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# Classification of parameter-dependent quantum integrable models, their parameterization, exact solution and other properties

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## Abstract

We study general quantum integrable Hamiltonians linear in a coupling constant and represented by finite  $N \times N$  real symmetric matrices. The restriction on the coupling dependence leads to a natural notion of nontrivial integrals of motion and classification of integrable families into *types* according to the number of such integrals. A type  $M$  family in our definition is formed by  $N-M$  nontrivial mutually commuting operators linear in the coupling. Working from this definition alone, we parameterize type  $M$  operators, i.e. resolve the commutation relations, and obtain an exact solution for their eigenvalues and eigenvectors. We show that our parameterization covers all type 1, 2 and 3 integrable models and discuss the extent to which it is complete for other types. We also present robust numerical observation on the number of energy-level crossings in type  $M$  integrable systems and analyze the taxonomy of types in the 1D Hubbard model.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Quantum integrability is usually defined as the existence of a sufficient number of nontrivial operators commuting with the Hamiltonian. Such operators are interchangeably referred to as ‘conserved currents’, ‘families of commuting Hamiltonians’, integrals of motion, ‘dynamical conservation laws’ or simply ‘dynamical symmetries’, and are distinct from the ordinary spacetime or internal space symmetries, e.g. momentum and spin conservation, particle–hole symmetry, etc. Quantum integrable systems are believed to have a range of properties, which are sometimes proposed as alternative notions or tests of integrability—an exact solution for their eigenstates, Poissonian energy level statistics, level crossings violating the Wigner–von Neumann non-crossing rule in parameter-dependent systems and nondiffractive scattering, see e.g. [1] for a review.

There are two problems with the above definition. First, it has proved difficult to precisely formulate what constitutes a nontrivial commuting operator. For instance, any Hamiltonian commutes with projectors onto its eigenstates, so the mere statement that there exist commuting operators is not meaningful. In classical mechanics, one requires functional independence of the integrals of motion [2]. This criterion is not similarly useful in quantum mechanics where any two commuting matrices are functionally dependent unless there are degeneracies in the spectra of both of them [3]. Second, in a system with a finite Hilbert space, we clearly cannot have an infinite number of independent integrals, so we need to specify how many conserved currents are needed to say that the system is integrable. This is especially problematic in systems with no well-defined classical limit, e.g. the Hubbard model on a finite lattice [4, 5], as, unlike the classical mechanics, there is no natural notion of the number of degrees of freedom. There is a similar lack of clarity in delineating the properties of quantum integrable systems. For example, there are poorly understood exceptions to the energy level statistics and level crossing tests [1, 6] as well as no convincing derivation of the exact solution and other properties *from* the notion of integrability. This is in sharp contrast to the situation in classical mechanics where the Liouville–Arnold theorem [2] demonstrates an exact solution by quadratures and quasi-periodic motion on invariant tori to be direct consequences of integrability.

In this paper, we show that most of these problems can be resolved if one considers Hamiltonians and conserved currents that depend on a real parameter in a certain fixed way. This requirement alone leads to a well-defined notion of quantum integrability allowing us to classify and explicitly construct various integrable models, obtain their exact spectra, as well as derive other properties, directly from their integrability. Examples of such a real parameter, call it  $u$ , in condensed matter integrable systems are Coulomb interaction in the Hubbard model, magnetic field in Gaudin magnets with a boundary term [7], anisotropy in the  $XXZ$  Heisenberg model, interaction strength in the BCS model [9], etc. Dynamical symmetries depend on the same parameter in contrast to ordinary symmetries which are  $u$ -independent. Cases with no such parameter dependence, e.g. the isotropic Heisenberg magnet and the Gaudin model in the zero field [9], often correspond to a more general integrable model at a specific value of the parameter. In the Gaudin and BCS models, the Hamiltonian and all dynamical symmetries are linear in  $u$  [7, 10, 11], while in the Hubbard and  $XXZ$  models, the Hamiltonian and the first conserved current are linear, while higher currents are higher order polynomials in  $u$  [12, 13–17].

Note that in each of the above examples, there is a subset of mutually commuting operators, including the Hamiltonian, linear in the parameter. Let us therefore consider  $N \times N$  Hermitian operators of the form  $H(u) = T + uV$  and *require* the existence of dynamical symmetries linear in  $u$ . A simple but important observation is that this restriction on  $u$ -dependence leads to a natural notion of a nontrivial commuting partner. A typical (‘non-integrable’)  $H(u)$  will commute only with a single (trivial) operator linear in  $u$ , namely with  $\tilde{H}(u) = (a + bu)I + cH(u)$ , where  $a, b$  and  $c$  are real numbers and  $I$  is an identity operator. The requirement that there exists a nontrivial commuting partner linear in  $u$  imposes severe restrictions on  $H(u)$ . Indeed, as we will see below, a real symmetric  $H(u)$  for which such a partner exists is uniquely specified by a choice of no more than  $4N$  real parameters, while for a generic  $H(u)$  one needs  $N(N + 3)/2$  real parameters to fix the matrix elements of  $T$  and  $V$ .<sup>1</sup> Thus, Hermitian operators with fixed parameter dependence separate into two distinct classes—those that have nontrivial commuting partners and those that do not.

<sup>1</sup> Up to an arbitrary  $u$ -independent orthogonal transformation (e.g. to a basis where  $T$  is diagonal), which adds  $N(N - 1)/2$  parameters in both cases.

Further, there is a natural classification of families of parameter-dependent commuting operators according to the number  $K$  of independent members they contain. Families with the maximum possible number  $K = N$  (see below) of such operators are termed type 1, with one less than the maximum—type 2, etc. The main results of this paper are as follows. First, we construct type  $M$  families of commuting operators  $H^i(u) = T^i + uV^i$  for arbitrary  $M$ , i.e. we solve commutation relations  $[H^i(u), H^j(u)] = 0$  for matrix elements of  $H^i(u)$ . Second, we obtain exact eigenvalues and eigenstates of each  $H^i(u)$ . We thus derive the exact solution from the integrability. Third, we observe that the energy levels of  $H^i(u)$  frequently cross and determine the number of level crossings as a function of  $N$  and  $M$ . Type 1 (maximal) operators were previously constructed in [6, 18]. Here we analytically obtain *all* type 2 and 3 and a sub-class of type  $M$  real symmetric  $N \times N$  operators for arbitrary  $M > 3$ . In our construction, a type  $M$  commuting family is parameterized by  $2N + M - 2$  real parameters, while, as we show, a generic type  $M$  involves  $2N + 2M - 5$  parameters.

There are typically  $u$ -independent symmetries common to the Hamiltonian and its dynamical symmetries, e.g. total momentum, particle number,  $z$  component of the total spin, etc. It makes sense to ‘factor’ all these out, i.e. to go to blocks that correspond to a complete set of quantum numbers. Matrices we construct essentially represent such ‘irreducible’ blocks of integrable Hamiltonians and their dynamical symmetries. As an example, we consider irreducible blocks for the Hubbard model with six sites, three spin-up and three spin-down electrons, and determine their types.

A toy version of the above approach to integrability based on studying parameter-dependent commuting matrices was discussed in [19] in the context of level crossings in the Hubbard model. A major breakthrough came with Shastry’s work [18] where he constructed type 1 matrices. The advantages of this approach are conceptual simplicity, clear meaning of independent integrals and freedom from specific details of particular integrable systems. A significant disadvantage is that the  $H^i(u)$  correspond to isolated sectors/blocks of the Hamiltonian and dynamical symmetries and it is difficult to establish a correspondence between matrices  $H^i(u)$  and underlying operators whose blocks they represent. Motivated by Shastry’s work, we proposed a vector space definition (see section 2) and the above classification of integrable matrices into types, which allowed us to derive an explicit parameterization for type 1 matrices, map them to Gaudin magnets and obtain an exact solution for their spectra [6]. Further, we demonstrated that  $u$ -dependent energy levels of type 1 matrices have at least one and at most  $(N - 1)(N - 2)/2$  crossings, while levels of integrable matrices of higher types do not have to cross, even though they frequently do.

In what follows, we first lay out the general strategy for resolving the commutation relations in section 2. In section 3, we briefly review the type 1 construction of [18, 6]. Next, we obtain all type 2 matrices in section 4. In section 5, we construct integrable matrices of arbitrary type  $M$ , which includes  $M = 1$  and 2 as particular cases. Exact eigenvalues and eigenstates of type  $M$  are derived in section 6. Next, we present in section 7 ‘gauge’ redundancies in the ansatz parameterization. In section 8, numerical methods for generating random commuting matrices and determining their type are presented and, thereby, the number of parameters necessary to specify a type  $M$  commuting family is determined. In this section, we will also discuss whether the construction of section 5 generates all commuting families. Next, in section 9, we look at the frequent level crossings that occur in type  $M$  matrices and relate their number to the properties of the discriminant of the characteristic polynomial of those matrices. We analyze the typology of various Hamiltonian blocks of the 1D Hubbard model in section 10. We conclude the main text with the summary and discussion of open questions. Finally, various proofs, details and tangential results are relegated to appendices A through C.

## 2. Resolving the commutation relations

We start with primary definitions and general observations. First, we define type  $M$  operators. As outlined in the introduction, the task of identifying commuting operators linear in a real parameter  $u$  in the matrix representation reduces to finding a certain number,  $K$ , independent  $N \times N$  Hermitian matrices  $H^i(u) = T^i + uV^i$  such that

$$[H^i(u), H^j(u)] = 0 \quad \text{for all } u \text{ and } i, j = 0, \dots, K-1. \quad (1)$$

Equating terms at all orders of  $u$ , we have

$$[T^i, V^j] = [T^j, V^i], \quad [T^i, T^j] = [V^i, V^j] = 0. \quad (2)$$

As discussed above, we also require that there be no  $u$ -independent symmetry common to all  $H^i(u)$ , i.e. there is no  $\Omega \neq aI$  such that  $[\Omega, H^i(u)] = 0$  for all  $u$  and  $i$ , where  $I$  is an  $N \times N$  identity matrix. When such symmetry is present,  $H^i(u)$  acquire block-diagonal structure and the problem reduces to that of smaller matrices. Note also that every commuting family contains the trivial member

$$H^0(u) = (a + bu)I, \quad (3)$$

where  $a$  and  $b$  are arbitrary real numbers.

In light of the restriction on the  $u$ -dependence, being independent of  $H^i(u)$  with  $i = 0, \dots, K-1$  simply means linear independence, i.e.  $\sum_i a_i H^i(u)$  with real  $a_i$  is zero if and only if all  $a_i = 0$ . Note that due to the absence of  $u$ -independent symmetries linear independence of  $H^i(u)$  is equivalent to that of  $V^i$  or  $T^i$  separately. Further,  $K$  is defined as the maximum number of independent commuting matrices in a given family. In other words, any  $H(u) = T + uV$  that commutes with all  $H^i(u)$  is linearly dependent on those same  $H^i(u)$ , i.e.

$$H(u) = \sum_{i=0}^{K-1} a_i H^i(u). \quad (4)$$

We see that a commuting family is a  $K$ -dimensional vector space  $\mathcal{V}_K$  where  $H^i(u)$  serve as basic vectors. There is a certain freedom in choosing a basis in the vector space  $\mathcal{V}_K$ , which will be exploited in sections 3 and 4.

It turns out that a typical matrix of the form  $T + uV$  has no nontrivial independent commuting partners linear in  $u$  [6]; it commutes only with  $H^0(u) = (a + bu)I$  and itself. Commutation relations (1) impose strong constraints on the matrix elements of *each*  $H^i(u)$  such that the probability of finding randomly generated matrices  $T + uV$  with nontrivial commuting partners is zero. For example, in the  $3 \times 3$  case, matrix elements of  $T$  and  $V$  must meet a single algebraic constraint for such a partner to exist [19] and the number of constraints increases with the size of the matrix. In our experience, finding an operator with even one nontrivial commuting partner is quite difficult. A brute force approach to generating all such operators numerically (see section 8) already becomes computationally prohibitive for  $N \geq 6$ .

Therefore, the minimum (and typical) number of independent  $H^i(u)$  in the family is  $K = 2$ . The maximum possible number can be shown [6] to be  $K = N$ , i.e.  $K$  ranges from 2 to  $N$ . Following [6], we term the commuting families with the maximum number  $N$  of  $H^i(u)$  type 1, those with  $N - 1$  commuting operators type 2, etc up to type  $N - 1$ . An arbitrary type  $M$  family is one with  $K = N - M + 1$  independent commuting  $N \times N$  matrices  $H^i = T^i + uV^i$ , which consist of the trivial independent element  $H^0(u) = (a + bu)I$  and  $K - 1$  nontrivial ones.

Thus, our task is to identify nontrivial solutions of commutation relations (2). An important observation, which considerably simplifies this task, is that we have freedom to choose bases in two distinct linear spaces without loss of generality. First, we can choose a basis in the ‘target’

space—the Hilbert space  $\mathcal{H}_N$  on which Hermitian operators  $T^i$  and  $V^i$  act. Indeed, commutation relations (2) are invariant with respect to transformations  $(T^i, V^i) \rightarrow (U^\dagger T^i U, U^\dagger V^i U)$ , where  $U$  is any  $u$ -independent unitary operator. Second, we have freedom to choose a basis in the vector space  $\mathcal{V}_K$  of commuting operators, i.e. to make linear transformations  $H^i(u) \rightarrow \sum_j a_j^i H^j(u)$  with any real,  $u$ -independent and nondegenerate matrix  $(a_j^i)$ .

It is convenient to choose the basis in the target space to be the common eigenbasis of the mutually commuting matrices  $V^i$ . In this basis,  $V^i$  are diagonal matrices whose nonzero entries are denoted by  $d_k^i$ , i.e.

$$\text{diagonal of } V^i = (d_1^i, d_2^i, \dots, d_N^i). \quad (5)$$

The commutation relation  $[T^i, V^j] = [T^j, V^i]$  in this basis reads  $(d_k^i - d_m^i)T_{km}^j = (d_k^j - d_m^j)T_{km}^i$ . This implies that an auxiliary antisymmetric matrix  $S$  whose matrix elements are

$$S_{km} \equiv \frac{T_{km}^i}{d_k^i - d_m^i} = \frac{T_{km}^j}{d_k^j - d_m^j} \quad (6)$$

does not depend on  $i$ , i.e. is the same for all members of the family. Moreover, it follows from equation (6) that

$$T^i = W^i + [V^i, S], \quad (7)$$

where  $W^i$  is a diagonal matrix (the diagonal of  $T^i$ ) [18]. Its nonzero elements are denoted by  $f_k^i$ ,

$$\text{diagonal of } T^i = (f_1^i, f_2^i, \dots, f_N^i). \quad (8)$$

Now the commutation relations  $[V^i, V^j] = 0$  and  $[T^i, V^j] = [T^j, V^i]$  are automatically satisfied and equations (2) reduce to a single equation  $[T^i, T^j] = 0$  or, equivalently, to

$$[[V^i, S], [V^j, S]] + [[V^i, S], W^j] - [[V^j, S], W^i] = 0. \quad (9)$$

We have, for fixed  $i$  and  $j$ , a single matrix ‘master’ equation involving  $4N + N(N - 1)/2$  variables— $4N$  corresponding to  $V^i, V^j, W^i, W^j$  and  $N(N - 1)/2$  corresponding to  $S$ —and  $N(N - 1)/2$  constraints. In addition to this master equation, we will find that the quantity

$$v_{ijkl} \equiv S_{ij}S_{jk}S_{kl}S_{li} + S_{ik}S_{kl}S_{lj}S_{ji} + S_{il}S_{lj}S_{jk}S_{ki} \quad (10)$$

is an important one for determining the matrix type. We will see that for type 1,  $v_{ijkl} = 0$ , and for type 2,  $v_{ijkl} = Gx_i x_j x_k x_l$ , where  $G$  is some real parameter and  $x_j, j = \{1, \dots, N\}$ , are related to the matrix elements of  $V^j$ .

Next, let us discuss the choice of basis in the vector space  $\mathcal{V}_K$  of  $K$  commuting operators. Note from equation (7) that a linear transformation  $H^i(u) \rightarrow \sum_j a_j^i H^j(u)$  translates into the same transformation on diagonal matrices  $V^i$  and  $W^i$  and does not affect the matrix  $S$ , i.e.  $S$  is independent of the choice of basis in  $\mathcal{V}_K$  and in this sense is a universal characteristic of the commuting family. Moreover, we will see that for types 1 and 2, the master equation (9) reduces to a single equation on matrix elements of  $S$ . Once  $S$  is fully parameterized,  $V^i$  and  $W^i$  can be determined from equation (9). For type  $M > 2$  families, we do not attempt to resolve equation (9) generally, but use type 1 and 2 resolutions to formulate a working ansatz that generates such families.

As mentioned above, linear independence of  $H^i(u)$  is equivalent to that of  $V^i$ . Given equation (5), this means that  $K = N - M + 1$  vectors  $\vec{d}^i$  are linearly independent. Then, via a linear transformation  $V^i \rightarrow \sum_j a_j^i V^j$ , we can go to a *canonical* basis in the operator space  $\mathcal{V}_K$  such that  $K - 1 = N - M$  of diagonal matrix elements  $d_k^i$  are zero for every  $i$ . This is the basis we will use for type 1 and type 2 families in sections 3 and 4, respectively.

All Hermitian type 1 matrices were explicitly parameterized in [6]. In constructing higher types, here we will restrict our analysis to the real symmetric as opposed to general Hermitian matrices for simplicity. Specifically, we will choose the defining parameters of type  $M$  commuting families to be real and such that the resulting matrices are real symmetric. We note however that all other properties of type  $M$  operators we construct in this paper—i.e. commutation relations (2) and exact solution for the spectra of  $H^i(u)$  (see below)—persist even when the parameters are unrestricted complex numbers.

### 3. Review of type 1 matrices

Here we review the main results for type 1 [18, 6] families of  $N \times N$  commuting operators, linear in a parameter  $u$  that have a maximum number ( $N$ ) of linearly independent members, which were also termed ‘maximal operators’ in [6]. As discussed in the previous section, we can go from a general operator basis to a convenient canonical basis by taking linear combinations of  $H^i(u) = T^i + uV^i$ . Since there are  $N$  linearly independent diagonal matrices  $V^i$  in the case of type 1, we can choose the basis so that all  $d_k^i$  in equation (5) are zero except one, i.e.  $V_{kk}^i \equiv d_k^i = \delta_{ik}$  for all  $i$ .

Consider equation (9) in the type 1 canonical basis. It turns out that the left-hand side of this equation is an explicitly mostly empty matrix. Evaluating its nonzero matrix elements, we obtain after some algebra

$$f_j^i - f_k^i = -\frac{S_{ij}S_{ik}}{S_{jk}}, \tag{11}$$

where  $i, j, k$  are distinct and  $f_k^i$  are the diagonal matrix elements of  $T^i$ , see equation (8). Consistency requires that a triangular sum of such relations itself vanish, i.e.  $(f_j^i - f_k^i) + (f_k^i - f_l^i) + (f_l^i - f_j^i) = 0$ , from which follows

$$v_{ijkl} = 0, \tag{12}$$

where  $v_{ijkl}$  is defined in equation (10). This is the main equation defining type 1 matrices [18]. The key insight in [6] is the understanding that equation (12) *necessarily* implies the parameterization

$$S_{jk} = \frac{\gamma_j \gamma_k}{\varepsilon_j - \varepsilon_k}, \tag{13}$$

with unrestricted real parameters  $\gamma_j$  and  $\varepsilon_j$  characteristic of the commuting family (see also [20] for an elegant alternative derivation of equation (13) with the help of Plücker relations).

Once the matrix  $S$  is known, it is straightforward to calculate  $f_k^i$  and matrices  $T^i$  with the help of equations (9) and (11). A general type 1 matrix  $H(u) = \sum_{i=1}^N d_i H^i(u)$  with arbitrary real  $d_i$  takes the form

$$\begin{aligned} [H(u)]_{mn} &= \gamma_m \gamma_n \left( \frac{d_m - d_n}{\varepsilon_m - \varepsilon_n} \right), \quad m \neq n, \\ [H(u)]_{mm} &= u d_m - \sum_{j \neq m} \gamma_j^2 \left( \frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} \right). \end{aligned} \tag{14}$$

Equation (14) provides a complete explicit parameterization of all type 1 matrices. Note also that the parameterization (14) is independent of the choice of basis in the operator space  $\mathcal{V}_K$ .

In [6], we further show that type 1 operators map onto a sector of Gaudin magnets in the presence of the external magnetic field and as a consequence of this correspondence, we use

the well-known exact solution of the Gaudin model [7, 9, 11] to furnish an exact solution for type 1. The components of eigenvector  $\vec{v}_m(u)$  of the type 1 matrix  $H(u)$  are

$$[\vec{v}_m(u)]_j = \frac{\gamma_j}{\lambda_m - \varepsilon_j}, \tag{15}$$

corresponding to the eigenvalue

$$E_m(u) = \sum_{k=1}^N \frac{d_k \gamma_k^2}{\lambda_m - \varepsilon_k}, \tag{16}$$

where the  $\lambda_i, i = 1, \dots, N$ , are subject to a single algebraic equation

$$\sum_{j=1}^N \frac{\gamma_j^2}{\lambda_m - \varepsilon_j} = u. \tag{17}$$

An intriguing consequence of this exact solution is that a type 1  $H(u)$  typically multiply violates the Wigner–von Neumann noncrossing rule [21–27] and, moreover, it is *necessary* that it do so at least once [6].

#### 4. Complete parameterization of type 2 matrices

In this section, we construct all type 2 families, i.e. linear vector spaces formed by  $N - 1$  mutually commuting, independent matrices  $H^i(u) = T^i + uV^i$ . The derivation is similar to that of type 1 in the previous section, but somewhat more involved. We first choose a convenient canonical basis in the operator space  $\mathcal{V}_K$  as discussed in the end of section 2 and evaluate matrix elements of equation (9) in this basis. This results in an equation constraining the matrix  $S_{kl}$  defining type 2 matrices, which we then solve. The final result—parameterization of type 2 operators—is given by equation (33).

There are  $N - 1$  linearly independent diagonal matrices  $V^i$  for any type 2 family. Therefore, by taking linear combinations of  $H^i(u)$ , we can go to a basis in the operator space where only two of the  $d_k^i$  in equation (5) are nonzero. It is convenient to choose and parameterize these nonzero matrix elements as  $d_j^i = 1/x_j$  for some fixed  $j$  and  $d_i^i = -1/x_i$  for  $i = 1, 2, \dots, j - 1, j + 1, \dots, N$ . We also impose the following constraint on parameters  $x_k$ :

$$\sum_{k=1}^N x_k = 0, \tag{18}$$

so that the identity can be formed by linear combinations of the  $H^i(u)$ , thereby ensuring that it is an element of the commuting family.

Equation (9) in this basis yields

$$f_k^{ij} - f_l^{ij} = -\frac{S_{jk}S_{jl}}{x_j S_{kl}} + \frac{S_{ik}S_{il}}{x_i S_{kl}}, \tag{19}$$

where  $k, l \neq \{i, j\}$  and the diagonal elements of  $T^i$  now depend on an additional index  $j$ . Note that equation (19) holds for all  $j$ .<sup>2</sup> Similar to the analog equation (11), consistency requires that a triangular sum of such terms vanish, i.e.  $(f_k^{ij} - f_l^{ij}) + (f_l^{ij} - f_m^{ij}) + (f_m^{ij} - f_k^{ij}) = 0$ , from which it follows that  $x_i v_{ijkl} = x_j v_{iklm}$ . Without loss of generality, we can define a quantity  $G_{ijkl}$  such that  $v_{ijkl} = G_{ijkl} x_i x_j x_k x_l$ , which implies that  $G_{jklm} = G_{iklm}$ . This relation is general

<sup>2</sup>  $N(N - 1)/2$  distinct choices of  $(i, j)$  produce  $N(N - 1)/2$  mutually commuting matrices  $H^{ij}(u) = T^{ij} + uV^{ij}$  that all belong to the same type 2 family, i.e. only  $N - 1$  of them are independent.

and must be true for all distinct  $i, j, k, l, m$ , and therefore consistently permuting indices yield  $G_{ijkl} = \text{const} = G$ , where  $G$  is some nonzero real parameter. We therefore obtain

$$v_{ijkl} = Gx_i x_j x_k x_l. \tag{20}$$

We will show that equation (20) can be solved generically. Let us simplify by dividing the  $x$ 's, i.e. let  $S_{ij} \equiv \tilde{S}_{ij} \sqrt{x_i} \sqrt{x_j}$ , from which it follows that

$$\tilde{v}_{ijkl} \equiv \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} + \tilde{S}_{ik} \tilde{S}_{kl} \tilde{S}_{lj} \tilde{S}_{ji} + \tilde{S}_{il} \tilde{S}_{lj} \tilde{S}_{jk} \tilde{S}_{ki} = G. \tag{21}$$

We are looking for a parameterization of  $\tilde{S}_{ij}$  that resolves equation (21), ideally in a manner that treats both of its indices similarly. Our strategy for pursuing this parameterization is to find another equation nontrivially related to equation (21) and use that to eliminate some of the  $S$ 's. More specifically, equation (21) involves four indices whereas the  $S$ 's are two index objects—implying that equation (21) overdetermines the  $S$ 's. We are looking for an equation involving fewer indices from which we derive a parameterization.

The  $\tilde{S}_{ij}$ , like the  $S_{ij}$ , are antisymmetric and, as it turns out, this property is sufficient to identically satisfy the equation

$$\tilde{v}_{mjkl} \tilde{S}_{ij} \tilde{S}_{ik} \tilde{S}_{il} - \tilde{v}_{mikl} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{jl} + \tilde{v}_{mijl} \tilde{S}_{ik} \tilde{S}_{jk} \tilde{S}_{kl} - \tilde{v}_{mijk} \tilde{S}_{il} \tilde{S}_{jl} \tilde{S}_{kl} = 0.$$

Given equation (21), we can remove an overall common factor  $G$  such that

$$\tilde{S}_{ij} \tilde{S}_{ik} \tilde{S}_{il} - \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{jl} + \tilde{S}_{ik} \tilde{S}_{jk} \tilde{S}_{kl} - \tilde{S}_{il} \tilde{S}_{jl} \tilde{S}_{kl} = 0. \tag{22}$$

Equation (22) is now a nontrivial equation constraining the  $\tilde{S}_{ij}$  beyond simple antisymmetry. We can use equations (21) and (22) to eliminate  $\tilde{S}_{kl}$ . This is the resulting equation

$$\rho_{ij} \equiv \frac{(\tilde{S}_{ij} \tilde{S}_{ik})^2 + (\tilde{S}_{ij} \tilde{S}_{jk})^2 - (\tilde{S}_{ik} \tilde{S}_{jk})^2 - G}{2\tilde{S}_{ik} \tilde{S}_{jk}} + \tilde{S}_{ij}^2 \tag{23}$$

$$= \frac{(\tilde{S}_{ij} \tilde{S}_{il})^2 + (\tilde{S}_{ij} \tilde{S}_{jl})^2 - (\tilde{S}_{il} \tilde{S}_{jl})^2 - G}{2\tilde{S}_{il} \tilde{S}_{jl}} + \tilde{S}_{ij}^2, \tag{23}$$

where we have chosen to separate terms involving index  $k$  and those involving index  $l$ . Note that both sides of the equation are identical but for the different indices, true for all  $k, l \neq \{i, j\}$ , and it is for this reason that we can define the object  $\rho_{ij}$  as a two-index quantity though it is defined through quantities involving three indices. From equation (23), we obtain

$$\tilde{S}_{ij}^2 (\tilde{S}_{ik} + \tilde{S}_{jk})^2 - (\tilde{S}_{ik} \tilde{S}_{jk} + \rho_{ij})^2 = G - \rho_{ij}^2 \tag{24}$$

for all  $k \neq \{i, j\}$ . The key result of our strategy is that equation (24) implies that

$$\tilde{S}_{ik} + \tilde{S}_{jk} = \frac{\sqrt{G - \rho_{ij}^2}}{\tilde{S}_{ij}} \cosh \chi_k, \tag{25}$$

$$\tilde{S}_{ik} \tilde{S}_{jk} = \sqrt{G - \rho_{ij}^2} \sinh \chi_k - \rho_{ij}$$

for some parameters  $\chi_k$ .

Again, what we have done here is use the identically satisfied equation (22) to reduce a four-index equation (21) to a number of three-index equations involving the index-less parameter  $G$  and the two-index object  $\rho_{ij}$ . Actually, at this point, there is a simple algorithm to completely determine all  $S$ 's. We can fix indices  $i$  and  $j$  and choose values for  $G, \rho_{ij}, \tilde{S}_{ij}$  and these new parameters  $\chi_k, k \neq \{i, j\}$ . Given these chosen values, the  $\tilde{S}_{ik}$  and  $\tilde{S}_{jk}$  can be determined using equation (25). Once we have values for  $\tilde{S}_{ik}$  and  $\tilde{S}_{jk}$ , substitution into equation (22) allows us to completely determine  $\tilde{S}_{kl}, k, l \neq \{i, j\}$ , i.e.

$$\tilde{S}_{kl} = -\tilde{S}_{ij} \frac{\tilde{S}_{ik} \tilde{S}_{il} - \tilde{S}_{jk} \tilde{S}_{jl}}{\tilde{S}_{ik} \tilde{S}_{jk} - \tilde{S}_{il} \tilde{S}_{jl}} = -\frac{\tilde{\Gamma}_l \cosh \chi_k + \tilde{\Gamma}_k \cosh \chi_l}{\sinh \chi_k - \sinh \chi_l}, \tag{26}$$

where

$$\tilde{\Gamma}_m \equiv \delta_m \sqrt{\frac{G - \rho_{ij}^2}{\tilde{S}_{ij}^2} \cosh \chi_m^2 - 4\sqrt{G - \rho_{ij}^2} \sinh \chi_m + 4\rho_{ij}},$$

and  $\delta_m = \pm 1$  accounts for an inherent ambiguity in sign.

The above algorithm explicitly favors two indices over the others, i.e. the fixed  $i$  and  $j$  above, and we would like to treat all indices in a unified way. Let us make the following change of variables:  $\sinh \chi_m = (q\varepsilon_m + r)/(s\varepsilon_m + t)$  for  $q, r, s, t$  such that  $qt - rs = 1$ ,<sup>3</sup> and  $m \neq \{i, j\}$ . By substituting this into equation (26) and removing an overall scale (again without loss of generality), we find that

$$\tilde{S}_{kl} = \frac{1}{2} \sqrt{(\lambda_1 - \varepsilon_k)(\lambda_2 - \varepsilon_k)(\lambda_1 - \varepsilon_l)(\lambda_2 - \varepsilon_l)} \frac{\Gamma_k + \Gamma_l}{\varepsilon_k - \varepsilon_l}, \tag{27}$$

given

$$\Gamma_m \equiv \Gamma(\varepsilon_m) = \delta_m \sqrt{1 + \frac{P_1}{\lambda_1 - \varepsilon_m} + \frac{P_2}{\lambda_2 - \varepsilon_m}}, \tag{28}$$

where the dependence of  $\lambda_1, \lambda_2, P_1, P_2$  on parameters  $q, r, s, t, G, \rho_{ij}, \tilde{S}_{ij}$  can be determined with some algebra. We are not particularly interested in this functional dependence, however, because equation (28) generally satisfies equation (21), i.e. if we choose parameters  $\lambda_1, \lambda_2, P_1, P_2$  and  $\varepsilon_m$  for  $1 \leq m \leq N$ , all  $\tilde{S}_{kl}$  given by equation (27),  $k, l = \{1 \dots N\}$ , satisfy equation (21) where in these preferred parameters  $G = -P_1 P_2 (\lambda_1 - \lambda_2)^2 / 16$ .

To define the type 2  $S_{ij} \equiv \tilde{S}_{ij} \sqrt{x_i} \sqrt{x_j}$  fully, we can simply choose  $N - 1$  of the  $x_j$  parameters and determine the remaining one so that equation (18) is satisfied. We can, however, write  $x_j$  in a more transparent form. Let us introduce new real parameters  $\gamma_j$  such that

$$x_j = \frac{\gamma_j^2}{(\lambda_1 - \varepsilon_j)(\lambda_2 - \varepsilon_j)}. \tag{29}$$

By partial fraction decomposition

$$\frac{\gamma_j^2}{(\lambda_1 - \varepsilon_j)(\lambda_2 - \varepsilon_j)} = -\frac{1}{\lambda_1 - \lambda_2} \left( \frac{\gamma_j^2}{\lambda_1 - \varepsilon_j} - \frac{\gamma_j^2}{\lambda_2 - \varepsilon_j} \right), \tag{30}$$

we see that we can impose equation (18) by requiring that  $\lambda_1$  and  $\lambda_2$  be among the  $N$  solutions to

$$\sum_{j=1}^N \frac{\gamma_j^2}{\lambda_i - \varepsilon_j} = B, \tag{31}$$

where  $B$  is an arbitrary real number and  $\varepsilon_m$  are understood to be distinct. Given equation (27), we find that type 2  $S_{kl}$  take the simple form

$$S_{kl} = \frac{1}{2} \frac{\gamma_k \gamma_l}{\varepsilon_k - \varepsilon_l} (\Gamma_k + \Gamma_l). \tag{32}$$

Fully parameterizing type 2  $S_{kl}$  is the lion's share of the type 2 parameterization effort and with a bit of algebraic manipulation we determine the  $f_k^i$  for all  $k$  using equations (9) and (19), which are linear in  $f_k^i$ . Linear combinations of the resulting  $H^i(u)$  yield a general type 2 matrix

$$\begin{aligned} [H(u)]_{mn} &= \gamma_m \gamma_n \left( \frac{d_m - d_n}{\varepsilon_m - \varepsilon_n} \right) \frac{\Gamma_m + \Gamma_n}{2}, \quad m \neq n, \\ [H(u)]_{mm} &= u d_m - \sum_{j \neq m} \gamma_j^2 \left( \frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} \right) \frac{1}{2} \frac{(\Gamma_m + \Gamma_j)(\Gamma_j + 1)}{\Gamma_m + 1}, \end{aligned} \tag{33}$$

<sup>3</sup> We choose this condition because with the four parameters there is a redundant overall scale. Choosing  $qt - rs = 1$  fixes that scale and, as it turns out, simplifies a number of calculations.

where

$$d_m = g_0 + \sum_{j=1}^{N-2} \frac{g_j}{\lambda_{j+2} - \varepsilon_m} \tag{34}$$

and  $g_j$  are arbitrary real numbers.

Equation (33) together with auxiliary equations (28), (31) and (34) provides a complete parameterization of all type 2 commuting families.  $(2N + 3)$ -independent real parameters  $B, P_1, P_2, \gamma_i$  and  $\varepsilon_i$  specify the commuting family;  $N - 1$  real numbers  $g_i$  further specify a particular matrix within the family. The choice of all these parameters is unconstrained, except simple explicit restrictions on  $P_1$  and  $P_2$  to ensure that equation (28) produces real  $\Gamma_m$ , which we discuss in detail in the next section for general type  $M$ .

The choice of parameters is not unique. For example, a uniform shift  $\varepsilon_i \rightarrow \varepsilon_i + c$  yields the same family. The entire group of ‘gauge’ transformations on the parameters that leave type 2 commuting families invariant is explored in section 7. Interestingly, it turns out that in the case of type 2, there is an additional gauge freedom not present for other types, which allows an alternative, particularly transparent parameterization, see equation (60).

### 5. Type $M$ —the ansatz parameterization

Explicitly constructing the most general type  $M \geq 3$  commuting families is difficult if at all possible (see the discussion in section 11 for more on this). However, it turns out that the type 2 formulas of the previous section can be generalized to obtain some type  $M$  families with arbitrary  $M$ . The idea is to use the same equations (32) and (33) to generate the matrices, but to extend equation (28) by including more poles in its radicand. Specifically, we take

$$\Gamma_m \equiv \Gamma(\varepsilon_m), \quad \Gamma(\sigma) = \delta(\sigma) \sqrt{1 + \sum_{j=1}^M \frac{P_j}{\lambda_j - \sigma}}, \tag{35}$$

$$d_m = g_0 + \sum_{j=1}^{N-M} \frac{g_j}{\lambda_{j+M} - \varepsilon_m}, \tag{36}$$

where  $\delta(\sigma) = \pm 1, P_j$  are real numbers and  $\lambda_m, m = \{1, \dots, N\}$  are solutions of equation (31).

In appendix A we prove that equation (33) with  $\Gamma_m$  and  $d_m$  given by equations (35) and (36), respectively, indeed produce type  $M$  matrices as defined in section 2, i.e. families of  $K = N - M + 1$  linearly independent mutually commuting matrices with no  $u$ -independent symmetry. Equations (33) and (35) contain  $2N + M + 1$  parameters characterizing the commuting family  $2N$   $\gamma_i$ 's and  $\varepsilon_i$ 's,  $M$   $P_i$ 's and parameter  $B$ . As discussed in detail in section 7, there are certain gauge transformations, such as a uniform scaling of  $\gamma_i$  and  $\varepsilon_i$  or a uniform shift of  $\varepsilon_i$ , that leave the commuting family invariant. This allows us to fix three degrees of freedom meaning that the number of parameters needed to uniquely specify a type  $M$  commuting family produced by our ansatz is  $2N + M - 2$ . In sections 8 and 11, we argue based on numerical evidence and other considerations that a *general* type  $M \geq 3$  family is uniquely specified by  $2N + 2M - 5$  parameters. This implies that our construction misses  $M - 3$  parameters, i.e. it does not yield all commuting families for types  $M > 3$ .

A natural way to choose a basis for the ansatz type  $M$  commuting family is to define the  $K - 1 = N - M$  nontrivial  $H^i(u)$  such that  $g_j = \delta_{ij}$  in equation (36) for  $1 \leq i \leq N - M$ . For  $i = 0$ , we take  $H^0(u)$  to be proportional to the identity matrix, as given in equation (3). In other words,

$$H^i(u) = T^i + uV^i \text{ is given by equations (33) and (35) with } d_k \rightarrow d_k^i = \frac{1}{\lambda_{i+M} - \varepsilon_k} \tag{37}$$

for  $i = 1, \dots, N - M$ . In particular,  $V^i$  is a diagonal matrix and

$$\text{diagonal of } V^i = (d_1^i, d_2^i, \dots, d_N^i) = \left( \frac{1}{\lambda_{i+M} - \varepsilon_1}, \frac{1}{\lambda_{i+M} - \varepsilon_2}, \dots, \frac{1}{\lambda_{i+M} - \varepsilon_N} \right). \quad (38)$$

A general member of the commuting family is

$$H(u) = \sum_{i=0}^{N-M} g_i H^i(u), \quad (39)$$

consistent with equations (33) and (36), where  $g_i$  are arbitrary real coefficients.

Especially interesting in the structure of the ansatz is the form of equation (35). The presence of these  $\Gamma_j$ 's in equation (33) is what essentially distinguishes type  $M$ ,  $M > 1$ , from the type 1 operators reviewed in section 3, and therefore what allows for the systematic decrease in the size of the commuting family, i.e. these  $\Gamma$ 's see us go from the maximal operator  $S_{kl} = \gamma_k \gamma_l / (\varepsilon_k - \varepsilon_l)$  to

$$S_{kl} = \frac{1}{2} \frac{\gamma_k \gamma_l}{\varepsilon_k - \varepsilon_l} (\Gamma_k + \Gamma_l) \quad (40)$$

for type  $M$ . The successive inclusion of poles in the radicand of equation (35), i.e. increasing numbers of  $P_i \neq 0$ , incrementally decreases the size of the corresponding commuting family. In particular, two such poles yield precisely the type 2 matrices of section 4 (see in particular equation (28)), i.e. for  $M = 2$ , the ansatz is precisely the complete parameterization of type 2 given in the previous section.

For  $M = 1$ , the ansatz offers an alternate, but equivalent parameterization of type 1 matrices to that given in equations (14). To see this, consider the off-diagonal elements of a general type 1 matrix given by the ansatz in light of the equation  $\Gamma_m^2 - \Gamma_n^2 = P_1 (\varepsilon_m - \varepsilon_n) / (\lambda_1 - \varepsilon_m)(\lambda_1 - \varepsilon_n)$ :

$$[H(u)]_{mn} = \gamma_m \gamma_n \frac{1}{2} \frac{d_m - d_n}{\Gamma_m - \Gamma_n} \frac{P_1}{(\lambda_1 - \varepsilon_m)(\lambda_1 - \varepsilon_n)}, \quad m \neq n. \quad (41)$$

We can re-express equation (41) in the equation (14) form, i.e.

$$[H(u)]_{mn} = \tilde{\gamma}_m \tilde{\gamma}_n \left( \frac{d_m - d_n}{\tilde{\varepsilon}_m - \tilde{\varepsilon}_n} \right), \quad m \neq n,$$

if we have

$$\begin{aligned} \tilde{\varepsilon}_j &= \frac{2}{P_1} \frac{w \Gamma_j + x}{y \Gamma_j + z} \\ \tilde{\gamma}_j &= \frac{\gamma_j}{(\lambda_1 - \varepsilon_j)(y \Gamma_j + z)}, \end{aligned} \quad (42)$$

where  $w, x, y, z$  are arbitrary real numbers satisfying  $wz - xy = 1$ . Note that this constitutes a threefold redundancy in the ansatz parameterization of type 1 matrices. In section 7, we will see that this redundancy is in addition to another threefold parametric redundancy that all ansatz matrices share. Demonstrating the equivalence of the diagonal elements of maximal matrices as defined by equations (14) to those of ansatz type 1 matrices, as given by equations (33), proceeds similarly.

The above is sufficient to show that type 1 matrices, as specified by the above ansatz, are precisely the type 1 matrices of section 3. Note, however, that there is a subtlety in the definition of  $d_m$ . In particular, the ansatz seemingly stipulates (i.e. through equation (36)) a restricted set of allowed  $V$ , even for type 1, whereas equations (14) take the  $V$  to be diagonal matrices with arbitrary real diagonal elements  $d_m$ . In fact, these are equivalent statements because for  $M = 1$  there are  $N g_j$  and  $N d_m$  in equation (36) related via an  $N \times N$  matrix. This

matrix is similar to the Cauchy matrix [28] and can be similarly shown to be nondegenerate. Therefore, for an arbitrary choice of  $d_m$ , there always exists the corresponding choice of  $g_j$  and vice versa.

Note that equations (14), (33) and (35),  $M = 1$ , are explicitly identical when  $P_1 = 0$  and all  $\Gamma_m = 1$ . However, the ansatz allows for some  $\Gamma$ 's to be  $-1$  when  $P_1 = 0$ . So for an ansatz type 1 matrix with  $P_1 = 0$ —which we might call a ‘type 0’ matrix—if  $\Gamma_i = \Gamma_j$ ,  $S_{ij}$  has the form of equation (13). If, however,  $\Gamma_i = 1$  and  $\Gamma_j = -1$ ,  $S_{ij}$  vanishes. It follows that if we take a type 1 commuting family and tune  $|P_1|$  ever smaller, while keeping all other parameters fixed, we find that when  $P_1 = 0$  we are left with pseudo-type 1 matrices with block diagonal structure and, consequently,  $u$ -independent symmetry. Indeed, these ‘type 0’ commuting families will be the direct sum of two different type 1 families.

### 5.1 Ensuring reality of ansatz matrices

Next, we discuss the conditions on ansatz parameters such that the resulting matrices are real symmetric. For unrestricted complex parameters  $\gamma_i, \varepsilon_i, P_i, B$  and  $g_i$ , the above construction, equations (33) and (35), produces complex symmetric matrices for which all properties of type  $M$  matrices save for reality, but including the exact spectra derived in the next section, hold.

Moreover, it can be shown that there exist complex parameters such that the corresponding matrices are real type  $M$ , and where the complexity cannot be removed simply by multiplication by some common factor. A block of the 1D Hubbard model in appendix C provides an interesting example of such a situation. However, the most general constraint on complex parameters  $\gamma_i, \varepsilon_i, P_i, B$  and  $g_i$  to ensure real type  $M$  matrices appears to be rather nontrivial and we will not attempt to derive it here. Instead, let us choose the parameters  $B, \gamma_j, \varepsilon_j, P_i$  and  $g_i$  to be real and constrain them so that equations (33) and (35) always yield real matrices. In this case, the only problematic component for producing real matrices is the  $\Gamma_j$ , we still need the radicand in equation (35) to be positive.

First, we rewrite the expression under the square root in equation (35) as follows:

$$1 + \sum_{j=1}^M \frac{P_j}{\lambda_j - \sigma} = \frac{\prod_{j=1}^M (\phi_j - \sigma)}{\prod_{j=1}^M (\lambda_j - \sigma)}, \quad (43)$$

i.e. we traded  $M$  parameters  $P_j$  for new  $M$  parameters  $\phi_j$ . Note that  $P_j$ 's can be expressed in terms of  $\phi_j$ 's by computing the residues at the corresponding poles

$$P_j = \frac{\prod_{k=1}^M (\phi_k - \lambda_j)}{\prod_{k \neq j} (\lambda_k - \lambda_j)}.$$

Let us order  $\varepsilon_j$  so that  $\varepsilon_j < \varepsilon_{j+1}$ . Recall that  $\lambda_j$  are solutions of equation (31). It can be shown (see e.g. [6]) that they are all real and lie between consecutive  $\varepsilon_j$ . We have  $\varepsilon_i < \lambda_i < \varepsilon_{i+1}$  for  $B > 0$  and  $\varepsilon_{i-1} < \lambda_i < \varepsilon_i$  for  $B < 0$ , where  $i = 1, \dots, N$ . If we now take  $\phi_j, j = 1, \dots, M$ , to be also real and such that

$$\begin{aligned} \varepsilon_j < \phi_j < \lambda_j & \text{ for } B > 0 \\ \lambda_j < \phi_j < \varepsilon_j & \text{ for } B < 0, \end{aligned} \quad (44)$$

all terms  $\frac{\phi_j - \varepsilon_j}{\lambda_j - \varepsilon_j} > 0$ , which is sufficient (though not necessary) to ensure the positivity of the radicand in equation (35) and reality of the matrix elements.

## 6. Exact solution

In this section, we present the exact solution of the type  $M$  matrices parameterized by the above ansatz, see section 5. The exact solution of the eigenvalue problem for type  $M$  matrices

proceeds in a manner reminiscent of that of the type 1 operators of [6], see also section 3. A particularly striking feature of this exact solution is that though these operators have explicitly more structure than type 1 operators, the exact solution is nevertheless essentially just as simple, in that the entire eigensolution derives directly from the solution of a *single* algebraic equation. This is in stark contrast to the situation typical with Bethe’s ansatz where there are a large number of coupled algebraic equations to be solved to derive a small portion of the spectrum.

We want to determine the exact solution  $H^i(u) \cdot \vec{v}^\sigma(u) = E_\sigma^i(u) \vec{v}^\sigma(u)$ , where  $H^i(u)$  is a basic operator specified in equation (37), and  $\vec{v}^\sigma(u)$  and  $E_\sigma^i(u)$  are its eigenvectors and corresponding eigenvalues, respectively. The ansatz eigensolution is as valid for type 1, as it is for all ansatz type  $M$  and so the components of eigenvector  $\vec{v}^\sigma(u)$  must bear some resemblance to that of our existing type 1 exact solution. Let us modify equation (15) somewhat and assume that the  $j$ th component of this eigenvector is given by

$$v_j^\sigma \equiv [\vec{v}^\sigma(u)]_j = \frac{1}{2} \frac{\gamma_j}{\sigma - \varepsilon_j} (\Gamma(\sigma) + \Gamma_j). \tag{45}$$

Expanding in terms of our parameters and using equations (37) and (40), we obtain, after some algebraic manipulation,

$$[H^i \cdot \vec{v}^\sigma]_j = v_j^\sigma \sum_m \frac{1}{\lambda_i - \varepsilon_m} \frac{\gamma_m^2}{2} \frac{\Gamma(\sigma) + \Gamma_m}{\sigma - \varepsilon_m} + \frac{v_j^\sigma}{\lambda_i - \varepsilon_j} \left[ u - \sum_m \frac{\gamma_m^2}{2} \frac{\Gamma(\sigma) + \Gamma_m}{\sigma - \varepsilon_m} - \frac{B}{2} (\Gamma(\sigma) - 1) \right].$$

Note that the second term is a  $j$ -dependent coefficient on the  $j$ th putative eigenvector component. This term must vanish in order for equation (45) to constitute an eigenvector. Thus, given an eigenvector of matrix  $H^i(u)$  specified by equation (45) the corresponding eigenvalue is

$$E_\sigma^i = \sum_m \frac{1}{\lambda_i - \varepsilon_m} \frac{\gamma_m^2}{2} \frac{1}{\sigma - \varepsilon_m} (\Gamma_m + \Gamma(\sigma)), \tag{46}$$

where  $\sigma$  is such that the following algebraic equation:

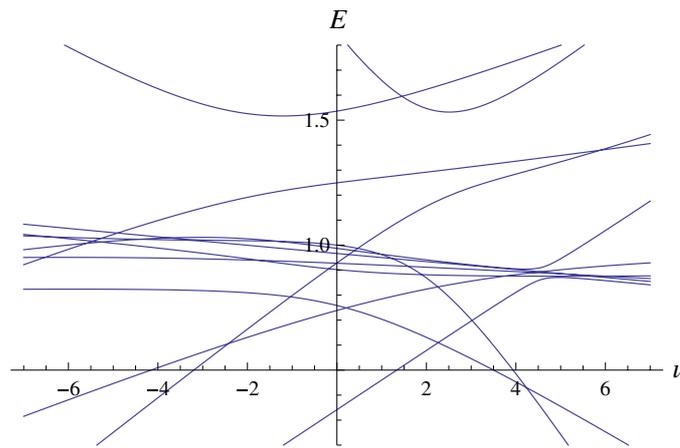
$$\sum_m \frac{1}{2} \frac{\gamma_m^2}{\sigma - \varepsilon_m} (\Gamma(\sigma) + \Gamma_m) - \frac{B}{2} (\Gamma(\sigma) - 1) = u, \tag{47}$$

is satisfied. We should note, however, that it remains to be proven that equation (47) generically yields  $N$  distinct solutions given both sign choices for  $\Gamma(\sigma)$ , see equation (35). Nevertheless, the numerical analysis has yielded no counterexample.

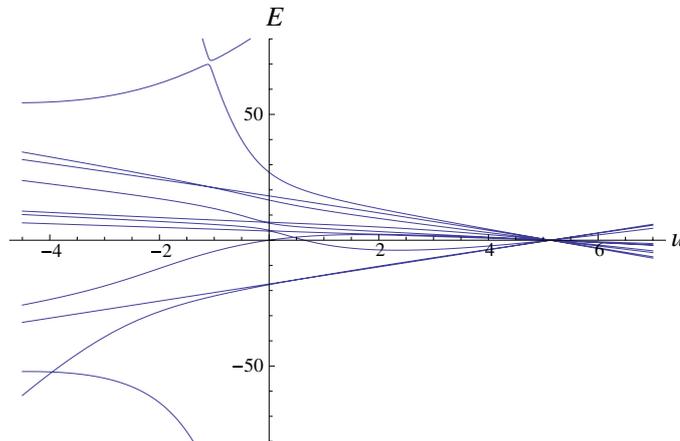
The behavior of the spectrum of a typical type  $M$  operator as a function of the parameter  $u$  is particularly interesting. As distinct from the spectra of random real symmetric operators linear in a parameter, random type  $M$  matrices—whose parameters  $B, \gamma_j, \varepsilon_j, \lambda_i, P_i$  are chosen from random uniform distributions—exhibit frequent violations of the Wigner–von Neumann noncrossing rule, see figure 1, which states that eigenvalues of the same symmetry do not cross as a function of a single coupling parameter.

It is interesting to note that the numerical observation of individual  $H^i(u)$  suggests that they undergo concurrent crossings of  $N - 2$  eigenvalues at some value of  $u$ , see figure 2. This is a surprising phenomenon that one would rightly expect from type 2 matrices<sup>4</sup> but, perhaps, not general type  $M$  matrices.

<sup>4</sup> As explained in section 2, in the canonical basis each type  $M$  basic operator at  $u = \infty$  is a diagonal matrix with  $N - M$  zero eigenvalues.



**Figure 1.** Numerical energy levels of a  $12 \times 12$  type 4 operator  $H(u) = \sum_{i=0}^8 g_i H^i(u)$  with parameters  $B, \gamma_j, \varepsilon_j, P_j$  and  $g_j$  randomly chosen over a uniform distribution of real numbers. The eigenvalues exhibit many level crossings, frequently violating the Wigner–von Neumann noncrossing theorem.



**Figure 2.** Numerical energy levels of a  $12 \times 12$  type 4 operator  $H^i(u)$ ,  $i = 3$ , with parameters  $B, \gamma_j, \varepsilon_j, \lambda_i, P_i$  identical to those in figure 1. Note that the eigenvalues exhibit  $N - 2 = 10$  concurrent level crossings at a particular value of  $u$ .

We can explain this phenomenon by considering the  $H^i(u)$  eigenvalues. At a given  $u$ , and for all  $\sigma$  satisfying equation (47) we find, through some algebraic manipulation,

$$E_\sigma^i = \sum_m \frac{1}{\lambda_i - \varepsilon_m} \frac{\gamma_m^2}{\sigma - \varepsilon_m} \frac{1}{2} (\Gamma_m + \Gamma(\sigma)) = \frac{1}{\lambda_i - \sigma} \left[ u - \frac{1}{2} \left( B + \sum_m \frac{\gamma_m^2}{\lambda_i - \varepsilon_m} \Gamma_m \right) \right]. \quad (48)$$

Thus, at  $\tilde{u} = \frac{1}{2} \left( B + \sum_m \frac{\gamma_m^2}{\lambda_i - \varepsilon_m} \Gamma_m \right)$  we find that, for  $\sigma \neq \lambda_i$ , the  $\sigma$ th eigenvalue of  $H^i(\tilde{u})$  is zero. Naturally, not all the eigenvalues can vanish, otherwise  $H^i(\tilde{u})$  is trivial. Indeed, setting  $\sigma = \lambda_i$  and  $u = \tilde{u}$  in equation (47), we find that it is solved for both sign choices for  $\Gamma(\sigma)$ , i.e. equation (47) has a doubly degenerate solution  $\sigma_-(\tilde{u}) = \sigma_+(\tilde{u}) = \lambda_i$ , such that  $\Gamma(\sigma_+) = -\Gamma(\sigma_-)$ .

Numerically, we find confirmation of this phenomenon wherein  $N - 2 \sigma$ 's correspond to zero eigenvalues, while the other two eigenvalues at  $\tilde{u}$  are nonzero.

### 7. Gauge redundancy in the ansatz parameters

Given a set of real parameters  $B, \gamma_j, \varepsilon_j, \lambda_i, P_i$  defining a type  $M$  commuting family, one might ask whether these parameters are unique, i.e. does there exist a set  $\tilde{B}, \tilde{\gamma}_j, \tilde{\varepsilon}_j, \tilde{\lambda}_i, \tilde{P}_i$  yielding the same type  $M$  commuting family?

On inspection, it is apparent that parameters are, in fact, not unique, e.g. a uniform shift of the  $\varepsilon$  (and  $\lambda$ ) parameters leaves equations (33) and (35) unchanged. Similarly, one can find a variety of uniform scaling schemes under which all matrix elements are invariant.

More interesting, however, is the following linear fractional parameter transformation:

$$\tilde{\varepsilon}_j = \frac{w\varepsilon_j + x}{y\varepsilon_j + z}, \quad \tilde{\lambda}_j = \frac{w\lambda_j + x}{y\lambda_j + z}, \quad \tilde{\gamma}_j = \frac{\gamma_j}{y\varepsilon_j + z}\Gamma(\zeta), \quad \tilde{P}_j = \frac{1}{\Gamma^2(\zeta)} \frac{P_j}{(y\varepsilon_j + z)^2}, \tag{49}$$

where  $w, x, y, z$  are arbitrary real parameters subject to constraint  $wz - xy = 1$ ,  $\zeta = -z/y$  and  $\Gamma(\zeta)$  is defined in equation (35). It can be shown then that the above transformation yields the corollary transformation:

$$\tilde{\Gamma}_j = \frac{\Gamma_j}{\Gamma(\zeta)}. \tag{50}$$

By direct computation it can be shown that the  $S_{ij}$  and, therefore, all off-diagonal matrix elements of the members of the ansatz commuting families are invariant under the above. Now consider the action of this transformation on diagonal matrix elements of the basic commuting matrix  $H^i(u) = T^i + uV^i$  given by equations (37) and (38). For  $V^i$  we have

$$\tilde{d}_j^i \equiv \left( \frac{\lambda_i}{1 - \zeta^{-1}\lambda_i} - \frac{\varepsilon_j}{1 - \zeta^{-1}\varepsilon_j} \right)^{-1} = d_j^i (1 - \zeta^{-1}\lambda_i)^2 + \zeta^{-1} (1 - \zeta^{-1}\lambda_i). \tag{51}$$

We see that the effect of the transformation on the diagonal matrix  $V^i$  is a uniform scaling by  $(1 - \zeta^{-1}\lambda_i)^2$  and shift by a trace,  $\zeta^{-1}(1 - \zeta^{-1}\lambda_i)I$ , i.e.

$$\tilde{V}^i = (1 - \zeta^{-1}\lambda_i)^2 V^i + \zeta^{-1} (1 - \zeta^{-1}\lambda_i) I. \tag{52}$$

Using partial fraction decomposition, see equation (30) and equation (35), it can be shown that

$$(1 - \zeta^{-1}\lambda_i)^{-2} \tilde{H}^i(u) = H^i \left[ \frac{u}{(1 - \zeta^{-1}\lambda_i)^2} + \sum_j \gamma_j^2 \frac{\Gamma(\zeta) + \Gamma_j}{\zeta - \varepsilon_j} - \frac{B}{2} (\Gamma(\zeta) - 1) \right] - I \sum_j \gamma_j^2 \frac{\Gamma_j + \Gamma(\zeta)}{(\lambda_i - \varepsilon_j)(\zeta - \varepsilon_j)},$$

i.e. the gauge transformation preserves the matrix elements of the type  $M$  matrices, up to a uniform rescaling, a shift in the coupling parameter  $u$  and the addition of a multiple of the identity matrix.

#### Type 2

With respect to the ansatz parameterization, in addition to the invariance under the above transformation, type 2 matrices have a parametric redundancy not shared by types  $M > 2$ . The numerical work undertaken to ‘reverse engineer’ ansatz parameters directly from the matrix elements of  $M > 2$  commuting matrices produced by the ansatz (see appendix B) yields back

the input parameters, modulo the transformation (49). However, type 2 matrices exhibit a 1D gauge freedom beyond that of equations (49).

To see where this extra redundancy comes from, consider again equation (27). Let us now allow  $\varepsilon_m$  to be a smooth function of a parameter  $t$  such that

$$\tilde{S}_{mn} = \frac{1}{2} \frac{\tilde{\Gamma}(\varepsilon_m(t), \varepsilon_n(t)) + \tilde{\Gamma}(\varepsilon_n(t), \varepsilon_m(t))}{\varepsilon_m(t) - \varepsilon_n(t)}, \tag{53}$$

where  $\tilde{\Gamma}(\alpha, \beta) \equiv \delta_m \sqrt{(\lambda_1 - \alpha)(\lambda_2 - \alpha)(\phi_1 - \beta)(\phi_2 - \beta)}$  and  $\phi_1, \phi_2$  are defined by the equation

$$\frac{(\phi_1 - \alpha)(\phi_2 - \alpha)}{(\lambda_1 - \alpha)(\lambda_2 - \alpha)} = 1 + \frac{P_1}{\lambda_1 - \alpha} + \frac{P_2}{\lambda_2 - \alpha}$$

for all  $\alpha$ . By direct computation, it can be shown that  $\tilde{S}_{mn}$  is  $t$ -independent if

$$\left(\frac{d\varepsilon_m}{dt}\right)^2 = (\lambda_1 - \varepsilon_m(t))(\lambda_2 - \varepsilon_m(t))(\phi_1 - \varepsilon_m(t))(\phi_2 - \varepsilon_m(t)), \tag{54}$$

i.e.  $\varepsilon_m(t)$  is the elliptic function of  $t$  corresponding to equation (54) with initial conditions  $\varepsilon_m(0) \equiv \varepsilon_m$ . Thus,  $\tilde{S}_{mn}$  is invariant under the transformation  $\varepsilon_m(0) \rightarrow \varepsilon_m(t)$ .

A general elliptic function—i.e. one defined by equation (54) wherein  $\lambda_1, \lambda_2, \phi_1, \phi_2$  are arbitrary complex numbers—is related to a specific Jacobi elliptic function through a linear fractional transformation. Concretely,

$$\varepsilon_j(t) = \frac{wk \operatorname{sn}((t_j + t)/\tau) + x}{yk \operatorname{sn}((t_j + t)/\tau) + z}, \tag{55}$$

where  $w, x, y, z$  are complex numbers such that  $wz - xy = 1$  and

$$\frac{w\lambda_1 + x}{y\lambda_1 + z} = k^{-1}, \quad \frac{w\lambda_2 + x}{y\lambda_2 + z} = -k^{-1}, \quad \frac{w\phi_1 + x}{y\phi_1 + z} = k, \quad \frac{w\phi_2 + x}{y\phi_2 + z} = -k, \tag{56}$$

$\tau = \pm k \sqrt{(-y^2 k^2 + w^2)(-y^2 k^{-2} + w^2)}$ , and  $k$  is any one of the four roots of the equation

$$\left(\frac{k - k^{-1}}{k + k^{-1}}\right)^2 = \frac{(\lambda_1 - \phi_1)(\lambda_2 - \phi_2)}{(\lambda_1 - \phi_2)(\lambda_2 - \phi_1)}. \tag{57}$$

Elliptic functions have a number of beautiful transformation properties. Interestingly, it can be shown, using the well-known ‘angle’ addition formulae involving Jacobi elliptic functions [29], that

$$\operatorname{sn}(t/\tau + \kappa/4) = \Gamma_j/\Gamma(-z/y), \tag{58}$$

where  $\Gamma(\sigma)$  is defined as in equation (28) and  $\kappa$  is one of the characteristic periods of these doubly periodic functions. This is a surprising intermingling of  $\varepsilon_j$  and  $\Gamma_j$ , the broader meaning of which remains mysterious.

The most compelling such exploitation of the behavior of Jacobi elliptic functions comes in reconsidering equation (53). By directly substituting equation (55) into equation (53), we find that

$$\tilde{S}_{mn} = \frac{1}{2\tau} \frac{\operatorname{cn}((t_m + t)/\tau) \operatorname{dn}((t_n + t)/\tau) + \operatorname{cn}((t_n + t)/\tau) \operatorname{dn}((t_m + t)/\tau)}{\operatorname{sn}((t_m + t)/\tau) - \operatorname{sn}((t_n + t)/\tau)}. \tag{59}$$

However,  $\tilde{S}_{mn}$  is independent of the elliptic parameter  $t$ . We can make this  $t$ -invariance manifest by using the addition law properties of Jacobi elliptic functions [29], whereby it can be shown that

$$\tilde{S}_{mn} = \frac{1}{2\tau} \frac{\operatorname{cn}\left(\frac{t_m - t_n}{2\tau}\right) \operatorname{dn}\left(\frac{t_m - t_n}{2\tau}\right)}{\operatorname{sn}\left(\frac{t_m - t_n}{2\tau}\right)}, \tag{60}$$

where  $\operatorname{sn}^2 + \operatorname{cn}^2 = 1$  and  $k^4 \operatorname{sn}^2 + \operatorname{dn}^2 = 1$ .

This parametric redundancy is peculiar to type 2—the naïve generalization to higher types, i.e. extending equations (53) and (54) by including  $\lambda_i, \phi_i, 1 \leq i \leq M$  and generating  $\varepsilon_j(t)$ 's that satisfy hyperelliptic differential equations does not yield an invariant analog of equation (53). That such a simple, geometric redundancy is embedded in the derived construction of type 2 matrices is both beautiful and compelling; however, its *meaning* is unclear.

### 8. Random generation of commuting matrices and completeness of the ansatz

In this section, we address the issue of completeness of the ansatz—whether *all* type  $M > 2$  commuting matrices are *also* ansatz matrices, i.e. for any pair of commuting matrices both linear in a parameter  $u$  do ansatz parameters  $B, \gamma_j, \varepsilon_j, \lambda_i, P_i$  and  $g_i$  exist that yield back these matrices? As yet, we have no analytic means to directly answer this question, but numerically we find that the answer is ‘Yes’ for  $M = 3$  and ‘No’ for  $M > 3$ . Specifically, numerical analysis shows that  $2N + 2M - 5$  continuous parameters are needed to uniquely specify a generic type  $M$  commuting family, i.e. the ansatz of section 5 is short by  $M - 3$  parameters.

Below we will detail this analysis, wherein we

- (1) generate two random commuting matrices and determine the type of the commuting family they belong to,
- (2) and then process one of these matrices through an algorithm designed to check whether a matrix belongs to an ansatz type  $M$  commuting family and extract ansatz parameters if it does.

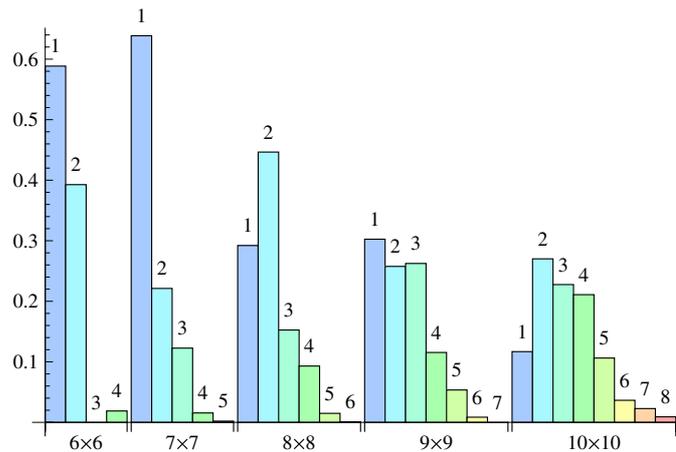
Recall from section 2 that any two commuting symmetric matrices  $H^1(u) = T^1 + uV^1$  and  $H^2(u) = T^2 + uV^2$  can be represented as  $H^1(u) = uV^1 + [V^1, S] + W^1$  and  $H^2(u) = uV^2 + [V^2, S] + W^2$ , where  $V^1, V^2, W^1, W^2$  are diagonal and  $S$  is an antisymmetric matrix.  $[H^1(u), H^2(u)] = 0$  is then equivalent to (see equation (9))

$$[[V^1, S], [V^2, S]] + [[V^1, S], W^2] - [[V^2, S], W^1] = 0. \tag{61}$$

It turns out that we can generate a pool of generic commuting matrices by brute forcing a solution to equation (61). Using Mathematica’s FindRoot [30] function—an algorithm designed to search a  $D$ -dimensional parameter space for solutions to a set of  $D$  equations—we input  $4N$  random real values for the nonzero elements of  $V^1, V^2, W^1, W^2$  and ask FindRoot to find a solution  $S$  to equation (61) whose matrix elements are close to those of a randomly generated antisymmetric seed matrix. Typically, FindRoot is able to quickly find such an antisymmetric matrix. This solution appears to be one of many in a large discrete set of compatible antisymmetric matrices, not all of them real. We know that there are many solutions because changing the random seed matrix, given the same inputs  $V^1, V^2, W^1, W^2$ , frequently sees FindRoot land on an entirely different antisymmetric matrix.

We believe that the solution set is discrete because when Mathematica’s NSolve—an algorithm designed to search for *many* numerical solutions to an arbitrary set of constraints—attempts to solve equation (61) given those same  $4N$  inputs, it manages to generate a discrete set of  $S$ . However, if we decrease the number of inputs, e.g. specify only  $4N - 1$  inputs, NSolve very quickly indicates that the system is under-constrained by at least one equation and randomly generates a linear constraint so as to proceed toward calculating a solution set.

In practice, we observe that the FindRoot algorithm can find at least one solution to equation (61), given  $4N$  inputs. Note, though, that this method of randomly generating



**Figure 3.** The distribution of various matrix types  $1 \leq M \leq N - 2$  among randomly generated integrable matrices of sizes  $N = 6, 7, 8, 9$  and  $10$ .  $4N$  random real values (flatly distributed between  $-1$  and  $1$ ) are given to the nonzero matrix elements of diagonal matrices  $V^1, V^2, W^1, W^2$  and equation (61) is solved numerically. At each iteration of random generation, one such  $S$  solution is chosen among a large set of such solutions, and matrices  $H^1(u)$  and  $H^2(u)$  are constructed and their type determined. We observe that random generation favors lower types, and we note with some surprise that there are types that appear with extremely low (albeit finite) frequency, e.g.  $8 \times 8$  type 6, or not at all,  $6 \times 6$  type 3.

commuting matrices is independent of the matrix type, i.e. nowhere in the procedure does one explicitly specify the size of the commuting family to which  $H^1(u)$  and  $H^2(u)$  belong. The initial  $4N$  parameters defining matrices  $V^1, V^2, W^1, W^2$  and the antisymmetric solution matrix  $S$  correspond to some type  $M$  commuting family, and we can determine the value of  $M$  by counting the number of linearly independent matrices in the commuting family.

In particular, we take a putative matrix  $H^k(u) = uV^k + W^k + [V^k, S]$ , where  $V^k$  and  $W^k$  are diagonal matrices whose nonzero entries,  $d_m^k$  and  $f_m^k$ , respectively, are undetermined real variables, and solve the commutation relation  $[H^1(u), H^k(u)] = 0$ . There will be  $N(N - 1)/2$  equations linear in these  $2N$  unknown  $d_m^k$  and  $f_m^k$ . Let  $p$  be the number of these equations found to be linearly independent. Then, it follows that because  $H^k(u)$  is a type  $M$  matrix and, by definition, we can only freely choose  $K = N - M + 1$  of the nonzero matrix elements of  $V^k$  and the overall trace of  $W^k$ ,  $p = 2N - (K + 1)$ , i.e. the commutator yields  $p = N + M - 2$  linearly independent constraints on  $V^k$  and  $W^k$ . Thus, counting the number  $p$  of independent equations shows the matrix type,  $M = p - N + 2$ .

We observe that this procedure for generating generic commuting matrices yields a distribution of types weighted toward the lower end, see figure 3. Interestingly, out of many hundreds of random generations, not one  $6 \times 6$  type 3 matrix was generated, even though we know that they exist and can be readily made by the ansatz.

We cannot as yet prove that given some random diagonal matrices  $V^1, V^2, W^1, W^2$ , a solution  $S$  must always exist for equation (61), but for the reasons stated above we find empirically that  $4N$  parameters are always sufficient and apparently necessary to generate two distinct commuting matrices. Accordingly, let us assume  $4N$  as the correct number of parameters necessary to specify two distinct, generic  $N \times N$  real symmetric commuting matrices, linear in parameter  $u$ , regardless of the type. If those two matrices are type  $M$ , of those  $4N$  inputs  $2(N - M + 2)$  are used to uniquely determine the two independent members

of the commuting family, i.e.  $2(N - M + 1)$  for both unique  $V$ 's and an extra two inputs to fix a trace on both  $W$ 's not constrained by the commutation relations. In addition, we can remove an overall scale on  $S$  matrix elements. This leaves  $2N + 2M - 5$  of our initial  $4N$  input parameters to uniquely specify a generic type  $M$  commuting family.

*Are all type  $M$  matrices also ansatz?*

We already know that all type 1 and 2 matrices conform to the ansatz—this is essentially what the ansatz was designed to do, see sections 4 and 5. Now let us consider ansatz type  $M \geq 3$  matrices. Counting ansatz parameters  $B, \gamma_j, \varepsilon_j, P_i$  and removing the parametric redundancies (see section 7), we are left with  $2N + M - 2$  parameters to uniquely specify an ansatz type  $M \geq 3$  family. Compared to the number of parameters needed to specify a generic type  $M$  commuting family, we see that ansatz type  $M$  families appear smaller by  $M - 3$  continuous parameters.

These parameter-counting arguments strongly suggest that a generic type  $M$  commuting matrix is not an ansatz matrix, i.e. the ansatz generates only a small subset of type  $M$  for  $M > 3$ . The case of type 3 appears marginal, i.e. on the one hand the preceding argument indicates that there are sufficient parameters for ansatz to cover all of type 3, and on the other hand the ansatz is a naïve extension on a full parameterization of type 2 and there is no obvious reason for the ansatz to extend fully to type 3.

To gain further insight into the completeness of the ansatz, we randomly generate generic commuting matrices of arbitrary type (see above) and subject them to an algorithm (described in detail in appendix B) that reverse engineers the ansatz parameters from the matrix elements if the matrix is ansatz type  $M$ . However, when we run generic commuting matrices through this algorithm, all observed matrices of type  $M > 3$  are seen not to be ansatz matrices. In particular, in these cases, the algorithm is unable to find nontrivial  $\Gamma_j$  satisfying equation (B.3).

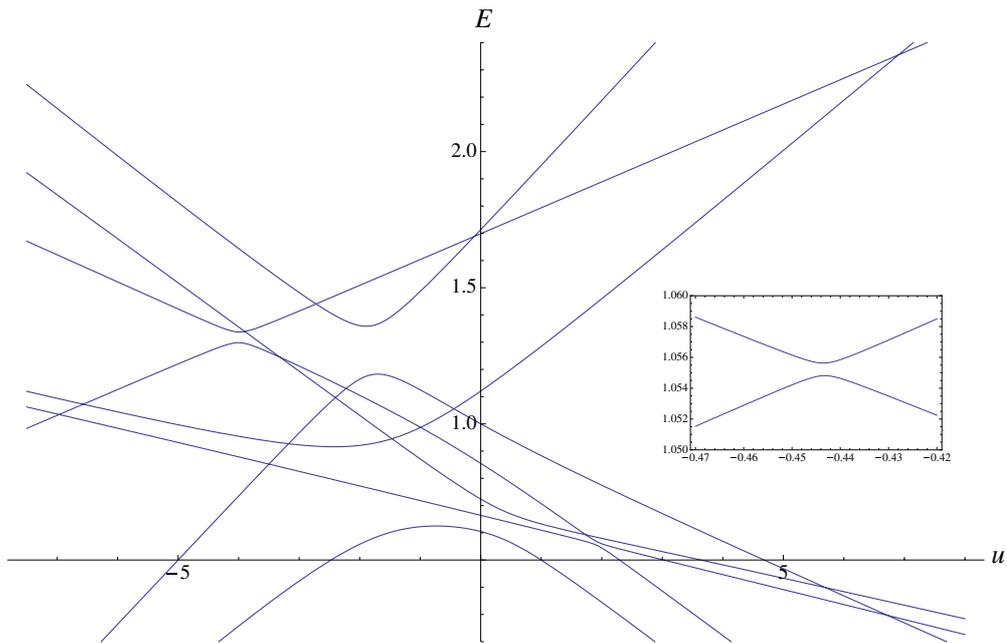
What is surprising, though, is that the ansatz carries no guarantee that it covers type 3, and yet *every* observed, randomly generated type 3 matrix has an ansatz parameterization. In view of this result, we conjecture that the ansatz is complete for types 1 and 2—which we have proved—and for type 3, while incomplete for general type  $M > 3$  by an unknown set of  $M - 3$  parameters, which we note empties for type 3. In section 11, we provide further arguments supporting this conjecture.

## 9. Level crossings

One of the signatures of parameter-dependent integrable models is the presence of level crossings in their spectra. A general  $u$ -dependent Hamiltonian,  $H(u)$ , obeys the Wigner–von Neumann noncrossing rule which states that when one plots the  $u$ -dependence of those energy levels sharing the same quantum numbers, the levels may approach one another closely, but they never cross [21–27]. The spectra of type  $M$  Hamiltonians, however, behave very differently—they carry frequent (though not necessary [6]) violations of the noncrossing rule, see figures 1, 2 and 4.

For type 1, we were able to offer a topological rationale for the necessary existence of at least one such crossing [6], but the analysis in type  $M$  is more difficult. We do observe, however, that the number of crossings varies with type as

$$\text{No of crossings} = \frac{(N - 1)(N - 2)}{2} - \tilde{M} + 1 - 2L, \quad (62)$$



**Figure 4.** The  $u$ -dependent spectrum of a randomly generated  $8 \times 8$  type 4 operator  $H(u)$ . Generally the discriminant of the characteristic polynomial of an  $N \times N$  matrix  $T + uV$  is a polynomial in  $u$  of order  $N(N - 1)$ . We observe that discriminants of type  $M$  matrices factor as  $\text{Disc}_{H(u)} = P(u)Q(u)^2$ . While the complex roots of  $P(u)$  correspond to values of  $u$  for which the spectral decomposition of  $H(u)$  involves Jordan blocks, the crossings occur at roots of the  $(N - 1)(N - 2) - M + 1 = 18$  order polynomial  $Q(u)$ . This type 4 matrix has 16 crossings at  $u = \{-6.97, -6.57, -3.90, -3.50, -3.32, -3.17, -2.72, -2.40, -2.22, -1.47, -0.99, -0.11, 1.74, 5.69, 6.72, 6.90\}$  and we numerically confirm that they are roots of  $Q(u)$ . Its other two roots are complex conjugates, corresponding to ‘complex crossings’ occurring at  $u = \{0.52 + 1.00i, 0.52 - 1.00i\}$ . Reliance on visual inspection alone suggests that there is a crossing at  $u = -0.44$  and  $E = 1.06$ . However, the inset—and analysis of the roots of the discriminant—shows that this is actually a ‘close’ level-repulsion.

where  $L$  is some non-negative integer bounded such that the above expression is non-negative, and where  $\tilde{M}$  is an integer such that

$$\tilde{M} \geq M.$$

In the overwhelming number of observed commuting matrices  $\tilde{M} = M$ . Specifically, out of several hundreds of matrices analyzed, only two nonansatz matrices were observed to be such that  $\tilde{M} \neq M$ .  $\tilde{M}$  then is overwhelmingly identical to the type, but there are apparent exceptions (see section 11). Note that we always find  $\tilde{M} = M$  for ansatz matrices.

The crossings’ story gets more interesting when one extends the notion of crossings to ‘complex crossings’, by which we mean complex values of  $u$  wherein at least two of the eigenvalues of an operator  $H(u)$  are equal. More concretely, complex crossings occur at values of  $u$  corresponding to the roots of the discriminant of the characteristic polynomial  $\mathcal{P}(\lambda, u)$  of  $H(u)$ . The discriminant is defined as  $\prod_{i < j} (E_i - E_j)^2$ , where  $E_i$  are the roots of  $\mathcal{P}(\lambda, u)$ , i.e. it is polynomial in the coefficients of  $\mathcal{P}(\lambda, u)$  and, therefore, polynomial in  $u$ . The order of this polynomial can be shown to be  $N(N - 1)$  and, therefore, there are exactly  $N(N - 1)$  complex

values of  $u$  (roots) where the discriminant vanishes, independent of type or even whether  $H(u)$  belongs to a commuting family.

However, from numerical analysis we see that for a general  $H(u)$ , these roots do not correspond to eigenvalues being equal to one another; they correspond to a situation in which there ceases to be a full eigen-decomposition of the operator. For a completely general, non-integrable matrix, the roots of the discriminant correspond to a spectral decomposition of  $H(u)$  involving Jordan blocks, rather than eigenspaces, see [32]. A type  $M$  matrix,  $H^i(u)$ , however, is rather special. Its discriminant—which we denote  $\text{Disc}_{H^i(u)}$ —factorizes as

$$\text{Disc}_{H^i(u)} = P(u)Q^i(u)^2,$$

where  $P(u)$  and  $Q^i(u)$  are polynomials. It turns out that  $Q^i(u)$  is a polynomial of order  $(N-1)(N-2)/2 - \tilde{M} + 1$  and its roots are often real, but not always. Its complex roots come in complex conjugate pairs, hence the  $2L$  term in equation (62). All of its roots correspond to values of  $u$  where at least two eigenvalues are equal. Moreover, if we consider two type  $M$  matrices,  $H^i(u)$  and  $H^j(u)$  belonging to the same commuting family, i.e.  $[H^i(u), H^j(u)] = 0$ , their discriminants will have distinct factors  $Q^i(u)$  and  $Q^j(u)$ . However, they will have the same factor  $P(u)$  and its  $2N + 2\tilde{M} - 4$  complex roots correspond to values of  $u$  where a breakdown of the eigenstructure to Jordan blocks occurs.

Interestingly we also observe that if we express  $H^j(u)$  as a polynomial in  $H^i(u)$ —which we can do for any pair of nondegenerate commuting matrices, see [3]—the coefficients of that polynomial are rational functions in  $u$  the denominators of which are *all*  $Q^i(u)$ , i.e.

$$H^j(u) = \sum_{m=0}^{N-1} \frac{R_m^j(u)}{Q^i(u)} [H^i(u)]^m, \quad (63)$$

where  $R_m^j(u)$  are polynomials in  $u$  of order  $(N-1)(N-2)/2 - \tilde{M} - m + 2$ . Note that the failure of polynomials of a degenerate matrix to span that same matrix commutant is reflected in the fact that the above polynomial's coefficients blow up at the roots of  $Q^i(u)$ , i.e. at those values of  $u$  where  $H^i(u)$  undergoes a level crossing.

## 10. Taxonomy of types in the 1D Hubbard model

In section 8, we explored the distribution of general and ansatz matrix types among randomly generated commuting matrices. Here, we study the prevalence of types in a well-known integrable system. As mentioned in the introduction (section 1), sectors of such systems corresponding to a complete set of quantum numbers are type  $M$  operators. Our aim is to determine the values of  $M$  for different sectors and to see if these operators conform to the ansatz of section 5.

As an example, we consider the 1D Hubbard model on six sites, three spin-up and three spin-down electrons. Periodic boundary conditions are assumed. The Hamiltonian reads

$$\hat{H} = T \sum_{j=1}^6 \sum_{s=\uparrow\downarrow} (c_{js}^\dagger c_{j+1s} + c_{j+1s}^\dagger c_{js}) + U \sum_{j=1}^6 \left( \hat{n}_{j\uparrow} - \frac{1}{2} \right) \left( \hat{n}_{j\downarrow} - \frac{1}{2} \right), \quad (64)$$

where  $c_{js}^\dagger$  and  $c_{js}$  are fermionic creation and annihilation operators, respectively,  $s$  denotes the spin projection,  $\hat{n}_{js} = c_{js}^\dagger c_{js}$  is the number operator and  $T$  and  $U$  are real parameters. There is a hierarchy of conserved currents commuting with each other and with the Hubbard Hamiltonian. The first nontrivial current is linear in both  $T$  and  $U$ , while higher currents are polynomials in  $U$  of order 3 and higher. Following [31], we choose energy units so that

**Table 1.** Blocks of 1D Hubbard model are matrices of various types, some of which are described by the ansatz of section 5. Each block corresponds to a complete set of parameter-independent symmetry quantum numbers. The number of nontrivial commuting partners is  $N - M - 1$ , where  $M$  is the type. For blocks in the tables this number ranges from 1 to 10, where 10 corresponds to the  $14 \times 14$  type 3 block.

Momenta $P = 1, 5$			
No of identical blocks	$N \times N$	Type	Is it ansatz?
1	$8 \times 8$	Type 3	Yes
2	$3 \times 3$	Type 1	Yes
2	$16 \times 16$	Type 12	No
1	$14 \times 14$	Type 3	Yes
2	$3 \times 3$	Type 1	Yes
Momenta $P = 2, 4$			
2	$12 \times 12$	Type 7	No
1	$14 \times 14$	Type 11	No
2	$4 \times 4$	Type 1	Yes
2	$2 \times 2$	—	—
1	$16 \times 16$	Type 6	No

$U - 4T = 1$ , which is equivalent to the replacement

$$U = u \quad T = (u - 1)/4,$$

where  $u$  is a dimensionless real parameter.

Hamiltonian (64) for three spin-up and three spin-down electrons can be represented by a square matrix of size  $(C_3^6)^2 = 400$ . It has numerous parameter-independent symmetries such as total momentum, spin, particle–hole symmetry, etc. Using these symmetries, one can bring the Hamiltonian to a block diagonal form with block sizes ranging from 1 to 16; see [19, 31] for details as well as the quantum numbers corresponding to block sizes shown in table 1. With the help of this procedure and algorithms described in section 8 and appendix B, we determine the smallest possible blocks, their type and whether they conform to our ansatz of section 5. The results are displayed in table 1. The total momentum takes integer values  $P = 0, \dots, 5$  in our notation. Spectra for momenta  $P$  and  $6 - P$  are identical due to a spatial reflection symmetry. For simplicity, we show the results only for momenta  $P = 1, 5$  and  $P = 2, 4$ .

We make the following observations.

- (1) Recall that  $N \times N$  type  $M$  matrix has  $N - M - 1$  nontrivial commuting partners linear in  $u$ . We note that blocks in table 1 have 1–10 such partners. This implies that the 1D Hubbard model has at least ten conserved currents linear in  $u$ , while only one such current has been previously identified. All ten currents can be written in terms of fermionic creation/annihilation operators  $c_{js}^\dagger, c_{js}$  with the help of projectors onto the corresponding sectors. The resulting expressions however might turn out to be nonlocal and rather cumbersome.
- (2) The exact solution of section 6 applies to all ansatz blocks in table 1. Therefore, in all these sectors Bethe’s ansatz solution for the 1D Hubbard model [5, 33]—a large number of coupled nonlinear equations—reduces to single equation (47). It is interesting to see explicitly how such a simplification becomes possible.

In appendix C, we explicitly write down one of the ansatz Hubbard blocks—the  $8 \times 8$  type 3 block with momentum  $P = 1$ .

## 11. Discussion

In this paper, we have defined quantum integrable systems, linear in a parameter  $u$ , to be operators for which there exist a number of mutually commuting operators similarly linear in  $u$ . We introduced a classification of these quantum integrable systems according to type, where we defined a type  $M$  operator as belonging to a commuting family with  $K = N - M + 1$  linearly independent members. For types 1 and 2, we were able to resolve the corresponding commutation relations, equation (9), yielding a complete parameterization. For higher types, we extended the type 2 parameterization to create an ansatz, see equations (33)–(36), that parameterizes a subset of all type  $M$  commuting matrices. Moreover, these ansatz matrices are exactly solvable through a *single* algebraic expression, equation (47).

In addition to this analytic approach to resolving commutation relations, we also brute-forced random numerical solutions. We observed that the number of parameters necessary to specify one of these randomly generated commuting families is  $2N + 2M - 5$ , whereas the ansatz involves  $2N + M - 2$ , i.e.

$$M - 3 = (\text{no of general type } M \text{ parameters}) - (\text{no of ansatz type } M \text{ parameters}).$$

When we attempt to find ansatz parameters that can reproduce a randomly generated commuting family's matrix elements, we find that this is possible for types 1, 2 and 3 only; types  $M > 3$  do not appear to be ansatz matrices, in general. Interestingly, when attempting to find ansatz parameters for the blocks of the 1D Hubbard model Hamiltonian, we find that a number of them are indeed ansatz. Additionally, we looked at the unusual frequent violations of the noncrossing rule in the spectra of type  $M$  matrices and found that the number of (possibly complex) crossings of a type  $M$  matrix is precisely the same as the order of a polynomial in  $u$  whose square is a factor of the discriminant of that matrix's characteristic polynomial. The other factor of that discriminant is a polynomial common to all members of the commuting family, and its roots correspond to complex values of  $u$  for which all  $H(u)$  cease to have a full eigen-decomposition, whereby these matrices have a nontrivial Jordan canonical form.

The ansatz constitutes a significant step toward a comprehensive theory of parameter-dependent finite-dimensional integrable models. However, there remains much to be understood about general type  $M$  operators. For example, it is believed that there is some to-be-determined equivalence between the existence of parameter-dependent conservation laws and the existence of an exact solution. This relationship is on full display with ansatz type  $M$  matrices. However, we could, for the sake of curiosity, consider a type  $N - 1$  'commuting family' by following the ansatz exactly as written, but taking  $M = N - 1$ . Note that this is a very degenerate family because the number of independent elements is  $K = 2$ , i.e. a single nontrivial operator and the identity. At first glance, such matrices do not seem to be different from most parameter-dependent matrices in that their only commuting partner is the identity. Nevertheless, these type  $N - 1$  matrices are exactly solved by the same equations (46) and (47). This suggests that conservation laws linear in the parameter are not necessary for an exact solution, though they may be sufficient. Interestingly, we find numerically that all ansatz type  $N - 1$  matrices have  $N$  commuting currents *quadratic* in  $u$ . The exact relationship between these two properties remains elusive.

Most interesting is the question of how the ansatz fails to completely parameterize all type  $M$  commuting matrices, i.e. why ansatz type  $M$  commuting families involve  $M - 3$  fewer parameters than generic type  $M$ . Parameter counting alone tells us that the ansatz parameterizes all type 3 matrices, which we have confirmed numerically. Nevertheless, nothing in the derivation of the ansatz necessitates this type 3 parametric completeness: this result is a complete (albeit pleasant) surprise. But why this surprise coverage of type 3? Why does the

ansatz completely parameterize type 2 with a redundancy that can be made manifest through elliptic functions?

Let us start with the last question first and let it be the foundation of the following speculation: type  $M$  commuting matrices *live* on compact Riemann surfaces of genus  $g = M - 1$ . From this point of view, type 2 matrices live on tori which, when equipped with a complex structure, become Riemann surfaces of genus 1. Working backward, type 1 lives on the Riemann sphere. Similarly, type 3 lives on a Riemann surface of genus 2, a complex 2-manifold with two handles, type 4 a genus 3 surface with three handles, etc.

What do we get for this idle speculation? First, the redundancy of type 2 ansatz parameters detailed in section 7 involves elliptic functions. One way to view such functions is as doubly periodic functions over the complex plane, but another way to understand them is as the basic functions that live on a fundamental parallelogram, where the identification of the parallelogram edges implies that these functions live on a complex torus. This alone is a middling justification for such speculation in that it is limited to type 2.

The Riemann surface conjecture becomes broadly instructive when we further conjecture that ansatz type  $M$  matrices live on the *hyperelliptic* subset of Riemann surfaces of genus  $g = M - 1$ . It is a well-known result of the study of Riemann surfaces that general surfaces of genus  $g$  have complex structures parameterized by  $3g - 3$  complex parameters, whereas hyperelliptic Riemann surfaces are much simpler and have complex structures parameterized by  $2g - 1$  complex parameters [35]. If we speculate that the ansatz consistently maps Riemann surfaces of genus  $g = M - 1$  to type  $M$  commuting families, but that the ansatz is presently restricted to that subset of surfaces that are hyperelliptic, parameter counting tells us that we should be short  $g - 2$  complex parameters, i.e. that same  $M - 3$  parameter gap between randomly generated versus ansatz derived type  $M$  commuting families. Moreover, it is well known that most Riemann surfaces are not hyperelliptic, but that all surfaces of genera 0, 1 and 2 are—much the same way that the ansatz does not cover most types, but is complete for types 1, 2 and (apparently) 3.

In future work [34], we will show that the eigenvalues of the ansatz type  $M$  matrices belong to the topologically restricted vector space of meromorphic functions on hyperelliptic Riemann surfaces of genus  $M - 1$ . Moreover, the components of the eigenvectors will also be given by meromorphic functions on these surfaces and we will attempt to generalize the manner in which these functions are determined to arbitrary, nonhyperelliptic Riemann surfaces. There we will see that the well-known Riemann–Roch theorem [35] justifies the correspondence between a Riemann surface’s genus and the commuting family’s type. In particular, this theorem will show that the more general genus-type relationship is given by  $g \geq M - 1$ , where the typical situation satisfies the lower bound. From this point of view, the fact that sometimes  $\tilde{M} \neq M$  (see section 9) is reflected in the fact that  $g = M - 1$ .

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## Appendix A. Does the ansatz really parameterize type $M$ ?

Here we prove that the ansatz of section 5 indeed yields type  $M$  families of commuting operators as defined in section 2. Specifically, we need to show that (1)  $H^i(u)$  defined by equation (37) admit no  $u$ -independent symmetry, (2)  $[H^i(u), H^j(u)] = 0$  for all  $i, j \leq N - M$

and (3) the linear space  $\mathcal{V}_K$  formed by  $H^i(u)$  has dimension  $K = N - M + 1$ , i.e. the *only* matrices that commute with all  $H^i(u)$  are linear combinations thereof.

*A.1. Absence of u-independent symmetry.*

Let us assume that such a symmetry  $\Omega \neq aI$  exists, i.e.  $[\Omega, H^i(u)] = 0$  for all  $u$  and  $i$ ; then,

$$[\Omega, V^i] = [\Omega, T^i] = 0.$$

However, for ansatz type  $M$  matrices, we can choose a basis (38) on  $\mathcal{V}_K$  such that  $d_k^i = 1/(\lambda_i - \varepsilon_k)$  where, by premise, all  $\varepsilon_k$  are distinct. Consequently, an ansatz type  $M$  commuting family has a basis in which all diagonal matrices  $V^i$  are nondegenerate. It follows then that the only  $\Omega$  that commutes with all these  $V^i$  is itself a diagonal matrix. By directly computing the commutator, we see that for  $\Omega$  to commute with all  $T^i$

$$\gamma_m \gamma_n (\omega_m - \omega_n) \left( \frac{d_m^i - d_n^i}{\varepsilon_m - \varepsilon_n} \right) \frac{\Gamma_m + \Gamma_n}{2} = 0,$$

where  $\omega_m \equiv \Omega_{mm}$  which is only generally satisfied if

$$\omega_m = \omega_n$$

for all  $m, n = \{1, \dots, N\}$ , i.e. if  $\Omega$  is a multiple of the identity.

*A.2. Mutual commutativity.*

Using the relation

$$(\Gamma_m^2 - \Gamma_k^2) (\Gamma_l^2 - 1) = (\varepsilon_m - \varepsilon_k) \sum_s \sum_t \frac{P_s P_t}{(\lambda_s - \varepsilon_m)(\lambda_s - \varepsilon_k)(\lambda_t - \varepsilon_l)},$$

and some algebraic manipulation we find that

$$\begin{aligned} [H^i(u), H^j(u)]_{kl} &= [[V^i, S], [V^j, S]] + [[V^i, S], W^j] - [[V^j, S], W^i]_{kl} \\ &= - \sum_m \sum_{t>s} \sum_{s=1}^M \frac{C_{ijkl} \gamma_m^2 P_s P_t (\lambda_s - \lambda_t)^2 (\varepsilon_k - \varepsilon_l) (\varepsilon_l - \varepsilon_m) (\varepsilon_k - \varepsilon_m)}{(\lambda_i - \varepsilon_m)(\lambda_j - \varepsilon_m)(\lambda_s - \varepsilon_k)(\lambda_t - \varepsilon_l)(\lambda_s - \varepsilon_l)(\lambda_s - \varepsilon_m)(\lambda_t - \varepsilon_k)(\lambda_t - \varepsilon_m)}, \end{aligned} \tag{A.1}$$

where

$$C_{ijkl} = \frac{\gamma_k \gamma_l (\lambda_i - \lambda_j)}{4(\lambda_i - \varepsilon_k)(\lambda_j - \varepsilon_k)(\lambda_i - \varepsilon_l)(\lambda_j - \varepsilon_l) (\Gamma_k + 1) (\Gamma_l + 1)}.$$

Recall that the  $\lambda_i$  satisfy equation (31). Given this, it can be shown, using partial fraction decomposition, that

$$\sum_m \frac{\gamma_m^2}{(\lambda_{i_1} - \varepsilon_m)(\lambda_{i_2} - \varepsilon_m) \dots (\lambda_{i_k} - \varepsilon_m)} = 0, \tag{A.2}$$

provided  $i_1, i_2, \dots, i_k$  are distinct. All terms summed over the index  $m$  in equation (A.1) contain a factor of the form (A.2) for  $k = 2, 3$  and  $4$ , where  $i_1, i_2, \dots, i_k$  are by construction distinct. Thus,  $[H^i(u), H^j(u)] = 0$ , for all  $i$  and  $j$  ranging from  $0$  to  $K - 1$ .

A.3. Dimension of the vector space formed by  $H^i(u)$ .

Next, we show that there are no members of this commuting family that are linearly independent of  $H^i(u)$ . Toward this end, let us consider a general matrix  $H(u) = T + uV$  and specifically the diagonal matrix  $V$  and its nonzero matrix elements  $d_m$ . First, we note that if there exists an  $H(u)$  in the commuting family that is linearly independent of the  $H^i(u) = T^i + uV^i$ ,  $i = 0, \dots, K - 1$ , but whose  $V$  is linearly dependent on the  $V^i$ , then there exists a linear combination of  $H^i(u)$  and  $H(u)$  that is both  $u$ -independent and in the commuting family—a situation precluded by the absence of  $u$ -independent symmetry demonstrated above.

More generally, in [18], Shastry shows that for arbitrary (not necessarily ansatz) commuting matrices—independent of type—there exists a necessary (but not sufficient) set of constraints on  $d_m$  such that  $H(u)$  commute with all elements of the commuting family. The constraint is

$$d_j \mu_{j;kl} + d_k \mu_{k;l j} + d_l \mu_{l;jk} + \sum_{i \neq j,k,l} d_i v_{ijkl} = 0, \tag{A.3}$$

where  $v_{ijkl}$  and  $\mu_{j;kl}$  are quantities characteristic of the commuting family as a whole and not specific to any particular one of its members.  $v_{ijkl}$  is defined as in equation (12) and is  $\mu_{j;kl}$  defined as follows:

$$\mu_{j;kl} \equiv \frac{f_j^i - f_k^i}{d_j^i - d_k^i} - \frac{f_j^i - f_l^i}{d_j^i - d_l^i} + S_{jk}^2 S_{jl}^2 - \sum_{m \neq j,k} S_{jm} S_{mk} S_{kl} S_{lj} \frac{d_m^i - d_k^i}{d_j^i - d_k^i} - \sum_{m \neq j,l} S_{jk} S_{kl} S_{lm} S_{mj} \frac{d_m^i - d_l^i}{d_j^i - d_l^i}.$$

We have used the matrix elements of  $H^i(u)$  to define  $\mu_{j;kl}$  above, but it turns out that  $\mu_{j;kl}$  is independent of index  $i$ . Let us now consider these general quantities as determined by matrix elements given by equations (33) and (35). By direct computation, it can be shown that given the section 5 ansatz for type  $M$  matrices

$$\mu_{j;kl} = - \sum_{t>s}^M \sum_{s=1}^M \left[ \frac{P_s P_t (\lambda_s - \lambda_t)^2}{16} \left( \frac{\gamma_j^2}{(\lambda_s - \varepsilon_j)(\lambda_t - \varepsilon_j)} \right)^2 \prod_{q=\{k,l\}} \frac{\gamma_q^2}{(\lambda_s - \varepsilon_q)(\lambda_t - \varepsilon_q)} \right],$$

$$v_{ijkl} = - \sum_{t>s}^M \sum_{s=1}^M \left[ \left( \frac{P_s P_t (\lambda_s - \lambda_t)^2}{16} \right) \prod_{q=\{i,j,k,l\}} \frac{\gamma_q^2}{(\lambda_s - \varepsilon_q)(\lambda_t - \varepsilon_q)} \right].$$

Now let us compute equation (A.3) expressing  $d_m$  as  $d_m = \sum_{s=1}^N \frac{g_s}{\lambda_s - \varepsilon_m}$  for some  $g_s$ . Note that this can always be done because the  $N \times N$  Cauchy matrix  $A_{sm} = (\lambda_s - \varepsilon_m)^{-1}$  is invertible [28]. With  $d_m$  so defined, we directly compute equation (A.3) to determine the necessary  $g_s$  such that  $[H^i(u), H(u)] = 0$ . We find that

$$\frac{1}{2} \sum_{t>s}^M \sum_{s=1}^M \sum_{i=1}^N \left[ \left( g_s \frac{\gamma_i^2}{(\lambda_s - \varepsilon_i)^2} \right) \times \left( \frac{P_s P_t (\lambda_s - \lambda_t) \gamma_j^2 \gamma_k^2 \gamma_l^2}{16 (\lambda_s - \varepsilon_j)(\lambda_t - \varepsilon_j)(\lambda_s - \varepsilon_k)(\lambda_t - \varepsilon_k)(\lambda_s - \varepsilon_l)(\lambda_t - \varepsilon_l)} \right) \right] = 0,$$

which requires that

$$\begin{vmatrix} 1 & 1 & 1 \\ \sum_t \frac{P_t}{\lambda_t - \varepsilon_j} & \sum_t \frac{P_t}{\lambda_t - \varepsilon_k} & \sum_t \frac{P_t}{\lambda_t - \varepsilon_l} \\ \sum_s \frac{r_s P_s}{\lambda_s - \varepsilon_j} & \sum_s \frac{r_s P_s}{\lambda_s - \varepsilon_k} & \sum_s \frac{r_s P_s}{\lambda_s - \varepsilon_l} \end{vmatrix} = 0, \tag{A.4}$$

where summations over  $s$  and  $t$  run from 1 to  $M$  and  $r_s = g_s \sum_{i=1}^N \gamma_i^2 / (\lambda_s - \varepsilon_i)^2$ . Note that terms containing  $g_s$  with  $s > M$  cancel from equation (A.3) by virtue of identity (A.2).

For the determinant to vanish, the rows in the matrix in equation (A.4) must be linearly dependent. By the uniqueness of partial fractional decomposition, it follows that  $r_s = r$ , i.e. the determinant vanishes if (and only if)  $r_s$  does not depend on  $s$ . Thus, we require that

$$g_s = r \left( \sum_{i=1}^N \frac{\gamma_i^2}{(\lambda_s - \varepsilon_i)^2} \right)^{-1}, \quad \text{for } 1 \leq s \leq M. \tag{A.5}$$

It turns out that the degree of freedom associated with  $r$  amounts to variation in an overall trace as it can be shown that

$$\tilde{d} \equiv \sum_{j=1}^N \frac{r}{\lambda_j - \varepsilon_m} \left( \sum_{i=1}^N \frac{\gamma_i^2}{(\lambda_j - \varepsilon_i)^2} \right)^{-1} = rB^{-1}, \tag{A.6}$$

independent of the index  $m$ . Thus, it follows that the only  $V$  allowed in an ansatz type  $M$  family are constrained to be of the form

$$V = \sum_{i=1}^K a_i V^i + \tilde{d}I, \tag{A.7}$$

where  $a_i = g_{i+M} - \tilde{g}_{i+M}$  and  $\tilde{g}_{i+M}$  is defined by the right-hand side of equation (A.5) with  $s = i + M$ , whereby the ansatz necessarily parameterizes matrices of *exactly* type  $M$ .

**Appendix B. Inverse problem: determining parameters given an ansatz matrix**

Here we detail an algorithm that, given an arbitrary  $N \times N$  symmetric matrix  $H(u) = T + uV$ , determines if it conforms to the ansatz of section 5 and returns the ansatz parameters when it does.

The algorithm is based on the central observation that the difference between a type 1  $S_{ij}$ , see equation (13), and that of an ansatz type  $M$  matrix, see equation (40), is the factor  $\eta_{ij} \equiv (\Gamma_i + \Gamma_j)/2$ . If a type  $M$  commuting family has an antisymmetric matrix  $S$  for which no such factor can be found, we know it does not conform to an ansatz parameterization. If, however, such a factor exists and it can be determined, then we know that

$$\Sigma_{ij} \equiv \frac{S_{ij}}{\eta_{ij}} = \frac{\gamma_i \gamma_j}{\varepsilon_i - \varepsilon_j}, \tag{B.1}$$

for some  $\gamma_j$  and  $\varepsilon_j$  and  $\Sigma_{ij}$  so defined satisfies the equation

$$\Sigma_{ij} \Sigma_{jk} \Sigma_{kl} \Sigma_{li} + \Sigma_{ik} \Sigma_{kl} \Sigma_{lj} \Sigma_{ji} + \Sigma_{il} \Sigma_{lj} \Sigma_{jk} \Sigma_{ki} = 0. \tag{B.2}$$

Moreover, the reverse is true, i.e. it can be shown that equation (B.2) implies equation (B.1) for some  $\gamma_j$  and  $\varepsilon_j$ . Consequently, if equation (B.2) can be solved for  $\Gamma_j$ , we can essentially strip an ansatz  $S_{ij}$  of its factor  $\eta_{ij}$ , and the ansatz parameters  $\gamma_j$  and  $\varepsilon_j$  can be determined. Without loss of generality, we can multiply equation (B.2) by an overall factor of  $16 \eta_{ij} \eta_{ik} \eta_{il} \eta_{jk} \eta_{jl} \eta_{kl}$  and look for solutions  $\Gamma_j$  to the resulting equation

$$(\Gamma_i + \Gamma_k)(\Gamma_j + \Gamma_l)S_{ij}S_{jk}S_{kl}S_{li} + (\Gamma_i + \Gamma_l)(\Gamma_j + \Gamma_k)S_{ik}S_{kl}S_{lj}S_{ji} \\ + (\Gamma_i + \Gamma_j)(\Gamma_k + \Gamma_l)S_{il}S_{lj}S_{jk}S_{ki} = 0. \tag{B.3}$$

This is a massively overdetermined set of  $N(N - 1)(N - 2)(N - 3)/4!$  equations, quadratic in the  $N$  terms  $\Gamma_j$ . We note that the equations are actually linear and homogenous in the terms  $B_{ij} \equiv \Gamma_i \Gamma_j$  and that it is possible to attempt solving for the  $\Gamma_j$  by first finding the minimal set of linearly independent linear equations in  $B_{ij}$  and then solving for  $\Gamma_j$ . In numerical

practice, however, Mathematica’s NSolve—programmed to find all sets  $\Gamma_j$  that satisfy the  $N(N - 1)(N - 2)(N - 3)/4!$  equations of equation (B.3)—quickly finds nontrivial values for the  $\Gamma_j$  directly from the overdetermined quadratic equations, if they exist. We find that for matrices with a generic, nonansatz  $S$ , solving equation (B.3) yields only trivial solutions, e.g.  $\Gamma_j = 0$ , for all  $j$  but one. For ansatz matrices where  $M \geq 3$ , however, there are two equivalent, inversely related nontrivial solutions to equation (B.3), i.e.

$$\Gamma_i = \{\alpha D_i, \beta D_i^{-1}\}$$

where  $\alpha, \beta$  are arbitrary complex numbers. Note that having real  $S_{ij}$  does not generally guarantee that the corresponding  $\Gamma_i$  are themselves real. Note also that despite having proved that type 1 and 2 matrices always have an ansatz parameterization, one can just as well attempt to determine their  $\Gamma$ ’s using solutions to equation (B.3). However, in both cases, the  $\Gamma_j$  cannot be determined uniquely. In the type 1 case, NSolve generates a warning indicating that there are not enough constraints for it to express a solution set and that it must generate three additional constraints in order to proceed; from section 5, we know that these additional constraints amount to fixing the threefold parametric redundancy unique to type 1. In the case of type 2, NSolve generates a warning that it must generate a single constraint to proceed; this corresponds to a single redundancy in the parameterization of type 2 matrices involving elliptic functions, which we detailed in section 7.

If nontrivial solutions  $\Gamma_j$  exist, the above procedure determines their values. It follows from equation (B.1) that the resulting  $\Sigma_{ij}$  satisfy a corollary equation

$$(\gamma_i \Sigma_{jk})^{-1} - (\gamma_j \Sigma_{ik})^{-1} + (\gamma_k \Sigma_{ij})^{-1} = 0 \tag{B.4}$$

for some  $\gamma_i$  to be determined. This we do by fixing  $i$  and  $j$  and choosing arbitrary values for  $\gamma_i$  and  $\gamma_j$  such that we can use equation

$$\gamma_k^{-1} = \Sigma_{jk}((\gamma_j \Sigma_{ik})^{-1} - (\gamma_i \Sigma_{jk})^{-1}) \tag{B.5}$$

to determine  $\gamma_k$  for all  $k \neq i, j$ . From here, determining  $\varepsilon_k$  is straightforward, i.e.

$$\varepsilon_k = \varepsilon_i - \frac{\gamma_i \gamma_k}{\Sigma_{ij}} \quad \text{for all } k \neq i \tag{B.6}$$

where we have yet another arbitrary degree of freedom in our choice of  $\varepsilon_i$ . These three parametric redundancies, unearthed by this algorithm, constitute an interesting gauge freedom in all ansatz type  $M$  commuting families, see section 7.

If and when the algorithm finds values  $\gamma_j, \varepsilon_j$  and  $\Gamma_j$  for a putative ansatz type  $M$  matrix  $H(u)$ , proceeding to determine ansatz parameters  $\lambda_j$  is a matter of determining whether there exists an  $\tilde{H}(u) = u\tilde{V} + \tilde{T}$  in the commuting family wherein the nonzero elements of the diagonal matrix  $\tilde{V}$  are of the form

$$\tilde{d}_j = \frac{1}{\tilde{\lambda} - \varepsilon_j}$$

for  $\tilde{\lambda}$  to be determined by the algorithm, see section 5. Recall from section 8 that in determining the size of a random type  $M$  commuting family, finding all matrices that commute with  $H(u)$  reduces to  $N + M - 2$  linearly independent equations linear in its commuting partner’s diagonal elements. Of these equations, we know that because there are only  $N - M + 1$  independent members of the commuting family, exactly  $M - 1$  of these equations can be found that involve the  $N \tilde{d}_j$  alone. Consequently, there will be  $M - 1$  constraints on  $\tilde{\lambda}$  of the form

$$\sum_{j=1}^N \frac{c_j^k}{\tilde{\lambda} - \varepsilon_j} = 0,$$

where coefficients  $c_j^k$  are determined by the matrix elements of  $H(u)$ . Solving these simultaneous equations reduces to finding all  $\tilde{\lambda}$  that satisfy  $M - 1$  polynomials of order  $N - 1$ . Generally, such simultaneous polynomial equations have no solution. These ones derived from ansatz matrices, however, have an  $M$ -element solution set  $\{\lambda_1, \lambda_2, \dots, \lambda_M\}$ . To be ansatz parameters, each one must correspond to the same parameter  $B$  satisfying equation (31).

Finding the rest of the  $M$  parameters  $P_j$  is a matter of solving  $N$  overdetermined linear equations

$$\Gamma_k^2 = 1 + \sum_{j=1}^M \frac{P_j}{\lambda_j - \varepsilon_k},$$

see equation (35). If the algorithm can determine these  $P_k$ ,  $H(u)$  is ansatz. Failure to determine any of these parameters uniquely (up to the aforementioned gauge redundancy) or any inconsistency with respect to the ansatz equations indicates that the matrix is not ansatz. For example, it is possible that the algorithm could find some  $\Gamma_j$  and parameters  $\varepsilon_j, \gamma_j$  given some matrix, and yet fail to have a consistent ansatz parameter  $B$ . In practice, however, all matrices tested through this algorithm failed to yield ansatz parameters at the  $\Gamma$ -stage in so much as the algorithm could not find nontrivial  $\Gamma_j$  consistent with equation (B.3). That is, if there are nonansatz type  $M$  matrices that satisfy equation (32) for some  $\Gamma_j$  not satisfying equation (35), they appear to be rather rare in a random ensemble of type  $M$  matrices.

### Appendix C. An example of a sector of the 1D Hubbard model described by the ansatz

Here we explicitly write down the  $8 \times 8$ , momentum  $P = 1$  block of the 1D Hubbard model, which is an ansatz type 3 matrix, see section 10. This block is of the form  $T + uV$ , where  $V$  is a diagonal matrix with the following entries:

$$\text{diagonal of } V = \frac{1}{4}(-3, 1, -3, 0, 3, 0, 3, -1) \tag{C.1}$$

and

$$T = \frac{1}{12} \begin{pmatrix} 7 & \sqrt{6} & 0 & 1 & -1 & 3 & -3 & \sqrt{6} \\ \sqrt{6} & -3 & \sqrt{6} & \sqrt{6} & -\sqrt{6} & -\sqrt{6} & -\sqrt{6} & 0 \\ 0 & \sqrt{6} & 11 & 3 & 3 & 1 & 1 & -\sqrt{6} \\ 1 & \sqrt{6} & 3 & -2 & -1 & 0 & 3 & \sqrt{6} \\ -1 & -\sqrt{6} & 3 & -1 & -11 & 3 & 0 & -\sqrt{6} \\ 3 & -\sqrt{6} & 1 & 0 & 3 & 2 & -1 & \sqrt{6} \\ -3 & -\sqrt{6} & 1 & 3 & 0 & -1 & -7 & \sqrt{6} \\ \sqrt{6} & 0 & -\sqrt{6} & \sqrt{6} & -\sqrt{6} & \sqrt{6} & \sqrt{6} & 3 \end{pmatrix}. \tag{C.2}$$

Using the procedure outlined in appendix B, we find the corresponding ansatz parameters

$$\varepsilon = \{-1, 0, 5, -5/2, 5, -5/2, -1, 0\},$$

$$\gamma = \frac{1}{2\sqrt{6}}\{1 - i, 3\sqrt{6} + i\sqrt{6}, -1 - 7i, 4 + 8i, -7 + i, -8 + 4i, 1 + i, \sqrt{6} - 3\sqrt{6}i\}$$

and

$$\Gamma_j^2 = \frac{5\varepsilon_j^3 - (1 - 37i)\varepsilon_j + 4 - 3i}{5\varepsilon_j^3 - (1 + 37i)\varepsilon_j + 4 + 3i}.$$

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