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Fine structure of matrix Darboux-Toda integrable mapping

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Abstract

The matrix Darboux-Toda mapping is represented as a product of a number of commutative mappings. The matrix Davey-Stewartson hierarchy is invariant with respect to each of these mappings. We thus introduce an entirely new type of discrete transformation for this hierarchy. The discrete transformation for the vector nonlinear Schrödinger system coincides with one of the mappings under necessary reduction conditions. © 1998 Published by Elsevier Science B.V.

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1. Introduction

Integrable mappings are an important tool for the investigation of integrable systems. It has been suggested that the theory of integrable systems is closely connected with the representation theory of the group of integrable mappings [1]. This viewpoint continues to get many independent confirmations. In an approach like this the classification of integrable mappings plays the key role. A mapping (V-mapping) for the vector nonlinear Schrödinger system (VNLSS) [2] has recently been introduced by Aratyn's group [3]. To find it, they considered the transformations that preserve the form of the corresponding Lax operator and equation (this technique can be applied to the $(1+1)$-dimensional case only).

In the present paper, we show a new discrete symmetry of the $(1+2)$-dimensional matrix nonlinear Schrödinger system (MNLSS) [4,5]. We also show that the V-mapping (generalized to two space dimensions) is a particular case of this symmetry. Additional reduction from two to one dimension gives the transformations considered in Ref. [3].

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2. Multi-soliton solutions of the MNLSS

In the next two sections we discuss one of the possible ways to derive discrete transformations for the MNLSS in the $(1+1)$-dimensional case. The V-mapping [3] is obtained as a particular case of these mappings. In Section 4 the results are generalized to the two-dimensional case. In this section we represent explicit expressions for multi-soliton type solutions of the MNLSS (we did not encounter this sort of expressions in the available literature on the subject.) Proofs and details can be found in Ref. [5]. The MNLSS reads

$$-v_t + v_{xx} + 2vuv = 0, \qquad u_t + u_{xx} + 2uvu = 0, \tag{1}$$

where u and v are $k \times k$ matrices of arbitrary rank. Particularly, when the ranks of the matrices equal 1, the non-zero part of v is a single column and the non-zero part of u is a single line, and the system (1) coincides with the VNLSS. The system (1) can be obtained with the help of the Maurer–Cartan identity as applied to the following equations,

$$g_x g^{-1} = \begin{pmatrix} \lambda E & u \\ v & -\lambda E \end{pmatrix}, \qquad g_t g^{-1} = \begin{pmatrix} 2\lambda^2 E - uv & u_x + 2\lambda u \\ -v_x + 2\lambda v & -2\lambda^2 E + vu \end{pmatrix}, \tag{2}$$

where E denotes the $k \times k$ unity matrix and λ is a spectral parameter. A soliton-like solution of (1) is described by a pair of vectors (n_i, m_j), where n_i and m_j are natural numbers, $n_i \geqslant -1$ and $m_j \geqslant -1$,

$$u_{i,j} = -\frac{[n_1, \dots, n_i + 1, \dots, n_k; m_1, \dots, m_j - 1, \dots, m_k]}{[n_1, \dots, n_k; m_1, \dots, m_k]},$$

$$v_{i,j} = \frac{[n_1, \dots, n_j - 1, \dots, n_k; m_1, \dots, m_i + 1, \dots, m_k]}{[n_1, \dots, n_k; m_1, \dots, m_k]}. \tag{3}$$

Here $[n_1, \dots, n_k; m_1, \dots, m_k]$ stands for the determinant of the matrix whose lines consist of sublines of lengths $n_1 + 1, \dots, n_k + 1; m_1 + 1, \dots, m_k + 1$, respectively. The sublines from the sth line, corresponding to $n_i + 1$ and $m_j + 1$, are

$$n_i: \quad e^{2\tau_s}, e^{2\tau_s}\lambda_s, \dots, e^{2\tau_s}\lambda_s^{n_i},$$

$$m_j: \quad 1, \lambda_s, \dots, \lambda_s^{m_j},$$

where $\tau_s = \lambda_s x/2 - \lambda_s^2 t/4 + c_s$, λ_s and c_s are sets of arbitrary parameters. For example,

$$[0;1] = \begin{vmatrix} e^{2\tau_1} & 1 & \lambda_1 \\ e^{2\tau_2} & 1 & \lambda_2 \\ e^{2\tau_3} & 1 & \lambda_3 \end{vmatrix}, \qquad [1;-1] = \begin{vmatrix} e^{2\tau_1} & e^{2\tau_1}\lambda_1 \\ e^{2\tau_2} & e^{2\tau_2}\lambda_2 \end{vmatrix}.$$

One can directly check that (3) are indeed solutions of (1) using the identity (9) from the Appendix. The solutions of VNLSS can be derived from (3) by inserting $m_j = -1$ for $j \geqslant 2$.

3. Discrete transformations for the MNLSS

We now consider solutions with $n_\alpha - 1$ and $m_\beta + 1$. Let us denote them by $\tilde{u}_{ij}$ and $\tilde{v}_{ij}$ and call them transformed solutions. First of all, from Eq. (3) we notice that $\tilde{u}_{\alpha\beta} = 1/v_{\beta\alpha}$. Using only the identity (9) from the Appendix, one can prove the following relations between the initial and transformed functions,

$$(\tilde{u}_{i\beta} v_{\beta\alpha})_x = -(uv)_{i\alpha}, \qquad (\tilde{u}_{\alpha j} v_{\beta\alpha})_x = -(vu)_{\beta j}, \qquad \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}}, \qquad \left(\frac{v_{\beta i}}{v_{\beta\alpha}}\right)_x = (\widetilde{uv})_{\alpha i},$$

$$
\begin{aligned}
&\left(\frac{v_{j\alpha}}{v_{\beta\alpha}}\right)_x = (\widetilde{vu})_{j\beta}, \qquad \tilde{v}_{ji} = v_{ji} - \frac{v_{\beta i}v_{j\alpha}}{v_{\beta\alpha}}, \qquad \tilde{u}_{ij} = u_{ij} + \tilde{u}_{i\beta}\tilde{u}_{\alpha j}v_{\beta\alpha}, \\
&v_{\beta\alpha}^2(\widetilde{uvu})_{\alpha\beta} - (vuv)_{\beta\alpha} = v_{\beta\alpha}(\ln v_{\beta\alpha})_{xx},
\end{aligned} \tag{4}
$$

where $i \neq \alpha$ and $j \neq \beta$. Note that there are k^2 basic commutative mappings since α and β are arbitrary. Relations (4) establish the connection between the various definite types of (soliton-like) solutions of the system (1). It turns out that the transformation (4) works not only for this definite type of solutions, but for arbitrary solutions. That is, if u and v obey the system (1), $\tilde{u}$ and $\tilde{v}$ obey it as well, no matter whether u and v are soliton-like or not. At the moment, we can check this only by direct substitution of (4) into (1). About the connection between (4) and the Darboux–Toda substitution see Section 5. The product of an arbitrary number of mappings (4) is, obviously, a discrete symmetry of (1) again.

4. Two-dimensional case

In this case, the MNLSS (the two-dimensional matrix Davey–Stewartson system) reads

$$
\begin{aligned}
&u_t + au_{xx} + bu_{yy} + 2au\int \mathrm{d}y\,(vu)_x + 2b\int dx\,(uv)_y u = 0, \\
&-v_t + av_{xx} + bv_{yy} + 2a\int \mathrm{d}y\,(vu)_x v + 2bv\int dx\,(uv)_y = 0,
\end{aligned} \tag{5}
$$

where a and b are arbitrary numerical parameters. The system (5) is the third term of the matrix nonlinear Schrödinger hierarchy (MNLSH) [6]. Now we generalize (4) to two space dimensions,

$$
\begin{aligned}
&(\tilde{u}_{i\beta}v_{\beta\alpha})_x = -(uv)_{i\alpha}, \qquad (\tilde{u}_{\alpha j}v_{\beta\alpha})_y = -(vu)_{\beta j}, \qquad \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}}, \qquad \left(\frac{v_{\beta i}}{v_{\beta\alpha}}\right)_x = (\widetilde{uv})_{\alpha i}, \\
&\left(\frac{v_{j\alpha}}{v_{\beta\alpha}}\right)_y = (\widetilde{vu})_{j\beta}, \qquad \tilde{v}_{ji} = v_{ji} - \frac{v_{\beta i}v_{j\alpha}}{v_{\beta\alpha}}, \qquad \tilde{u}_{ij} = u_{ij} + \tilde{u}_{i\beta}\tilde{u}_{\alpha j}v_{\beta\alpha}, \\
&v_{\beta\alpha}^2(\widetilde{uvu})_{\alpha\beta} - (vuv)_{\beta\alpha} = v_{\beta\alpha}(\ln v_{\beta\alpha})_{xy}.
\end{aligned} \tag{6}
$$

Within the scope of the present paper the above form of the two-dimensional mapping is a suggestion that should be checked independently. Substituting the transformed functions $\tilde{u}$ and $\tilde{v}$ into (5), we directly prove that the system (5) is invariant with respect to the mapping (6). In this paper, we do not consider the problem of constructing the hierarchy corresponding to the *isolated* mapping from (6). But finding the hierarchy invariant with respect to *all* mappings (6) is not a problem. Indeed, the matrix Darboux–Toda substitution can be represented as a product of mappings (6) (see the next section). Hence, it commutes with any transformation from (6). Therefore, all systems of MNLSH are invariant with respect to any transformation from (6).

5. Different mappings and the connection between them

First of all, we easily derive the V-mapping from the transformation (4). Let us take

$$
\begin{aligned}
&\alpha = \beta = r, \qquad u_{ir} \equiv u_i, \qquad v_{ri} \equiv v_i, \\
&u_{ij} = v_{ji} = 0, \qquad j \neq r.
\end{aligned}
$$

Now we consider the connection between the various discrete transformations corresponding to MNLSH. In Ref. [6], the $(1+2)$-dimensional MNLSH has been constructed as a consequence of its invariance with regard to the matrix Darboux–Toda transformation

$$\tilde{u} = v^{-1}, \qquad \tilde{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1}v_y)_x], \tag{7}$$

where u and v are invertible $k \times k$ matrices. Denote this transformation by M_k and the mapping (6) by $T_{\alpha\beta}$. The equality $T_{11}T_{22} \times \ldots \times T_{kk} = M_k$. holds. The operators T_{ij} are related by $T_{ij}T_{ji} = T_{ii}T_{jj}$, $T_{ij}T_{jk}T_{ki} = T_{ii}T_{jj}T_{kk}$, and so on. The algebra of the T_{ij} generators may appear to be an important instrument to investigate MNLSS solutions.

In the one-dimensional case, there is another substitution corresponding to the MNLSS (i.e., the MNLSS is invariant with respect to that mapping),

$$\tilde{u}_x = u - \tilde{u}v\tilde{u}, \qquad v_x = v\tilde{u}v - \tilde{v}. \tag{8}$$

We have checked that in the scalar case (when u and v are scalar functions) the solutions produced by this mapping are the same as those produced by the Darboux–Toda transformation. The only difference is in choosing initial (starting) functions. However, it is not clear whether the substitution (8) has a two-dimensional analogue.

6. Outlook

The main result of the present paper consists in new discrete transformations for the $(1+2)$-dimensional MNLSS. Whereas the Darboux–Toda mapping (7) requires $\operatorname{Det} v \neq 0$, the mappings (4) are free from that restriction. This expands the possibility of investigating the MNLSS. Especially, when the ranks of the matrices are equal to 1, we get $(1+2)$-dimensional generalizations of the VNLSS and the corresponding mapping [3]. But these are by far not the only possible partial cases. At the moment, we do not know how to solve the symmetry equation for an isolated $T_{\alpha\beta}$ mapping from (6). Obviously, the conventional Lax technique does not work in the two-dimensional case. Solution of the symmetry equation is the most intriguing unsolved problem of the present paper. We hope to return to it in future publications.

Appendix

Consider a square matrix

$$F = \begin{pmatrix} A & a_1 & b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{pmatrix},$$

where A is a square matrix; a_1, b_1 and a_2, b_2 are columns and rows of the corresponding dimension, respectively, and $c_{1,2}$ and $d_{1,2}$ are scalars. We have

$$\operatorname{Det} F = \operatorname{Det} A \operatorname{Det} \begin{pmatrix} E & A^{-1}a_1 & A^{-1}b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{pmatrix} = \operatorname{Det} A \operatorname{Det} \begin{pmatrix} c_1 - a_2A^{-1}a_1 & d_1 - a_2A^{-1}b_1 \\ c_2 - b_2A^{-1}a_1 & d_2 - b_2A^{-1}a_1 \end{pmatrix},$$

where E is the corresponding unit matrix. Using this, we readily prove the following identity:

$$|\Pi 12||\Pi 34| + |\Pi 23||\Pi 14| = |\Pi 24||\Pi 13|. \tag{9}$$

Here | | stands for the determinant, Π denotes a $k \times (k-2)$ matrix and $1, 2, 3$ and 4 are columns of dimension k.

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