 \]

- Maximally pumped. All atoms excited. No emission.
- Exists for all values of $\delta$ and $W$

Nonequilibrium phase diagram for two ensembles of atoms in a bad cavity (mean-field)

$W$ – pumping, $\delta$ – detuning from the cavity mode
Trivial steady state:
\[ s_{x,y}^{A,B} = 0, \quad s_z^{A,B} = 1 \]

- Maximally pumped. All atoms excited. No emission.
- Exists for all values of \( \delta \) and \( W \)
- Loses stability via supercritical pitchfork (PF) and supercritical Hopf (H+) bifurcations.

Nonequilibrium phase diagram for two ensembles of atoms in a bad cavity (mean-field)

\( W \) – pumping, \( \delta \) – detuning from the cavity mode
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: fixed points

Non-trivial steady state [NTSS] (synchronized, superradiant):

$$s^A_x = s^B_x = \frac{J}{2}$$
$$s^A_y = -s^B_y = \frac{\delta}{W} \frac{J}{2}$$
$$s^A_z = s^B_z = \frac{\delta}{2} \left( \frac{\delta}{W} + \frac{W}{\delta} \right)$$

$$J = \sqrt{2} \sqrt{1 - (W - 1)^2 - \delta^2}$$

+ rotation around z-axis

Nonequilibrium phase diagram for two ensembles of atoms in a bad cavity (mean-field)

$W$ – pumping, $\delta$ – detuning from the cavity mode
Stability analysis of NTSS: Coexistence

- Loses stability via sub-critical Hopf bifurcation on the boundary between green and blue regions

- Unstable limit cycle exists in the blue region. Acts as the separatrix between the NTSS and the already existing limit cycle from the super-critical Hopf bifurcation of the TSS. Hence coexistence
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: limit cycle

$\mathbb{Z}_2$ symmetric limit cycle

\[
\begin{align*}
  s^A_x &= s^B_x , \\
  s^A_y &= -s^B_y , \\
  s^A_z &= s^B_z .
\end{align*}
\]

- $W$ – pumping, $\delta$ – detuning from the cavity
- Trivial steady state (TSS)
- Constant superradiance (NTSS)

+ rotation around z-axis

(1-parameter family of limit cycles)
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: limit cycle

$\mathbb{Z}_2$ symmetric limit cycle

$$s_x^A = s_x^B, \quad s_y^A = -s_y^B, \quad s_z^A = s_z^B$$

Superradiance emission spectrum:

Equidistant peaks, odd harmonics of the limit cycle frequency $\Omega \sim 100$ Hz, periodically oscillating superradiance (cf. single peak at $\omega_0 \sim 1$ GHz for NTSS)

Can generate unusual, orders of magnitude different frequencies in this superradiant laser!
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: limit cycle

$\mathbb{Z}_2$ symmetric limit cycle – analytic solution in several limits:

$$s^A_x = a \cos(\Omega t - \alpha), \quad s^A_y = a \sin \Omega t,$$

$$s^A_z = W - \frac{a^2}{2\delta} \sin(2\Omega t - \alpha)$$

$$2\Omega = \sqrt{\delta^2 - W^2}, \quad a = \sqrt{2W(1 - W)},$$

$$\tan \alpha = \frac{W}{2\Omega}$$

$W$ – pumping, $\delta$ – detuning from the cavity
Phase diagram: $\mathbb{Z}_2$ symmetric attractors: limit cycle

$\mathbb{Z}_2$ symmetric limit cycle – analytic solution near triple point $\delta = W = 1$

$s_x = s_y = a \text{cn}(bt, k), \quad s_z = 1 - s_y^2$

Jacobi elliptic function $\text{cn}$

$$b^2 = \frac{\delta - 1}{2(1 - 2k^2)}, \quad a^2 = \frac{2k^2(\delta - 1)}{1 - 2k^2}$$

Period: $T = \frac{4K(k)\sqrt{2|1 - 2k^2|}}{\sqrt{|\delta - 1|}}$

$$\frac{4}{5(1 - 2k^2)} \frac{2(k^4 - k^2 + 1)E(k) - (2 - k^2)(1 - k^2)K(k)}{(2k^2 - 1)E(k) + (1 - k^2)K(k)} = \frac{1 - W}{\delta - 1}$$

$W$ – pumping, $\delta$ – detuning from the cavity
Phase diagram: $\mathbb{Z}_2$ symmetry breaking: asymmetric limit cycles

![Phase diagram](image)

Trivial steady state

Constant superradiance

$\mathbb{Z}_2$ symmetry breaking line

$\mathbb{Z}_2$ symmetric limit cycle

Asymmetric limit cycles (yellow region)

Projections of the asymmetric limit cycle

Note: for symmetric limit cycle $s_z^A = s_z^B$, $s_\perp^A = s_\perp^B$

$\delta = 0.38$, $W = 0.055$
Floquet stability analysis of the symmetric limit cycle

Change to symmetric and antisymmetric variables

\[ s_x = \frac{s_x^A + s_x^B}{2}, \quad s_y = \frac{s_y^A - s_y^B}{2}, \quad s_z = \frac{s_z^A + s_z^B}{2} \]

\[ m_x = s_x^A - s_x^B, \quad m_y = s_y^A + s_y^B, \quad m_z = s_z^A - s_z^B \]

For symmetric limit cycle: \( m(t) = 0, \quad s(t) = s^A(t) \) periodic
Floquet stability analysis of the symmetric limit cycle

Change to symmetric and antisymmetric variables

\[ s_x = \frac{s_x^A + s_x^B}{2}, \quad s_y = \frac{s_y^A - s_y^B}{2}, \quad s_z = \frac{s_z^A + s_z^B}{2} \]

\[ m_x = s_x^A - s_x^B, \quad m_y = s_y^A + s_y^B, \quad m_z = s_z^A - s_z^B \]

For symmetric limit cycle: \( \mathbf{m}(t) = 0, \quad \mathbf{s}(t) = \mathbf{s}^A(t) - \text{periodic} \)

Linearize eqs of motion in \( \mathbf{m} \) around the symmetric limit cycle

\[
\begin{align*}
\dot{m}_x &= \frac{\delta}{2} m_y - \frac{W}{2} m_x + s_x(t) m_x \\
\dot{m}_y &= \frac{\delta}{2} m_x - \frac{W}{2} m_y + s_z(t) m_y \\
\dot{m}_z &= -W m_z - s_x(t) m_x - s_y(t) m_y
\end{align*}
\]

Linear diff eqs with periodic coefficients

– use Floquet theorem

\[
\mathbf{m}(t) = \sum_{k=1}^{3} c_k e^{\mu_k t} \mathbf{p}_k(t)
\]

Floquet exponents
Phase diagram: $\mathbb{Z}_2$ symmetry breaking: asymmetric limit cycles

$$m(t) = \sum_{k=1}^{3} c_k e^{\mu_k t} p_k(t)$$

$\mu_1 = 0$ due to symmetry w.r.t. rotations around $z$-axis

$\mu_{2,3} < 0$ symmetric limit cycle stable

$\mu_2$ or $\mu_3 > 0$ – unstable

$\mathbb{Z}_2$ symmetric limit cycle

Asymmetric limit cycles (yellow region)

Trivial steady state

$\mathbb{Z}_2$ symmetry breaking line
Signatures of asymmetric limit cycle in the emission spectrum

- Even harmonics of the limit cycle $\Omega$ frequency appear
- Shift of the carrier frequency
- Stokes – anti-Stokes asymmetry

Asymmetric LC for $\delta = .42 , W = .056$

Symmetric LC for $\delta = .44 , W = .056$
Phase diagram: “funny” region

W – pumping, δ – detuning from the cavity mode

Trivial steady state

Constant superradiance

$\mathbb{Z}_2$ symmetric limit cycle
Phase diagram: “funny” region

As we move along the white arrow in the $W\delta$-plane another fundamental frequency emerges – quasiperiodic motion with 2 frequencies. Attractor: circle $\rightarrow$ 2D torus

$\delta = .115, \ W = .055$

$W$ – pumping, $\delta$ – detuning from the cavity mode

yellow – asymmetric LCs
green – symmetric LC
dark blue – quasiperiodic
orange – chaos
red – synchronized chaos
Quasiperiodic motion with 2 frequencies. Attractor – 2D torus

$|E(\omega)|^2$

$\delta = .115, \ W = .055$

Max Lyapunov exponent = 0

$W$ – pumping, $\delta$ – detuning from the cavity mode
Phase diagram: quasiperiodic (RTN) route to chaos

As we move a bit further, chaos ensues.

Ruelle–Takens–Newhouse (RTN) theorem: quasiperiodic motion with 3 or more frequencies not robust against small perturbations giving rise to chaos. RTN route to chaos is typical for strongly coupled systems.

In our case, chaotic and 2-frequency quasiperiodic behaviors alternate.

\(W\) – pumping, \(\delta\) – detuning from the cavity mode

**Colors:**
- yellow – asymmetric LCs
- green – symmetric LC
- dark blue – quasiperiodic
- orange – chaos
- red – synchronized chaos
Phase diagram: chaos

Continuous emission spectrum

Max Lyapunov exponent $> 0$

$\delta = 0.1, \quad W = 0.055$

$W$ – pumping, $\delta$ – detuning from the cavity mode
Phase diagram: synchronized chaos

As we move further chaos synchronizes

Dynamics of each ensemble are individually chaotic, but they copy each other most of the time

$\mathbb{Z}_2$ symmetry approximately restored in chaotic dynamics

$W$ – pumping, $\delta$ – detuning from the cavity mode

red – chaos
orange – quasiperiodic
yellow – asymmetric LCs
green – symmetric LC
dark blue – synchronized chaos
Synchronized chaos

\[ \delta = 0.079, \quad W = 0.055 \]

W – pumping, \( \delta \) – detuning from the cavity mode

Max Lyapunov exponent > 0
Synchronized chaos via on-off intermittency

Fixed $W = .05$ in all plots

$\delta = .0802$

$\delta = .0801$

$\delta = .0800$

$\delta = .0800$

$t \times 10^{-4}$
Dynamics of each ensemble are individually chaotic, but they copy each other most of the time.

Typically, directional coupling between the two parts – “drive” and “response”. Here – synchronization due to mutual coupling between two ensembles.
Potential application of synchronized chaos: steganography

Cryptography = hide the meaning of the message

Steganography = hide the existence of the message

Figure 1.3 Schematics of the concept of optical communication with synchronized chaotic lasers.

Figure from A. Uchida: “Optical Communication with Chaotic Lasers: Applications of Nonlinear Dynamics and Synchronization” 2012.
Nonequilibrium phase diagram for two ensembles of atoms in a bad cavity (mean-field)

$W$ – pumping, $\delta$ – detuning from the cavity mode

Collaborators:

Aniket Patra  
*Rutgers University*

Boris Altshuler  
*Columbia University*

Thanks to Sergey Denisov & Yuri Rubo for discussions