Integrable Time-Dependent Quantum Hamiltonians

Nikolai A. Sinitsyn,[1] Emil A. Yuzbashyan,[2] Vladimir Y. Chernyak,[3] Aniket Patra,[1,2] and Chen Sun[1,4]
[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Center for Materials Theory, Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08854, USA*
[3]*Department of Chemistry and Department of Mathematics, Wayne State University, 5101 Cass Avenue, Detroit, Michigan 48202, USA*
[4]*Department of Physics, Texas A&M University, Texas 77840, USA*

(Received 23 November 2017; revised manuscript received 22 March 2018; published 11 May 2018)

We formulate a set of conditions under which the nonstationary Schrödinger equation with a time-dependent Hamiltonian is exactly solvable analytically. The main requirement is the existence of a non-Abelian gauge field with zero curvature in the space of system parameters. Known solvable multistate Landau-Zener models satisfy these conditions. Our method provides a strategy to incorporate time dependence into various quantum integrable models while maintaining their integrability. We also validate some prior conjectures, including the solution of the driven generalized Tavis-Cummings model.

DOI: 10.1103/PhysRevLett.120.190402

Quantum coherent dynamics controlled by strong time-dependent fields can be realized and explored nowadays in systems of considerable complexity [1–8]. Time-dependent parameters play a critical role in NMR [9], quantum information processing [10–20], molecular dynamics [21–23], and cold atom experiments [24–26]. On the theory side, the quantum dynamics of time-dependent many-body Hamiltonians, especially their exact analytical description, presents considerable challenges. In contrast, exact solutions of significant relevance to experiments inform our understanding of stationary states, e.g., Bethe's ansatz solution of paradigmatic models [27–29]. Nontrivial exact results have been also obtained for quantum quenches, such as the generalized Gibbs ensemble description of the dynamics of the spin-$1/2$ Heisenberg chain [30] and quantum quench phase diagrams of BCS superconductors [31]. Such methods, unfortunately, do not apply to a Hamiltonian with continuous time dependence.

In this Letter, we propose an approach for solving the nonstationary Schrödinger equation exactly for a broad class of time-dependent Hamiltonians. This approach allows us to make parameters of a quantum integrable model, e.g., the BCS and generalized Tavis-Cummings Hamiltonians, vary in time in such a way that the resulting dynamics are exactly solvable. While here we primarily focus on the scattering problem for Hamiltonians linear in time, our method applies to a much broader class of problems, including Floquet Hamiltonians [32] and models with other nonpolynomial time dependence.

Important examples of driven systems are matrix Hamiltonians of the form $H(t) = A + Bt$, where A and B are time-independent Hermitian $N \times N$ matrices. The problem of finding the scattering matrix that relates the state of the system at $t = +\infty$ to that at $t = -\infty$ is called then the multistate Landau-Zener problem. The 2×2 problem was solved by Landau, Zener, Majorana, and Stückelberg in 1932 [33–36]. For $N \geq 3$, the solution is known only for special choices of A and B. The earliest examples include Demkov-Osherov [37,38], bow-tie [39], generalized bow-tie [40,41], composite [42], and infinite chain [43] models. In a more recent work [44], it was shown that nontrivial solvable models belong to families of mutually commuting Hamiltonians linear or quadratic in t. It was therefore conjectured that quantum integrability understood as the existence of nontrivial time-dependent commuting partners [45–48] is a necessary condition for the multistate Landau-Zener solvability. In a parallel development, empirical methods to solve and search for new models were discovered [49–51], and since then the number of such models has grown rapidly [52–55]. However, these results still lack a rigorous justification.

Our approach provides a framework to justify exact solutions for all these models and to identify and solve new ones. It also supports the conjecture made in Ref. [44]. Below, we first formulate our approach and then discuss various many-body and matrix models that fit into it. To illustrate our technique, we solve the scattering problem for two nontrivial models—the generalized Tavis-Cummings Hamiltonian with a linear drive and a new four-state Hamiltonian linear in t. Moreover, we present a new class of solvable interacting spin systems with time-dependent couplings. We conclude with several general observations and an outline of the idea of the solution for arbitrary t.

Consider a Hamiltonian $\hat{H}(t, \vec{x})$ that, in addition to time, depends on M real parameters $(x^1, \ldots, x^M) = \vec{x}$. For example, in the multistate Landau-Zener problem these can be certain matrix elements of A and B. The main idea is to embed the nonstationary Schrödinger equation for $\hat{H}(t,\vec{x})$ into a set of multitime Schrödinger equations

$$i\partial_j\Psi(\boldsymbol{x}) = \hat{H}_j\Psi(\boldsymbol{x}), \qquad j = 0,1,\ldots,M, \tag{1}$$

where $\boldsymbol{x} = (t,\vec{x})$, $\partial_j \equiv \partial/\partial x^j$, $x^0 = t$, $\hat{H}_0 \equiv \hat{H}(t,\vec{x})$, and Hamiltonians $\hat{H}_j$ are independent. In other words, the first equation ($j = 0$) is the original nonstationary Schrödinger equation, while the rest are auxiliary Schrödinger equations that help us solve it exactly. Taking the derivative of Eq. (1) with respect to x^k, we derive consistency conditions

$$\partial_j\hat{H}_k - \partial_k H_j - i[\hat{H}_k,\hat{H}_j] = 0, \qquad k,j = 0,\ldots,M. \tag{2}$$

These conditions are sufficient and necessary for system (1) to possess a joint solution for any initial condition [56,57]. We may view them as a generalization of the notion of integrals of motion for time-dependent quantum Hamiltonians.

A formal solution of Eq. (1) along a path in the space of real parameters $\boldsymbol{x}$ that starts at a reference point $\boldsymbol{x_0}$ is an ordered exponential:

$$\Psi(\boldsymbol{x}) = T\exp\left(-i\int_{\mathcal{P}}\hat{H}_j dx^j\right)\Psi(\boldsymbol{x_0}), \tag{3}$$

where we assume summation over repeated indices. Treating Hamiltonians $\hat{H}_j$ as matrix components of a non-Abelian gauge field $\mathcal{A}(\boldsymbol{x})$, $\mathcal{A}_j = -i\hat{H}_j$, we interpret Eq. (2) as the zero curvature condition $\mathcal{F}_{jk} \equiv \partial_j\mathcal{A}_k - \partial_k\mathcal{A}_j - [\mathcal{A}_j,\mathcal{A}_k] = 0$, so that $\Psi(\boldsymbol{x})$ in Eq. (3) is independent of the integration path $\mathcal{P}$ as long as its end points are fixed. It is precisely this freedom to choose a suitable path that enables us to explicitly solve the scattering problem. A similar zero curvature integrability condition is also well known in soliton physics [58].

Furthermore, consider a path $\mathcal{P}_\tau$ parametrized by a variable τ:

$$\mathcal{P}_\tau\colon\ x^j(\tau) = v^j\tau + x_0^j, \qquad j = 0,\ldots,M, \tag{4}$$

where v^j and x_0^j are constants. The state vector $\Psi(\tau) = \Psi[\boldsymbol{x}(\tau)]$ along this path satisfies

$$i\frac{d\Psi(\tau)}{d\tau} = \hat{h}(\tau)\Psi(\tau), \tag{5}$$

$$\hat{h}(\tau) = \sum_j v^j\hat{H}_j[\boldsymbol{x}(\tau)]. \tag{6}$$

Solutions of Eq. (5) follow from those of Eq. (1). Therefore, $\hat{h}(\tau)$—an arbitrary linear combination of $\hat{H}_j$—is also a solvable time-dependent model just like a linear combination of integrals of motion of a time-independent model is also an integral. Note, however, that the coefficients v^j of this linear combination dictate the time dependence of $\hat{h}(\tau)$. More generally, one can choose nonlinear $x^j(\tau)$ instead of the linear path in Eq. (4), leading to, e.g., integrable Hamiltonians $\hat{h}(\tau)$ with an exponential or oscillatory behavior of the couplings.

An important observation is that complex-looking Eq. (1) simplifies considerably when the matrix elements of the Hamiltonians are real. Then, the real and imaginary parts of Eq. (2) yield two separate conditions:

$$[\hat{H}_j,\hat{H}_k] = 0, \tag{7}$$

$$\partial_j\hat{H}_k - \partial_k\hat{H}_j = 0, \qquad j,k = 0,1,\ldots,M. \tag{8}$$

These equations suggest a strategy for identifying solvable time-dependent models. First, we note that Eq. (7) is to be supplemented with a notion of a nontrivial commuting partner that weeds out trivial partners (e.g., projectors onto the eigenstates of $\hat{H}$). One way is to consider parameter-dependent $\hat{H}_j$ and to fix their dependence on the parameter [59]. This leads to a systematic classification and explicit construction of commuting families of parameter-dependent matrix Hamiltonians [44–47], which are interesting candidates for our approach. More generally, any quantum integrable model that contains two or more real parameters is a potential candidate. Such models have an extensive number of integrals of motion that satisfy Eq. (7). If no initial subset of integrals satisfies Eq. (8), we attempt to redefine them by taking various combinations and similarly redefine the parameters to make Eq. (8) work for at least $M = 1$. Note that, once we declare one of the variables x^j to be the physical time, commuting partners $\hat{H}_j$ cease to be integrals of motion.

For example, take the generalized Tavis-Cummings model

$$\hat{H}_{\mathrm{TC}} = \sum_{j=1}^{N_s}\varepsilon_j\hat{s}_j^z - \omega\hat{a}^\dagger\hat{a} + g\sum_{j=1}^{N_s}(\hat{a}^\dagger\hat{s}_j^- + \hat{a}\hat{s}_j^\dagger), \tag{9}$$

where $\hat{a}$ is the boson annihilation operator and $\hat{s}_j^z$ and $\hat{s}_j^\pm$ are spin-1/2 operators. Its commuting partners are [60]

$$\hat{H}_j = (\varepsilon_j + \omega)\hat{s}_j^z + g(\hat{a}^\dagger\hat{s}_j^- + \hat{a}\hat{s}_j^\dagger) + 2g^2\sum_{k\neq j}\frac{\hat{\mathbf{s}}_j\cdot\hat{\mathbf{s}}_k}{\varepsilon_j - \varepsilon_k}. \tag{10}$$

Equations (7) and (8) hold with $M = N_s$, $\hat{H}_0 = \hat{H}_{\mathrm{TC}}$, and $\boldsymbol{x} = (\omega,\varepsilon_1,\ldots,\varepsilon_{N_s})$. To derive a new solvable time-dependent model, consider the BCS Hamiltonian. In terms of Anderson pseudospin-1/2 operators, it reads

$$\hat{H}_{\rm BCS} = \sum_{j=1}^{N_s} 2\varepsilon_j \hat{s}_j^z - \frac{1}{2B}\sum_{j,k} \hat{s}_j^+ \hat{s}_k^-, \tag{11}$$

where $(2B)^{-1}$ stands for the BCS coupling constant. Its commuting partners are Gaudin magnets [27,61]:

$$\hat{H}_j = 2B\hat{s}_j^z - \sum_{k\neq j} \frac{\hat{\mathbf{s}}_j \cdot \hat{\mathbf{s}}_k}{\varepsilon_j - \varepsilon_k}. \tag{12}$$

Now $\hat{H}_0 = \hat{H}_{\rm BCS}$ and $x^0 = B$. Thus, the BCS Hamiltonian with coupling $\propto 1/t$ fits into our construction. Similarly, using the commuting partners derived in Ref. [44], we verified that many known solvable models, including the Demkov-Osherov, bow-tie, and generalized bow-tie as well as Landau-Zener-Coulomb models [62–65], also fit into our construction.

A key point of this Letter is that zero curvature condition (1) leads to an explicit exact solution of the scattering problem. Consider, e.g., the multistate Landau-Zener model $\hat{H}(t,\vec{x}) = \hat{A}(\vec{x}) + t\hat{B}(\vec{x})$ for which we need to determine the matrix of transition probabilities P with elements $P_{nn'} \equiv P_{n'\to n} \equiv |S_{nn'}|^2$. Here S is the scattering matrix between eigenstates at $t=-\infty$ and $t=+\infty$ at some fixed values of the parameters, $\vec{x}=\vec{c}$ [38]. As discussed above, we are free to choose any path in the space $\boldsymbol{x} = (t,\vec{x})$ that connects the points $(-\infty,\vec{c})$ and $(+\infty,\vec{c})$. It is convenient to choose a path $\mathcal{P}_\infty$, such that $|\boldsymbol{x}|$ is always large and the time evolution is adiabatic everywhere, except the neighborhood of isolated points, where scattering takes place. The corresponding scattering problem is typically simple thanks to a large $|\boldsymbol{x}|$; e.g., it reduces to a 2×2 Landau-Zener problem in the two nontrivial examples we consider below. In general, Eqs. (7) and (8) enable one to construct a multidimensional version of WKB with simple scattering matrices connecting adiabatic (WKB) solutions in different adiabatic regions [66].

Our first example is the Tavis-Cummings model (9) with linear drive, $\omega = t$. Let $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{N_s}$. We are interested in the evolution along the path $\mathcal{P}_\omega$ shown in Fig. 1. On this path $\varepsilon_j = \text{const}$, while ω changes from $-R$ to R. At the end, we take the limit $R\to\infty$. This scattering problem was solved in Ref. [53] under the assumption that ε_j are well separated, i.e., $\varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_{N_s}$. It was further conjectured in Ref. [53] that this is the general solution. We are now in the position to prove this conjecture. To do so, consider the path $\mathcal{P}_\infty$ in Fig. 1 that has the same end points as $\mathcal{P}_\omega$. On the first vertical leg of $\mathcal{P}_\infty$, ε_j evolve, keeping the ordering of ε_j, until the condition $\varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_{N_s}$ is met. On the second vertical leg, they evolve back to their initial values. Since $|\omega|$ is large and ε_j are distinct, this evolution is purely adiabatic and does not affect the transition probabilities. On the horizontal leg of $\mathcal{P}_\infty$, the problem is

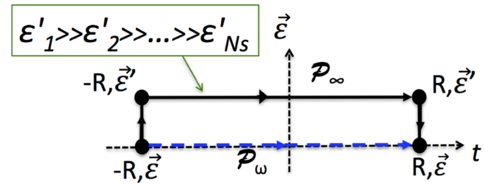

FIG. 1. Paths in the space of parameters ε_j and time used to evaluate transition probabilities for the driven generalized Tavis-Cummings model (9), $\omega = t$. On $\mathcal{P}_\omega$, ω changes from $-R$ to $+R$, and all ε_j are fixed. Since this model is a part of a commuting family of $\hat{H}_j$ that satisfies the zero curvature condition, we can deform $\mathcal{P}_\omega$ into a new path $\mathcal{P}_\infty$ without modifying the scattering matrix.

precisely the one solved in Ref. [53]. This proves the above conjecture.

In our second example, we take a previously solved 4×4 multistate Landau-Zener problem [51,52], which was originally introduced to describe hysteresis in molecular nanomagnets [67], and derive from it a new, more general Hamiltonian by the prescription outlined below Eq. (6). We then proceed to determine the transition probabilities for this new model. Let

$$H(t,e) = \begin{pmatrix} b_1 t + e & 0 & g & -\gamma \\ 0 & -b_1 t + e & \gamma & g \\ g & \gamma & b_2 t & 0 \\ -\gamma & g & 0 & -b_2 t \end{pmatrix}, \tag{13}$$

where b_1, b_2, e, g, and γ are constants. To determine if this Hamiltonian fits into our approach, we first search for a nontrivial commuting partner H_1 linear in t. This reduces to a set of linear algebraic equations for parameters of H_1 [47]. We find three linearly independent commuting operators. Two of them are trivial—the unit matrix and H itself. Therefore, there is a single nontrivial commuting partner, which we determine explicitly. When both $H_0 \equiv H$ and H_1 are linear in t, Eq. (7) implies that their time-dependent parts are diagonal in the same basis. So, to satisfy (8), the parameter x^1 must be constructed from diagonal time-independent elements of H. A natural candidate is $x^1 = e$. Searching then for H_1 that satisfies (8) in the form of a linear combination of the three commuting operators, we find

$$H_1(t,e) = \begin{pmatrix} t + \frac{b_1 e}{b_1^2 - b_2^2} & 0 & \frac{g}{b_1 - b_2} & \frac{-\gamma}{b_1+b_2} \\ 0 & t - \frac{b_1 e}{b_1^2-b_2^2} & \frac{-\gamma}{b_1+b_2} & \frac{-g}{b_1-b_2} \\ \frac{g}{b_1-b_2} & \frac{-\gamma}{b_1+b_2} & -\frac{b_2 e}{b_1^2-b_2^2} & 0 \\ \frac{-\gamma}{b_1+b_2} & \frac{-g}{b_1-b_2} & 0 & \frac{b_2 e}{b_1^2-b_2^2} \end{pmatrix}. \tag{14}$$

Let the evolution path be

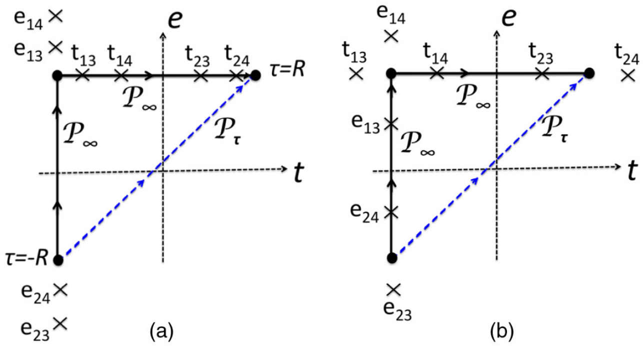

FIG. 2. Paths in the space (t, e) for evaluating transition probabilities for the model (16). On $\mathcal{P}_\tau$, τ changes from $-R$ to $+R$; all other parameters are fixed. We deform $\mathcal{P}_\tau$ into $\mathcal{P}_\infty$ without affecting the scattering matrix. Points e_{ij} and t_{ij} marked with crosses indicate nonadiabatic Landau-Zener transitions between levels i and j for (a) $v < b_1 - b_2$ and (b) $b_1 + b_2 > v > b_1 - b_2$.

$$\mathcal{P}_\tau: \ t = \tau, \qquad e = v\tau + e_0, \tag{15}$$

with constant v and e_0. The Hamiltonian (5) for $\mathcal{P}_\tau$ is

$$h(\tau) = \begin{pmatrix} \beta_1\tau + e_1 & 0 & g(1+x) & -\gamma(1+y) \\ 0 & \beta_2\tau + e_2 & \gamma(1-y) & g(1-x) \\ g(1+x) & \gamma(1-y) & \beta_3\tau + e_3 & 0 \\ -\gamma(1+y) & g(1-x) & 0 & \beta_4\tau + e_4 \end{pmatrix}, \tag{16}$$

$$x = \frac{v}{b_1 - b_2}, \qquad y = \frac{v}{b_1 + b_2},$$
$$\beta_{1,2} = 2v \pm b_1(1 + xy), \qquad \beta_{3,4} = \pm b_2(1 - xy),$$
$$e_{1,2} = e_0(1 \pm b_1 xy/v), \qquad e_{3,4} = \mp e_0 b_2 xy/v. \tag{17}$$

This is a new, previously unsolved model more general than (13); e.g., off-diagonal matrix elements in (16) are distinct. We proceed to solve it with our method.

Let $b_1 > b_2 > 0$ and $v > 0$. We are interested in the evolution matrix for $h(\tau)$ along the path $\mathcal{P}_\tau$ from $\tau = -R$ to $\tau = R$ [see Fig. 2(a)] in the limit $R \to \infty$. Because $H_0(t, e)$ and $H_1(t, e)$ satisfy the zero curvature condition, the evolution matrix is the same as that for the path $\mathcal{P}_\infty$. The latter has two pieces. In the vertical one, we set $t = -R$ and vary e from $-vR + e_0$ to $vR + e_0$. In the horizontal piece, we fix $e = vR + e_0$ and vary t from $-R$ to R. According to Eq. (3), only H_1 contributes on the first piece and only H_0 on the second. Along $\mathcal{P}_\infty$, diagonal matrix elements of H_0 and H_1 (diabatic levels) are large compared to the couplings. Therefore, the levels are well separated, except on disjoined small segments of $\mathcal{P}_\infty$ near points where a pair of the diagonal elements is degenerate. These segments connect adiabatic parts of $\mathcal{P}_\infty$ where the adiabatic approximation is exact in the limit $R \to \infty$. Let us write the state of the system as $\Psi(t, e) = \sum_k a_k |k\rangle$, where $|k\rangle$ are the eigenstates of the diagonal parts of H_0 and H_1 (diabatic eigenstates). Diabatic and adiabatic (instantaneous) eigenstates coincide in adiabatic parts of $\mathcal{P}_\infty$ when $R \to \infty$. In the adiabatic approximation, absolute values of a_k remain the same, while their phases evolve with t and e.

In the vicinity of degeneracy points, two levels come close and transitions between them become locally possible. The other two levels, however, remain far remote and do not affect these nonadiabatic transitions. Suppose $v < b_1 - b_2$. For this case, we mark the points of diabatic level crossings with crosses in Fig. 2(a). Along $\mathcal{P}_\infty$, adiabatic approximation breaks near four points that all have $e = vR + e_0$ and

$$t_{13/24} = \mp \frac{vR + e_0}{b_1 - b_2}, \qquad t_{14/23} = \mp \frac{vR + e_0}{b_1 + b_2}.$$

The distances between these points are $\propto R$, which means that regions of pairwise nonadiabatic transitions along $\mathcal{P}_\infty$ are well apart. Consider, e.g., the evolution of the amplitudes a_1 and a_3 near t_{13} that is governed by H_0. Writing $t = t' + t_{13}$ and disregarding the other two levels, we find

$$i\frac{da_1}{dt'} = b_1 t' a_1 + g a_3, \qquad i\frac{da_3}{dt'} = b_2 t' a_3 + g a_1, \tag{18}$$

which is a 2×2 Landau-Zener problem, whose scattering matrix is known explicitly [33–36]. Since the other two levels do not experience nonadiabatic transitions here, they produce only diagonal unit entries in the scattering matrix for evolution through t_{13}. The total evolution matrix S for the path $\mathcal{P}_\infty$ factorizes into an ordered product of such pairwise scattering matrices S^{ab}, where a and b label states experiencing nonadiabatic transitions and diagonal matrices $U^{\alpha,\beta}$ describe adiabatic evolution between points α and β on this path, i.e.,

$$S = U^{R,t_{24}} S^{24} U^{t_{24},t_{23}} S^{23} U^{t_{23},t_{14}} S^{14} U^{t_{14},t_{13}} S^{13} U^{t_{13},-R}.$$

Trivial phases resulting from the adiabatic evolution drop out from the matrix of transition probabilities, and we obtain [66]

$$P^{v<b_1-b_2} = \begin{pmatrix} p_1p_2 & 0 & p_2q_1 & q_2 \\ 0 & p_1p_2 & q_2 & p_2q_1 \\ p_2q_1 & q_2 & p_1p_2 & 0 \\ q_2 & p_2q_1 & 0 & p_1p_2 \end{pmatrix},$$
$$p_1 = e^{-2\pi g^2/(b_1-b_2)}, \qquad p_2 = e^{-2\pi\gamma^2/(b_1+b_2)},$$
$$q_{1,2} = 1 - p_{1,2}. \tag{19}$$

This result does not depend on v, so it coincides with the solution for the model (13) found in Refs. [51,52].

The situation changes for $b_2 + b_2 > v > b_1 - b_2$. Now the points of adiabaticity violation e_{24} and e_{13} lie on the first leg of the path $\mathcal{P}_\infty$ as shown in Fig. 2(b). Pairwise transitions near these points are now governed by the Hamiltonian H_1, and the transition probability matrix in this case is different:

$$P^{v>b_1-b_2}_{v<b_1+b_2} = \begin{pmatrix} p_1p_2 & q_1q_2 & p_2q_1 & p_1q_2 \\ q_1q_2 & p_1p_2 & p_1q_2 & p_2q_1 \\ p_2q_1 & p_1q_2 & p_1p_2 & q_1q_2 \\ p_1q_2 & p_2q_1 & q_1q_2 & p_1p_2 \end{pmatrix}. \quad (20)$$

For $v > b_1 + b_2$, all four points with Landau-Zener transitions lie on the first leg of $\mathcal{P}_\infty$, and

$$P^{v>b_1+b_2} = \begin{pmatrix} p_1p_2 & 0 & q_1 & p_1q_2 \\ 0 & p_1p_2 & p_1q_2 & q_1 \\ q_1 & p_1q_2 & p_1p_2 & 0 \\ p_1q_2 & q_1 & 0 & p_1p_2 \end{pmatrix}. \quad (21)$$

We see that our approach not only reproduces the previously known solution for the Hamiltonian (13) but also solves a more complex model (16).

Thus, we have identified a symmetry—multitime evolution with commuting Hamiltonians—that leads to the integrability of unitary dynamics with time-dependent Hamiltonians. Our approach generates numerous new solvable multistate Landau-Zener models. As examples, we solved a four-state model (16) and proved the previously conjectured solution of a combinatorially complex driven Tavis-Cummings Hamiltonian (9), which models detuning sweeps in atom-molecule condensates [68,69]. The time-dependent BCS Hamiltonian (11) describes the effect of turning off or on the superconducting coupling as t varies from 0^+ to $+\infty$ or vice versa, which can be experimentally realized in cold atomic fermions with tunable interactions. We believe that this integrability is behind most if not all nontrivial exactly solvable multistate Landau-Zener and Landau-Zener-Coulomb models [62–65]. It explains why in such problems the scattering matrix factorizes into a product of two-state scattering matrices [49]—since Eq. (1) allows a choice of an integration path that bypasses the region of complex nonadiabatic dynamics. It also explains why basic known solvable models have commuting partners with a simple linear or quadratic dependence on t [44]. Indeed, pairs of such operators that also satisfy Eq. (8) lead to relatively simple versions of the WKB approximation necessary to determine scattering matrices. Furthermore, Eq. (6) shows how certain distortions of the parameters [50] give rise to entire families of solvable models.

Finally, we note that, when $\hat{H}_j$ are isotropic Gaudin magnets [Eq. (12) at $B = 0$], the $j = 1, \ldots, M$ subsystem of Eq. (1) is the famous Knizhnik-Zamolodchikov equation [70]. Its solutions have been obtained using the off-shell Bethe's ansatz [71]. This was generalized to $B \neq 0$ in Ref. [72] (see also [73]) and exploited in Ref. [74] to obtain the dynamics of an isotropic Gaudin magnet with time-dependent ε_i. We believe that solutions to Eq. (1), i.e., exact inexplicit solutions of the nonstationary Schrödinger equation at arbitrary t, for all time-dependent Hamiltonians discussed in this Letter can be obtained by further extending this technique.

We thank A. Kamenev and S. L. Lukyanov for useful discussions. This work was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396 (N. A. S. and C. S.). It was also supported by National Science Foundation (NSF): DMR-1609829 (E. A. Y. and A. P.) and Grant No. CHE-1111350 (V. Y. C.). N. A. S. also thanks the support from the LDRD program at Los Alamos National Laboratory (LANL).

[1] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton's cradle, Nature (London) **440**, 900 (2006).

[2] R. Monaco, M. Aaroe, J. Mygind, R. J. Rivers, and V. P. Koshelets, Experiments on spontaneous vortex formation in Josephson tunnel junctions, Phys. Rev. B **74**, 144513 (2006).

[3] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Dynamical Control of Matter-Wave Tunneling in Periodic Potentials, Phys. Rev. Lett. **99**, 220403 (2007).

[4] C. N. Weiler, T. W. Neely, D. R. Scherer, A. S. Bradley, M. J. Davis, and B. P. Anderson, Spontaneous vortices in the formation of Bose-Einstein condensates, Nature (London) **455**, 948 (2008).

[5] A. Widera, S. Trotzky, P. Cheinet, S. Fölling, F. Gerbier, I. Bloch, V. Gritsev, M. D. Lukin, and E. Demler, Quantum Spin Dynamics of Mode-Squeezed Luttinger Liquids in Two-Component Atomic Gases, Phys. Rev. Lett. **100**, 140401 (2008).

[6] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. Adu Smith, E. Demler, and J. Schmiedmayer, Relaxation and prethermalization in an isolated quantum system, Science **337**, 1318 (2012).

[7] S. Trotzky, Y-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwock, J. Eisert, and I. Bloch, Probing the relaxation towards equilibrium in an isolated strongly correlated one-dimensional Bose gas, Nat. Phys. **8**, 325 (2012).

[8] R. Matsunaga, Y. I. Hamada, K. Makise, Y. Uzawa, H. Terai, Z. Wang, and R. Shimano, Higgs Amplitude Mode in BCS Superconductors $Nb_{1-x}Ti_xN$ Induced by Terahertz Pulse Excitation, Phys. Rev. Lett. **111**, 057002 (2013).

[9] M. Foroozandeh, R. W. Adams, N. J. Meharry, D. Jeannerat, M. Nilsson, and G. A. Morris, Ultrahigh-resolution NMR spectroscopy, Angew. Chem., Int. Ed. **53**, 6990 (2014).

[10] G. Cao, H.-O. Li, T. Tu, L. Wang, C. Zhou, M. Xiao, G.-C. Guo, H.-W. Jiang, and G.-P. Guo, Ultrafast universal quantum control of a quantum-dot charge qubit using Landau-Zener-Stückelberg interference, Nat. Commun. **4**, 1401 (2012).

[11] J. M. Nichol, S. P. Harvey, M. D. Shulman, A. Pal, V. Umansky, E. I. Rashba, B. I. Halperin, and A. Yacoby, Quenching of dynamic nuclear polarization by spin-orbit coupling in GaAs quantum dots, Nat. Commun. **6**, 7682 (2015).

[12] L. Wang, C. Zhou, T. Tu, H.-W. Jiang, G.-P. Guo, and G.-C. Guo, Quantum simulation of the Kibble-Zurek mechanism using a semiconductor electron charge qubit, Phys. Rev. A **89**, 022337 (2014).

[13] L. Gaudreau, G. Granger, A. Kam, G. C. Aers, S. A. Studenikin, P. Zawadzki, M. Pioro-Ladriere, Z. R. Wasilewski, and A. S. Sachrajda, Coherent control of three-spin states in a triple quantum dot, Nat. Phys. **8**, 54 (2012).

[14] M. N. Kiselev, K. Kikoin, and M. B. Kenmoe, SU(3) Landau-Zener interferometry, Europhys. Lett. **104**, 57004 (2013).

[15] F. Forster, G. Petersen, S. Manus, P. Hänggi, D. Schuh, W. Wegscheider, S. Kohler, and S. Ludwig, Characterization of Qubit Dephasing by Landau-Zener-Stückelberg-Majorana Interferometry, Phys. Rev. Lett. **112**, 116803 (2014).

[16] H. Ribeiro, J. R. Petta, and G. Burkard, Interplay of charge and spin coherence in Landau-Zener-Stückelberg-Majorana interferometry, Phys. Rev. B **87**, 235318 (2013).

[17] M. Wubs, K. Saito, S. Kohler, P. Hänggi, and Y. Kayanuma, Gauging a Quantum Heat Bath with Dissipative Landau-Zener Transitions, Phys. Rev. Lett. **97**, 200404 (2006).

[18] W. Wernsdorfer, R. Sessoli, A. Caneschi, D. Gatteschi, and A. Cornia, Nonadiabatic Landau-Zener tunneling in Fe_8 molecular nanomagnets, Europhys. Lett. **50**, 552 (2000).

[19] S. Puri, C. K. Andersen, A. L. Grimsmo, and A. Blais, Quantum annealing with all-to-all connected nonlinear oscillators, Nat. Commun. **8**, 15785 (2017).

[20] J. Dziarmaga, Dynamics of a Quantum Phase Transition: Exact Solution of the Quantum Ising Model, Phys. Rev. Lett. **95**, 245701 (2005).

[21] H. Nakamura, *Nonadiabatic Transition*, 2nd ed. (World Scientific, Singapore, 2012).

[22] M. S. Child, *Molecular Collision Theory* (Dover, New York, 2010).

[23] E. E. Nikitin and S. Y. Umanskii, *Theory of Slow Atomic Collisions* (Springer, New York, 2011).

[24] C. A. Regal, M. Greiner, and D. S. Jin, Observation of Resonance Condensation of Fermionic Atom Pairs, Phys. Rev. Lett. **92**, 040403 (2004).

[25] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Condensation of Pairs of Fermionic Atoms near a Feshbach Resonance, Phys. Rev. Lett. **92**, 120403 (2004).

[26] Y.-A. Chen, S. D. Huber, S. Trotzky, I. Bloch, and E. Altman, Many-body Landau-Zener dynamics in coupled one-dimensional Bose liquids, Nat. Phys. **7**, 61 (2011).

[27] M. Gaudin, *The Bethe Wavefunction* (Cambridge University Press, Cambridge, England, 2014).

[28] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).

[29] B. Sutherland, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems* (World Scientific, Singapore, 2004).

[30] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete Generalized Gibbs Ensemble in an Interacting Theory, Phys. Rev. Lett. **115**, 157201 (2015).

[31] E. A. Yuzbashyan, M. Dzero, V. Gurarie, and M. S. Foster, Quantum quench phase diagrams of an s-wave BCS-BEC condensate, Phys. Rev. A **91**, 033628 (2015).

[32] P. Weinberg, M. Bukov, L. D'Alessio, A. Polkovnikov, Sz. Vajna, and M Kolodrubetz, Adiabatic perturbation theory and geometry of periodically-driven systems, Phys. Rep. **688**, 1 (2017).

[33] L. Landau, Zur Theorie der Energieubertragung. II, Phys. Z. Sowjetunion **2**, 46 (1932).

[34] C. Zener, Non-adiabatic crossing of energy levels, Proc. R. Soc. **137**, 696 (1932).

[35] E. Majorana, Atomi orientati in campo magnetico variabile, Nuovo Cimento **9**, 43 (1932).

[36] E. C. G. Stückelberg, Theorie der unelastischen Stösse zwischen Atomen, Helv. Phys. Acta **5**, 370 (1932).

[37] Yu. N. Demkov and V. I. Osherov, Stationary and nonstationary problems in quantum mechanics that can be solved by means of contour integration, Zh. Exp. Teor. Fiz. **53**, 1589 (1967) [Sov. Phys. JETP **26**, 916 (1968)].

[38] S. Brundobler and V. Elser, S-matrix for generalized Landau-Zener problem, J. Phys. A **26**, 1211 (1993).

[39] V. N. Ostrovsky and H. Nakamura, Exact analytical solution of the N-level Landau-Zener-type bow-tie model, J. Phys. A **30**, 6939 (1997).

[40] Y. N. Demkov and V. N. Ostrovsky, Multipath interference in a multistate Landau-Zener-type model, Phys. Rev. A **61**, 032705 (2000).

[41] Y. N. Demkov and V. N. Ostrovsky, The exact solution of the multistate Landau-Zener type model: The generalized bow-tie model, J. Phys. B **34**, 2419 (2001).

[42] N. A. Sinitsyn, Multiparticle Landau-Zener model: Application to quantum dots, Phys. Rev. B **66**, 205303 (2002).

[43] V. L. Pokrovsky and N. A. Sinitsyn, Landau-Zener transitions in a linear chain, Phys. Rev. B **65**, 153105 (2002).

[44] A. Patra and E. A. Yuzbashyan, Quantum integrability in the multistate Landau-Zener problem, J. Phys. A **48**, 245303 (2015).

[45] B. S. Shastry, A class of parameter-dependent commuting matrices, J. Phys. A **38**, L431 (2005).

[46] H. K. Owusu, K. Wagh, and E. A. Yuzbashyan, The link between integrability, level crossings and exact solution in quantum models, J. Phys. A **42**, 035206 (2009).

[47] H. K. Owusu and E. A. Yuzbashyan, Classification of parameter-dependent quantum integrable models, their parameterization, exact solution and other properties, J. Phys. A **44**, 395302 (2011).

[48] E. A. Yuzbashyan and B. S. Shastry, Quantum integrability in systems with finite number of levels, J. Stat. Phys. **150**, 704 (2013).

[49] N. A. Sinitsyn, Exact transition probabilities in a 6-state Landau-Zener system with path interference, J. Phys. A **48**, 195305 (2015).

[50] N. A. Sinitsyn and V. Y. Chernyak, The quest for solvable multistate Landau-Zener models, J. Phys. A **50**, 255203 (2017).

[51] N. A. Sinitsyn, J. Lin, and V. Y. Chernyak, Constraints on scattering amplitudes in multistate Landau-Zener theory, Phys. Rev. A **95**, 012140 (2017).

[52] N. A. Sinitsyn, Solvable four-state Landau-Zener model of two interacting qubits with path interference, Phys. Rev. B **92**, 205431 (2015).

[53] N. A. Sinitsyn and F. Li, Solvable multistate model of Landau-Zener transitions in cavity QED, Phys. Rev. A **93**, 063859 (2016).

[54] C. Sun and N. A. Sinitsyn, Landau-Zener extension of the Tavis-Cummings model: Structure of the solution, Phys. Rev. A **94**, 033808 (2016).

[55] C. Sun and N. A. Sinitsyn, A large class of solvable multistate Landau-Zener models and quantum integrability, arXiv:1707.04963.

[56] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Yu. Novokshenov, *Painlevé Transcendents: The Riemann-Hilbert Approach*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, 2006).

[57] S. Petrat and R. Tumulka, Multi-time Schrödinger equations cannot contain interaction potentials, J. Math. Phys. (N.Y.) **55**, 032302 (2014).

[58] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, New York, 1987).

[59] For example, in the multistate Landau-Zener problem this parameter is the time t, while in the Tavis-Cummings model below it is the bosonic frequency ω. We set $\omega = t$ motivated by physical applications. Other choices, e.g., $\omega = e^{-t}$ lead to integrable Hamiltonians $\dot{\omega}\hat{H}_{\mathrm{TC}}(\omega)$ nonpolynomial in t.

[60] E. A. Yuzbashyan, V. B. Kuznetsov, and B. L. Altshuler, Integrable dynamics of coupled Fermi-Bose condensates, Phys. Rev. B **72**, 144524 (2005).

[61] M. C. Cambiaggio, A. M. F. Rivas, and M. Saraceno, Integrability of the pairing Hamiltonian, Nucl. Phys. **A624**, 157 (1997).

[62] N. A. Sinitsyn, Exact results for models of multichannel quantum nonadiabatic transitions, Phys. Rev. A **90**, 062509 (2014).

[63] V. N. Ostrovsky, Nonstationary multistate Coulomb and multistate exponential models for nonadiabatic transitions, Phys. Rev. A **68**, 012710 (2003).

[64] N. A. Sinitsyn, Nonadiabatic Transitions in Exactly Solvable Quantum Mechanical Multichannel Model: Role of Level Curvature and Counterintuitive Behavior, Phys. Rev. Lett. **110**, 150603 (2013).

[65] J. Lin and N. A. Sinitsyn, The model of a level crossing with a Coulomb band: Exact probabilities of nonadiabatic transitions, J. Phys. A **47**, 175301 (2014).

[66] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.190402 for details of the WKB approach.

[67] S. Miyashita and N. Nagaosa, Quantum tunneling in half-integer spin systems, Prog. Theor. Phys. **106**, 533 (2001).

[68] A. Altland and V. Gurarie, Many Body Generalization of the Landau-Zener Problem, Phys. Rev. Lett. **100**, 063602 (2008).

[69] A. Altland, V. Gurarie, T. Kriecherbauer, and A. Polkovnikov, Nonadiabaticity and large fluctuations in a many-particle Landau-Zener problem, Phys. Rev. A **79**, 042703 (2009).

[70] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions, Nucl. Phys. **B247**, 83 (1984).

[71] H. M. Babujian, Off-shell Bethe ansatz equations and N-point correlators in the SU(2) WZNW theory, J. Phys. A **26**, 6981 (1993).

[72] T. A. Sedrakyan and V. Galitski, Boundary Wess-Zumino-Novikov-Witten model from the pairing Hamiltonian, Phys. Rev. B **82**, 214502 (2010).

[73] G. Sierra, Conformal field theory and the exact solution of the BCS Hamiltonian, Nucl. Phys. **B572**, 517 (2000).

[74] D. Fioretto, J.-S. Caux, and V. Gritsev, Exact out-of-equilibrium central spin dynamics from integrability, New J. Phys. **16**, 043024 (2014).