

The link between integrability, level crossings and exact solution in quantum models

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Abstract

We investigate the connection between energy level crossings in integrable systems and their integrability, i.e. the existence of a set of non-trivial integrals of motion. In particular, we consider a general quantum Hamiltonian linear in the coupling u , $H(u) = T + uV$, and require that it has the maximum possible number of nontrivial commuting partners also linear in u . We demonstrate how this commutation requirement *alone* leads to: (1) an exact solution for the energy spectrum and (2) level crossings, which are *always* present in these Hamiltonians in violation of the Wigner–von Neumann non-crossing rule. Moreover, we construct these Hamiltonians explicitly by resolving the above commutation requirement and show their equivalence to a sector of Gaudin magnets (central spin Hamiltonians). In contrast, fewer than the maximum number of conservation laws does not guarantee level crossings.

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1. Introduction

Level crossings—the emergence of degeneracies in a physical system at a certain value of some tuned system coupling—underlie a myriad of compelling phenomena, including anomalies in relaxation rates [1], the onset of quantum chaos [2], quantum phase transitions [3], Berry's phase [4, 5] etc. It is widely believed that these degeneracies can often be understood in terms of a certain underlying symmetry. However, in many cases this connection between symmetry and degeneracy remains mysterious. This is especially true for quantum integrable systems, e.g. the 1D Hubbard, anisotropic Heisenberg, reduced BCS models etc. These systems are long known to display an abundance of level crossings [6–9], see figure 1, in violation of the famous Wigner–von Neumann non-crossing rule [10–16] and with no convincing symmetry explanation.

In this paper we *derive* the existence of level crossings and an exact solution for a general parameter-dependent quantum Hamiltonian *from* its integrability. Our work has been inspired

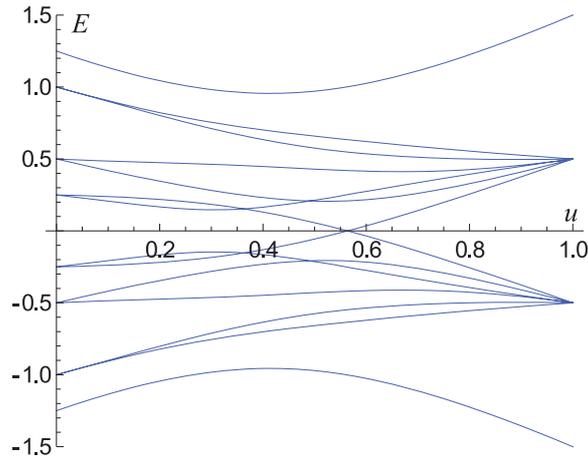


Figure 1. Energies of 1d Hubbard model on six sites characterized by a complete set of quantum numbers, i.e. all levels have the same u -independent symmetry, see [6, 18]. The energies are in units of $U - 4T$ plotted as functions of $u = U/(U - 4T)$, where U is the strength of the Coulomb repulsion and $T < 0$ is the hopping matrix element. The parameter u varies from 0 to 1 as U goes from 0 to ∞ . Note that, in violation of the Wigner-von Neumann non-crossing rule, we see a profusion of level crossings for states of the same symmetry.

in part by Gaudin and Yuzbashyan *et al* [17, 18] and especially by Shastry’s paper [19], which opened up a new, purely algebraic perspective on quantum integrable models independent of Bethe’s ansatz. In Hamiltonian mechanics the integrability of a system with n degrees of freedom is usually understood as the existence of a maximum number (n) of Poisson commuting independent invariants. Then, a well-known theorem due to Liouville and Arnold guarantees that the equations of motion can be solved by quadratures [20]. There is no similarly accepted notion of quantum integrability, especially in finite-dimensional systems, e.g. discrete lattice models in condensed matter physics where the state space is generally finite. In particular, it is often unclear what constitutes an independent integral and what is the natural notion of the number of degrees of freedom. Nevertheless, it turns out that these difficulties can be circumvented if one restricts the manner in which the integrals of motion depend on the coupling.

For concreteness, let us consider Hamiltonians linear in the coupling u . As we are interested in discrete energy spectra, we assume that the Hamiltonian can be represented by an $N \times N$ matrix. Following the classical notion of integrability, we require the existence of the *maximum* possible number of independent (see below) mutually commuting integrals, $[H^i(u), H^j(u)] = 0$, where $H^i(u) = T^i + uV^i$ are Hermitian operators. One of them is the Hamiltonian itself, e.g. $H^1(u) \equiv H(u)$. Using this commutation requirement *alone*, we derive an exact solution for the spectrum of each $H^i(u)$, which can be viewed as an extension of the Liouville–Arnold theorem to quantum Hamiltonians. Moreover, we are able to demonstrate that the eigenvalues of $H^i(u)$ are necessarily degenerate at a discrete set of values of u .

First, we solve the nonlinear commutation relations $[H^i(u), H^j(u)] = 0$ to obtain each $H^i(u)$ explicitly, see below. Interestingly, it turns out that these *maximally commuting* (or simply maximal) operators $H^i(u)$ can be mapped to exactly solvable Gaudin magnets [17, 21] (central spin Hamiltonians). The latter describes a localized spin in a magnetic field $B = u$ interacting with $N - 1$ ‘environmental’ spins and have a variety of physical applications

[22–27]. The mapping to Gaudin magnets allows us to obtain the exact solution for the eigenvalues and eigenfunctions of $H^i(u)$ and analyze its level crossings. The total number of pairwise crossings varies as $M_c = (N - 1)(N - 2)/2 - 2K > 0$, where K is a positive integer such that $M_c \geq 1$. For instance, $N = 5$ maximal operators display 2, 4 and 6 instances of level crossings. On the other hand, we demonstrate that Hamiltonians having less than the maximum number of u -dependent integrals can have no level crossings. Further, we show that the coupling-dependent commuting matrices obtained within the approach developed by Shastry in [19] (where it was also conjectured that these matrices always have crossings, see also [18]) are maximal, even though our constructions are quite different.

Pairwise crossings of energy levels are usually understood in the context of the Wigner–von Neumann non-crossing rule. This rule initially suggested by Hund [10] and justified by Wigner and von Neumann [11] has thereafter seen restatements and refinements by a number of authors [12–16]. It states that eigenstates of the same symmetry do not cross as a function of a single coupling parameter. This can be seen, for example, from the following argument. Suppose two energy levels $E_1(u)$ and $E_2(u)$ of $H(u)$ come close at a certain $u = u_0$. Expanding in a vicinity of u_0 : $H(u) \approx H(u_0) + (u - u_0)V(u_0)$ and using ordinary perturbation theory, we obtain [28]

$$\frac{d^2 \Delta}{du_0^2} = \frac{4V_{12}^2(u_0)}{\Delta(u_0)} + F(u_0) \quad (1)$$

where $\Delta(u_0) = E_1(u_0) - E_2(u_0)$, $V_{12}(u_0)$ is the matrix element of the perturbation $V(u_0)$ between states $|1\rangle$ and $|2\rangle$, and $F(u_0)$ represents the contribution of the remaining states. We see from equation (1) that as the two levels approach, $\Delta(u_0) \rightarrow 0$, infinite repulsion sets in, preventing them from crossing. This is indeed what takes place in the absence of symmetry—energy levels repel (see figure 2). The situation changes if the Hamiltonian $H(u)$ possesses a u -independent symmetry S , i.e. $[H(u), S] = 0$. This can be a spatial rotation, translational invariance, internal space reconfiguration, etc. Because S does not depend on the coupling u , it commutes with $H(u_0)$ and $V(u_0)$ individually. Evaluating the matrix element of $[V(u_0), S]$ between states $|1\rangle$ and $|2\rangle$, we obtain $V_{12}(u_0) = 0$ for any u_0 as long as $|1\rangle$ and $|2\rangle$ have different symmetry, $s_1 \neq s_2$, where $S|1\rangle = s_1|1\rangle$ and $S|2\rangle = s_2|2\rangle$. Thus, while levels of different symmetry can cross, crossings of levels of the same symmetry are prohibited.

Unfortunately, this basic argument does not extend to quantum integrable Hamiltonians $H(u)$, which typically violate the non-crossing rule. Indeed, these systems show crossings of energy levels that have the same quantum numbers for all u -independent symmetry [6–9], see e.g. figure 1. Integrable Hamiltonians are known to have special coupling dependent conserved currents, ‘dynamical symmetries’, in addition to u -independent symmetries. It is tempting to attribute these crossings to such symmetries. On the other hand, it is crucial for the validity of the non-crossing rule that the symmetry S be u -independent. Indeed, consider an integrable Hamiltonian acting on a finite-dimensional space, e.g. a lattice model with a finite number of sites. Let $H(u)$ be one of its blocks characterized by the same quantum numbers for a complete set of mutually commuting u -independent symmetries and let $\tilde{H}(u)$ be the corresponding block of one of the conserved currents

$$[H(u), \tilde{H}(u)] = 0 \quad \text{for all } u. \quad (2)$$

Due to the u -dependence, $\tilde{H}(u_0)$ no longer commutes with $V(u_0)$ separately and, therefore, the above argument lifting the level repulsion does not hold. At the same time, given a crossing one can always artificially engineer a ‘conserved current’ that commutes with $H(u)$. Therefore, restrictions on the form of $\tilde{H}(u)$ are necessary to make meaningful contact between symmetries and degeneracies.

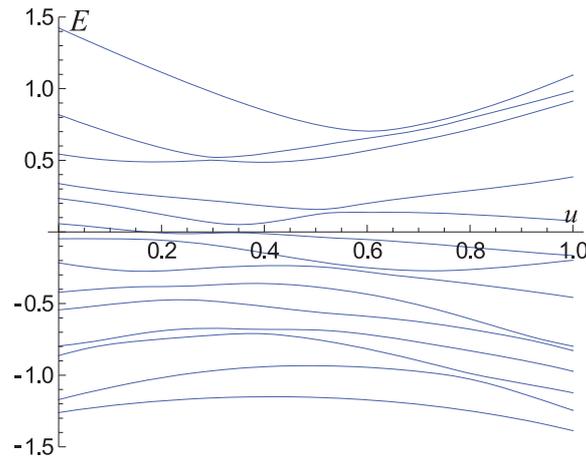


Figure 2. Energy levels of a 14×14 Hamiltonian $H(u) = A + uB$, where independent matrix elements of A and B are uniformly distributed random numbers. Note that though levels do approach one another closely, they never cross. A typical Hamiltonian with no u -independent symmetry exhibits such level repulsion, see equation (1), in contradistinction to what takes place in integrable systems, see figure 1. Here and throughout this paper we use *Mathematica* program to perform numerical tests and plot the results.

To be specific, let $H(u) = T + uV$ and $\tilde{H}(u) = \tilde{T} + u\tilde{V}$ be Hermitian operators acting on an N -dimensional space, i.e. they can be represented by $N \times N$ matrices. Equation (2) implies

$$[T, \tilde{T}] = [V, \tilde{V}] = 0, \quad [T, \tilde{V}] = [\tilde{T}, V]. \tag{3}$$

For any linear $H(u)$ there are always trivially related commuting partners $\tilde{H}(u) = aH(u) + (b + cu)I$, where I is an identity matrix. However, the requirement that equation (3) has nontrivial solutions leads to a set of nonlinear constraints that severely restricts the matrix elements of both $H(u)$ and $\tilde{H}(u)$. For example, for $N = 3$ eliminating \tilde{T} and \tilde{V} from equation (3), one obtains a single nonlinear constraint on the matrix elements of $H(u)$ [18]. In view of the preceding discussion regarding the prevalence of level crossings in integrable models, a natural question is whether these constraints, i.e. the existence of a nontrivial $\tilde{H}(u)$, imply crossings in the spectrum of $H(u)$ and vice versa. This is indeed the case for $N = 3$. Specifically, one can show that 3×3 matrices $H(u) = T + uV$ that have nontrivial commuting partners also have a level crossing and vice versa [18]. However, this is no longer true for $N \geq 4$ —equation (3) does not necessarily lead to level crossings. Moreover, crossings occur even in the absence of nontrivial partners and u -independent symmetries, see below. We see that a single dynamical symmetry is insufficient to explain level crossings. On the other hand, quantum integrable Hamiltonians typically have more than one coupling dependent commuting operator. In fact, as we show below, the maximum possible number—which turns out to be N —of integrals is necessary to ensure level crossings.

We define the set of maximally commuting Hamiltonians as a vector space, \mathcal{M} , formed by $N \geq 3$ Hermitian, mutually commuting $N \times N$ matrices $H^i(u) = T^i + uV^i$ together with the $N \times N$ identity matrix I ,

$$[H^i(u), H^j(u)] = 0, \quad \text{for all } u \text{ and } i, j = 1, \dots, N, \tag{4}$$

where u is a real parameter. Operators $H^i(u)$ are assumed to be independent in that matrices V^i are linearly independent, i.e. $\sum_{i=1}^N c_i V^i = 0$ iff $c_i = 0$ for all i (equivalently one can require that T^i be linearly independent). In addition, $H^i(u)$ are taken to have no u -independent

symmetry common to all $H^i(u)$ ¹,

$$\nexists \Omega \neq aI \text{ such that } [\Omega, H^i(u)] = 0 \text{ for all } u \text{ and } i. \tag{5}$$

Therefore, an arbitrary element $H(u) = T + uV$ of the vector space \mathcal{M} has the form

$$H(u) = \sum_{i=1}^N d_i H^i(u) + aI, \tag{6}$$

where d_i and a are real numbers. The addition of multiples of the identity affects neither commutation relations nor level crossings of $H(u)$ and we will often omit the term aI in equation (6). Note also that equation (6) implies that operators $H^i(u)$ together with I provide a basis in the vector space \mathcal{M} of maximal Hamiltonians.

The set \mathcal{M} is maximal in the sense that any Hermitian $H(u) = T + uV$ that commutes with all $H^i(u)$ can be written in the form (6). Indeed, since V and all V^i mutually commute, see equation (3), we can go to their common eigenbasis. In this basis, the N diagonal matrices V^i are N linearly independent N -dimensional vectors and, therefore, there exist real numbers d_i such that $V = \sum_{i=1}^N d_i V^i$. The matrix $H(u) - \sum_{i=1}^N d_i H^i(u)$ is u -independent and, since it also commutes with all $H^i(u)$, it must be of the form aI according to equation (5). Thus, $H(u)$ is of the form (6). By a similar argument one can show that one of the basic matrices $H^i(u)$ can be chosen as $H^i(u) = (a + ub)I$ with real coefficients a and b . We see that there are $N - 1$ nontrivial independent commuting operators. Therefore, the first nontrivial dimensionality is $N = 3$.

In what follows we begin with the explicit construction of a general, maximally commuting Hamiltonian $H(u)$. This is done in section 2 by choosing a convenient basis in the vector space \mathcal{M} and solving equation (4). In section 3 we establish some useful algebraic properties of $H(u)$. Interestingly, it turns out that the product of any two maximally commuting Hamiltonians can be written as a linear superposition of such Hamiltonians, i.e. the set \mathcal{M} has a certain quasi-ring structure.

Our parametrization of the maximally commuting Hamiltonians makes it transparent that they are related to the Gaudin magnets [17, 21], see section 4. The latter are N quantum spin Hamiltonians

$$\hat{h}^i(B) = -B\hat{s}_i^z + \sum_{k=1}^N \prime \frac{\hat{s}_i \cdot \hat{s}_k}{\varepsilon_i - \varepsilon_k}, \quad i = 1, \dots, N, \tag{7}$$

where the prime indicates that the summation is over $k \neq i$, B is the z -aligned magnetic field, \hat{s}_i is an operator of spin of magnitude s_i , and $1/(\varepsilon_i - \varepsilon_k)$ is the coupling between spins \hat{s}_i and \hat{s}_k . The Hamiltonians \hat{h}^i form a mutually commuting family

$$[\hat{h}^i(B), \hat{h}^j(B)] = 0 \quad \text{for any } B. \tag{8}$$

Note also that each $\hat{h}^i(B)$ is invariant under rotations around the z -axis, which means the z component of the total spin $\hat{J}^z = \sum_{i=1}^N \hat{s}_i^z$ is conserved

$$[\hat{h}^i, \hat{J}^z] = 0, \quad i = 1, \dots, N. \tag{9}$$

As we will see in section 4, the maximally commuting Hamiltonians (6) correspond to the sector of Gaudin magnets with $J^z = J_{\max}^z - 1$, where $J_{\max}^z = \sum_{i=1}^N s_i$ is the maximum eigenvalue of \hat{J}^z .

In section 5 we employ the mapping to the Gaudin magnets to obtain the exact solution for the spectra of maximally commuting Hamiltonians. Using this solution, we analyze the asymptotic behavior of the eigenstates in the limits $u \rightarrow \pm\infty$ in section 6. Matching the

¹ Any u -independent symmetry common to all $H^i(u)$ can always be removed by going to smaller blocks.

two limits allows us to establish the presence of level crossings and to count them. Next, we consider Hamiltonians having less than the maximum number of commuting partners. In section 7, we construct a family of 4×4 Hermitian operators linear in the coupling u that have a single nontrivial partner as opposed to two partners for the $N = 4$ maximal set. It turns out that these submaximal Hamiltonians often display no degeneracies at all. Finally, in appendix we review Shastry's approach to constructing commuting real symmetric operators and show that the resulting operators are always maximal.

2. The Parametrization of maximally commuting Hamiltonians

We begin our analysis by choosing a convenient basis in the vector space of maximally commuting Hamiltonians, which allows us to solve equation (4) explicitly. The solution yields a convenient parametrization for a general maximal Hamiltonian (6) and elucidates the algebraic structure of these operators. It also makes transparent the relationship between these operators and Gaudin magnets (7).

Consider the maximal operators $H^i(u) = T^i + uV^i$ defined in equations (4) and (5). It follows from equation (4) that all V^i commute with each other, see equations (3) and (2). By a u -independent unitary transformation we go to a basis where all V^i are diagonal. Since V^i are also linearly independent, we can bring them to the following 'canonical' form by a linear transformation:

$$D^k = \sum_i v_i^k V^i, \quad (10)$$

where v_i^k are real numbers, D^k are diagonal with a single nonzero matrix element $[D^k]_{jj} \equiv D_j^k = \delta_{jk}$. Next, we introduce a 'canonical' basis in the space of maximally commuting operators

$$h^i(u) = E^i + uD^i = \sum_{j=1}^N v_j^i H^j(u), \quad i = 1, \dots, N. \quad (11)$$

The operators $h^i(u)$ have all the properties of maximally commuting Hamiltonians defined in equations (4) and (5) as long as $H^i(u)$ do. In particular,

$$[h^i(u), h^j(u)] = 0, \quad \text{for all } u \text{ and } i, j = 1, \dots, N. \quad (12)$$

It follows from equation (6) that a general maximally commuting operator can be written as

$$H(u) \equiv T + uV = \sum_{k=1}^N d_k h^k(u) + aI, \quad (13)$$

where d_k and a are real numbers. Note that with our choice of D^k , d_k are the eigenvalues of V .

To determine $H(u)$ explicitly, we need to solve equation (12). In terms of D^i and E^i these equations read

$$[D^i, D^j] = 0, \quad [D^i, E^j] = [D^j, E^i], \quad [E^i, E^j] = 0. \quad (14)$$

The first equation holds since D^i are diagonal. The second equation in terms of matrix elements is

$$(D_m^i - D_n^i)E_{mn}^j = (D_m^j - D_n^j)E_{mn}^i \quad (15)$$

where E_{mn}^j is the mn th matrix element of E^j . By construction, the only nonzero matrix element of D^i is $D_i^i = 1$. We see that $E_{mn}^j = 0$ as long as $m \neq n$ and m and n do not equal j . Thus, matrix E^j only has nonzero elements of the form $E_{jm}^j = (E_{mj}^j)^*$ and E_{mm}^j , where z^*

denotes the complex conjugate of z . Note also by setting $m = i$ and $n = j$ in equation (15) that $E_{ij}^i = -E_{ij}^j$ for $i \neq j$.

It remains to solve the last equation in (14). Using the above properties of matrix elements of E^i , we rewrite this equation as

$$\begin{aligned} E_{ii}^n - E_{mm}^n &= \frac{E_{nm}^m E_{mn}^m}{E_{nn}^m - E_{ii}^m}, & i \neq m, n, \\ E_{ni}^n &= \frac{E_{nm}^m E_{mi}^m}{E_{nn}^m - E_{ii}^m}, & i \neq m, n, \\ E_{nn}^n - E_{mm}^n &= E_{mm}^m - E_{nn}^m + \sum_{j \neq m, n} \frac{E_{mj}^m E_{jm}^m}{E_{nn}^m - E_{jj}^m}. \end{aligned} \tag{16}$$

By direct computation, one can show that the following ansatz satisfies equations (16):

$$\begin{aligned} E_{mj}^m &= e^{i(\theta_m - \theta_j)} \frac{\gamma_m \gamma_j}{\varepsilon_m - \varepsilon_j}, \\ E_{jj}^m &= -\frac{\gamma_j^2}{\varepsilon_m - \varepsilon_j} + \psi^m, & i \neq m, \\ E_{mm}^m &= -\sum_{i \neq m} \frac{\gamma_i^2}{\varepsilon_m - \varepsilon_i} + \psi^m \end{aligned} \tag{17}$$

where $\gamma_j \neq 0$, ε_j , θ_m and ψ^m are real parameters. A nonzero value of ψ^m corresponds to an overall shift of the diagonal of E^m , which yields a (nonessential) contribution $\psi^m I$ to $h^m(u)$. Note that $\gamma_j = 0$ also satisfies equations (16) but generates matrices with block diagonal structure and, therefore, u -independent symmetry.

Furthermore, any solution of equations (16) admits parametrization (17). To establish this, it is sufficient to show that any choice of the $3N - 2$ matrix elements $E_{mn}^m = (E_{nm}^m)^*$ and E_{nn}^m for a certain m compatible with equations (16) corresponds to a set of $3N + 1$ real parameters, θ_j , γ_j , ε_j and ψ^m . Then, equations (16) ensure that all E^j are of the form (17). The extra three parameters are an overall scale for ε_j and γ_j , a shift $\varepsilon_i \rightarrow \varepsilon_i + \text{const}$, and a shift $\theta_j \rightarrow \theta_j + \text{const}$, which do not affect equations (17). To see the correspondence between $E_{mn}^m = (E_{nm}^m)^*$ and E_{nn}^m and θ_j , γ_j , ε_j , and ψ^m , note that equations (17) yield

$$\sum_{n \neq m} \frac{E_{mn}^m E_{nm}^m}{E_{nn}^m - \psi^m} = E_{mm}^m - \psi^m,$$

which, for given $E_{mn}^m = (E_{nm}^m)^*$ and E_{nn}^m , can be solved for ψ^m . In seeking a common denominator, we see that it constitutes an N th order polynomial

$$\prod_j (E_{jj}^m - \psi^m) - \sum_{n \neq m} \prod_{j \neq m, n} E_{mn}^m E_{nm}^m (E_{jj}^m - \psi^m) = 0, \tag{18}$$

yielding N solutions ψ^m . By considering the form of matrix E^m , i.e. that all matrix elements are zero save for a row, the corresponding column, and the diagonal, we find that the left-hand side of equation (18) is the characteristic polynomial given by $\det(E^m - \psi^m I)$. Consequently, all N solutions of equation (18) are guaranteed to be real as they are the eigenvalues of an explicitly Hermitian matrix. Once ψ^m is determined, we can calculate ratios

$$\gamma_j = -e^{i(\theta_j - \theta_m)} \frac{E_{mj}^m}{E_{jj}^m - \psi^m} \quad j \neq m,$$

where $e^{2i(\theta_j - \theta_m)} \equiv E_{jm}^m / E_{mj}^m$, θ_m is arbitrary and by a choice of an overall scale we set $\gamma_m = 1$. Lastly, letting $\varepsilon_m = 0$ (by shifting ε_i), we have

$$\varepsilon_j = \frac{1}{E_{jj}^m - \psi^m}.$$

Each of the N solutions ψ^m to equation (18) will yield a distinct set $\{\gamma_j, \varepsilon_j\}$, but by construction (see equation (18)) corresponds to the same set $\{E_{mn}^m, E_{nm}^m, E_{jj}^m\}$.

Now, consider E^m as defined in equation (17). The matrix E^m with complex matrix elements $E_{mj}^m = (E_{jm}^m)^*$, E_{jj}^m , and E_{mm}^m is conjugate to a matrix \tilde{E}^m , i.e.

$$E^m = \Sigma \tilde{E}^m \Sigma^{-1},$$

where Σ is a diagonal matrix with entries $\Sigma_{jj} \equiv e^{i\theta_j} = e^{i\theta_m} \sqrt{\frac{E_{jm}^m}{E_{mj}^m}}$, $(\tilde{E}_{mj}^m)^2 = E_{mj}^m E_{jm}^m$, $\tilde{E}_{jj}^m = E_{jj}^m$, and θ_m is an arbitrary real number. Given a Hermitian E^m , we find that \tilde{E}^m is necessarily real symmetric matrix and Σ is a conjugating diagonal matrix whose matrix elements are complex phases. Thus, all Hermitian maximally commuting operators are matrix conjugate to some real symmetric such. Operator spectra are invariant under matrix conjugation and, therefore, it is convenient to henceforth limit our discussion to maximally commuting real symmetric matrices, and we do so without loss of generality.

Moreover, as noted below equation (17), nonzero ψ^i contributes only a multiple of the identity, $\psi^i I$ to each $h^i(u)$, which affects neither the commutation relations nor the level crossings of $h^i(u)$ and their linear combinations. Henceforth, we adopt a convenient ‘gauge choice’ $\psi^i = 0$ for all $i = 1, \dots, N$ and $\theta_m = 0$ for all $m = 1, \dots, N$. With this choice of ψ^i and θ_m , we derive from equations (11),(17) and the definition of D^i the nonzero matrix elements of basic operators $h^i(u)$

$$\begin{aligned} [h^i(u)]_{ij} &= \frac{\gamma_i \gamma_j}{\varepsilon_i - \varepsilon_j}, & i \neq j, \\ [h^i(u)]_{jj} &= -\frac{\gamma_i^2}{\varepsilon_i - \varepsilon_j}, & i \neq j, \\ [h^i(u)]_{ii} &= u - \sum_{j \neq i} \frac{\gamma_j^2}{\varepsilon_i - \varepsilon_j}. \end{aligned} \tag{19}$$

Note that

$$\sum_{i=1}^N h^i(u) = uI. \tag{20}$$

Expressions (19) for matrix elements constitute a complete, explicit solution of commutation relations (12) or equivalently (4) for maximally commuting Hamiltonians. Different choices of parameters γ_j and ε_j (factoring out overall scale of γ_j and ε_j , a total shift of all ε_j , and the ‘gauge freedom’ discussed above) yield distinct families of such Hamiltonians. Equation (19) also determines matrix elements² of a general maximally commuting operator (6)

$$\begin{aligned} [H(u)]_{mn} &= \gamma_m \gamma_n \left(\frac{d_m - d_n}{\varepsilon_m - \varepsilon_n} \right), & m \neq n, \\ [H(u)]_{mm} &= u d_m - \sum_{j \neq m} \gamma_j^2 \left(\frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} \right). \end{aligned} \tag{21}$$

² We note that the restriction to Hermitian operators in the definition of the maximal set is not essential for this construction. Specifically, following the above arguments, one can show that an arbitrary maximally commuting and not necessarily Hermitian operator can be written in the form $\Sigma H(u) \Sigma^{-1}$, where Σ is diagonal and $H(u)$ is a symmetric matrix parametrized as in equation (19) but with arbitrary complex parameters $\Sigma_{jj}, d_j, \gamma_j$, and ε_j .

Let us also note that a convenient approach to producing nontrivial solutions of equation (2) was developed by Shastry in [19]. Interestingly, these solutions turn out to be essentially equivalent to the maximally commuting set constructed in this section, see appendix for details.

3. Algebraic properties of maximal Hamiltonians and an upper bound on the number of level crossings

The above parametrization makes transparent a beautiful property of maximal Hamiltonians—the product of two maximal operators is itself the u -dependent sum of maximal operators. This property, as we demonstrate in this section, allows one to express a general maximal Hamiltonian $\tilde{H}(u)$ as a polynomial in another such Hamiltonian $H(u)$. We employ this polynomial expansion to determine the maximum number of level crossings in the eigenvalue spectrum of $H(u)$.

First, we express the product of two basic maximally commuting operators $h^i(u)$ and $h^j(u)$ in terms of u -dependent linear combinations of $h^k(u)$. Using equation (19), one can show that

$$\begin{aligned} h^i(u) \cdot h^j(u) &= \frac{\gamma_j^2}{\varepsilon_i - \varepsilon_j} h^i(u) + \frac{\gamma_i^2}{\varepsilon_j - \varepsilon_i} h^j(u), \quad i \neq j, \\ h^i(u) \cdot h^i(u) &= \sum_{k=1}^N \left[u - \sum_{m \neq k} \frac{\gamma_m^2}{\varepsilon_k - \varepsilon_m} \right] h^k(u). \end{aligned} \tag{22}$$

Now consider two general maximally commuting Hamiltonians (6)

$$H(u) = \sum_{k=1}^N d_k h^k(u), \quad \tilde{H}(u) = \sum_{k=1}^N \tilde{d}_k h^k(u), \tag{23}$$

where without loss of generality we dropped multiples of identity in equation (6). From equation (22) we derive

$$H(u) \cdot \tilde{H}(u) = \sum_{k=1}^N \left[u d_k \tilde{d}_k - \sum_{m \neq k} \frac{\gamma_m^2 (d_k - d_m) (\tilde{d}_k - \tilde{d}_m)}{\varepsilon_k - \varepsilon_m} \right] h^k(u). \tag{24}$$

This quasi-ring structure—so called because, while the sum of maximal Hamiltonians is maximal, the product is a u -dependent sum of such and, therefore, not generally linear in u and not strictly a maximal operator—suggests a means of representing an element of a commuting maximal family by any other, see equation (25) below.

A typical maximal Hamiltonian $H(u)$ can be degenerate only at discrete values of u . Note that the only alternative to the discrete (possibly empty) set is a permanent degeneracy—when two eigenvalues of $H(u)$ coincide at all u [29]. Permanent degeneracies do not occur for a generic choice of d_k in equation (23). Indeed, recall that d_k are the eigenvalues of V (see below equation (6)). Since the eigenvalues of $H(u) = T + uV$ tend to those of uV for large u , the spectrum of $H(u)$ is not degenerate as long as d_k are distinct and V is itself nondegenerate.

Consider $H(u)$ at any u where it is nondegenerate. Any element of its commutant—the set of all real symmetric operators that commute with $H(u)$ —can be expressed as a polynomial in $H(u)$ of the order $N - 1$, i.e.

$$\tilde{H}(u) = \sum_{\alpha=0}^{N-1} P_\alpha(u) H^\alpha(u), \tag{25}$$

where $H^\alpha(u) \equiv [H(u)]^\alpha$ and, as we will see shortly, $P_\alpha(u)$ are rational functions of u . To see that $\tilde{H}(u)$ can be indeed written in terms of powers of $H(u)$, consider equation (25) in the common eigenbasis of commuting operators $H(u)$ and $\tilde{H}(u)$ at a given u . Since eigenvalues ω_m of $H(u)$ are N distinct real numbers, one can always find a polynomial $R_{N-1}(\omega) = \sum_{\alpha=0}^{N-1} P_\alpha \omega^\alpha$ of order $N - 1$ with N real coefficients P_α so that $R_{N-1}(\omega_m) = \tilde{\omega}_m$, where $\tilde{\omega}_m$ are the eigenvalues of $\tilde{H}(u)$. Indeed, the equations $R_{N-1}(\omega_m) = \tilde{\omega}_m$ are linear in P_α with a nonzero determinant.

Next, we observe from equations (23) and (24) that

$$H^\alpha(u) = \sum_{k=1}^N Q_k^\alpha(u) h^k(u), \tag{26}$$

where for $\alpha \geq 1$ $Q_k^\alpha(u)$ is an $\alpha - 1$ order polynomial in u determined by recursively applying equation (24) and $Q_k^0 = 1/u$ as follows from equation (20). Plugging equation (26) into equation (25) and using the second equation in (23), we obtain

$$\sum_{k=1}^N \sum_{\alpha=0}^{N-1} P_\alpha(u) Q_k^\alpha(u) h^k(u) = \sum_{k=1}^N \tilde{d}_k h^k(u). \tag{27}$$

Since $h^k(u)$ are linearly independent at any $u \neq 0$, i.e. $\sum_{k=1}^N f_k(u) h^k(u) = 0$ if and only if $f_k(u) \equiv 0$ for all $u \neq 0$,³ equation (27) becomes

$$\sum_{\alpha=0}^{N-1} P_\alpha(u) Q_k^\alpha(u) = \tilde{d}_k, \quad k = 1, \dots, N. \tag{28}$$

Note that because $Q_k^\alpha(u)$ are rational functions in u , $P_\alpha(u)$ are also rational functions.

Because $\tilde{H}(u)$ is arbitrary, equation (28) should have solutions for $P_\alpha(u)$ for any \tilde{d}_k as long as $H(u)$ is nondegenerate. On the other hand, solutions cease to exist if and only if $\det [Q_k^\alpha(u)] = 0$, where $Q_k^\alpha(u)$ is regarded as the αk th matrix element of an $N \times N$ matrix. Using the fact that $Q_k^\alpha(u)$ is a polynomial in u of degree $\alpha - 1$ for $\alpha \geq 1$ and $Q_k^0 = 1/u$, one can show that $\det [Q_k^\alpha(u)] = \mathcal{P}(u)/u$, where $\mathcal{P}(u)$ is a polynomial in u of order $\sum_{m=0}^{N-2} m$. The real roots of the equation $\mathcal{P}(u_\gamma) = 0$ are the values of $u = \{u_\gamma\}$ where $H(u)$ is degenerate. Thus, the maximum possible number of level crossings in the eigenvalue spectrum of an $N \times N$ maximally commuting Hamiltonian is

$$M_c^{\max} = \frac{(N - 1)(N - 2)}{2}. \tag{29}$$

The polynomial $\mathcal{P}(u)$ is of real coefficients and, therefore, its complex roots come in conjugate pairs. Consequently, the number of real roots of $\mathcal{P}(u)$ falls from the maximum M_c^{\max} in decrements of two. This enforces a parity such that the number of real roots is odd for integers of the form $4m, 4m + 1$ and even for integers $4m + 2, 4m + 3, m \in \mathbb{N}$. Ostensibly, when real roots of $\mathcal{P}(u)$ are degenerate their number need not correspond to the number of distinct crossings. In principle, a multiple real root of $\mathcal{P}(u)$ could correspond to a single pairwise crossing. Numerically, however, we have observed that such multiplicities occur only when more than two levels cross simultaneously, i.e. at the same value of u .

³ This can be seen by evaluating mn^{th} matrix elements of $\sum_{k=1}^N f_k(u) h^k(u) = 0$ for $m \neq n$ and using equations (19) and (20).

4. Mapping to the Gaudin magnets

In this section, we show that maximally commuting Hamiltonians $h^i(u)$ are equivalent to the Gaudin magnets,

$$\hat{h}^i(B) = -B\hat{s}_i^z + \sum_{k=1}^N \frac{\hat{s}_i \cdot \hat{s}_k}{\varepsilon_i - \varepsilon_k}, \quad i = 1, \dots, N, \quad (30)$$

in the next to highest weight sector, $J^z = J_{\max}^z - 1$, where J^z is the z projection of the total spin, $\hat{s}_i = \{\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z\}$ is the i th spin 3-vector of magnitude s_i and $[\hat{s}_i^\alpha, \hat{s}_j^\beta] = \epsilon_{\alpha\beta\gamma} \hat{s}_i^\gamma \delta_{ij}$. This mapping is very useful as Gaudin magnets (central spin Hamiltonians) have been extensively studied [17, 21, 30–34]. For example, an exact solution for the eigenstates and eigenvalues is available [17, 21]. We employ it in subsequent sections to obtain the spectra of maximally commuting Hamiltonians and to analyze their level crossings. This mapping also implies that all our conclusions regarding maximal Hamiltonians, e.g. the presence and the number of level crossings, quasi-ring structure (22) etc. can be immediately transferred to the corresponding sector of Gaudin magnets and their derivative models, such as the reduced BCS model [35–37]. At the same time, other sectors of the Gaudin model as well as more general models [33] of which it is a particular case can provide examples of Hamiltonians with less than the maximum number of commuting partners.

Since Gaudin magnets (30) commute with the z projection of the total spin \hat{J}^z , see equation (9), they are block-diagonal in any basis where \hat{J}^z has a definite value. Different blocks can be labeled by the eigenvalues of \hat{J}^z . Consider the sector $J^z = J_{\max}^z - 1$, where $J_{\max}^z = \sum_{i=1}^N s_i$ is the maximum eigenvalue of \hat{J}^z . It is populated by N basic states

$$|k\rangle = \frac{\hat{s}_k^- |0\rangle}{\sqrt{2s_k}}, \quad k = 1, \dots, N, \quad (31)$$

where $|0\rangle$ is the highest weight state $J^z = J_{\max}^z$, i.e. $\hat{s}_k^+ |0\rangle = 0$ for all k , and the highest weight s_k for each spin \hat{s}_k is given by $\hat{s}_k^z |0\rangle = s_k |0\rangle$. Therefore, Gaudin Hamiltonians (30) are N commuting real symmetric $N \times N$ matrices in this sector. Since there is also no obvious B -independent symmetry ($\hat{J}^z \propto I$ within a given sector), the $\hat{h}^i(B)$ appear to be good candidates for a maximally commuting set.

To check this, let us evaluate the nonvanishing matrix elements of $\hat{h}^i(B)$ given by equation (30) in the normalized basis (31). We obtain

$$\begin{aligned} \langle i | \hat{h}^i(B) | j \rangle &= \frac{\sqrt{s_i s_j}}{\varepsilon_i - \varepsilon_j}, \quad j \neq i, \\ \langle j | \hat{h}^i(B) | j \rangle &= -\frac{s_i}{\varepsilon_i - \varepsilon_j} + \left[-Bs_i + \sum_{k \neq i} \frac{s_i s_k}{\varepsilon_i - \varepsilon_k} \right], \quad j \neq i, \\ \langle i | \hat{h}^i(B) | i \rangle &= B - \sum_{k \neq i} \frac{s_k}{\varepsilon_i - \varepsilon_k} + \left[-Bs_i + \sum_{k \neq i} \frac{s_i s_k}{\varepsilon_i - \varepsilon_k} \right]. \end{aligned} \quad (32)$$

Comparing these expressions to matrix elements of $h^i(u)$ in equation (19), we observe that with the identifications $B = u$ and $s_k = \gamma_k^2$ the two matrices differ only by a multiple of an identity matrix $\psi^i I$, where

$$\psi^i = -Bs_i + \sum_{k \neq i} \frac{s_i s_k}{\varepsilon_i - \varepsilon_k}. \quad (33)$$

Recall that we arbitrarily selected a ‘gauge’ $\psi^i = 0$ for maximally commuting Hamiltonians $h^i(u)$, see equation (17) and the text above equation (19). This constant overall shift of all eigenvalues of $h^i(u)$ affects neither its eigenstates nor the degeneracies.

Thus, we see that Gaudin Hamiltonians (30) in the next to highest weight sector $J^z = J_{\max}^z - 1$ are equivalent to basic maximal Hamiltonians $h^i(u)$ with

$$u = B, \quad \gamma_k^2 = s_k, \quad (34)$$

and vice versa. Note that the magnitudes of quantum spins, s_k , take half-integer values for finite-dimensional representations of the spin $su(2)$ algebras, while γ_k are arbitrary real numbers. We believe that this restriction can be lifted by moving to an appropriate infinite-dimensional representations of the $su(2)$ s, where the highest weight states are still well defined but s_k take arbitrary real values [38]. Indeed, we have verified that, at least in our sector $J^z = J_{\max}^z - 1$, in all expressions for the eigenvalues and eigenstates of $\hat{h}^i(B)$ (see below) the replacements $B \rightarrow u$ and $s_k \rightarrow \gamma_k^2$ with arbitrary real γ_k produce the correct corresponding eigenvalues and eigenstates of $h^i(u)$.

5. Exact solution for the spectra of maximal Hamiltonians

A particularly useful consequence of the mapping (34) between Gaudin magnets $\hat{h}^i(B)$ and maximally commuting Hamiltonians $h^i(u)$ is that one can obtain the exact solution for $h^i(u)$ by importing the known exact solution for the spectra of $\hat{h}^i(B)$ [17, 21]. The latter has been derived both from the properties of the Gaudin algebra [17] and by Bethe’s ansatz [33].

The exact eigenvalues of the Gaudin Hamiltonian (30), $\hat{h}^i(B)$, in the next to highest weight sector $J^z = J_{\max}^z - 1$ are

$$(\lambda_m^i)_G = \frac{s_i}{x_m^G - \varepsilon_i} + \psi^i, \quad (35)$$

where ψ^i is the overall shift of all eigenvalues given by equation (33) and x_m^G are the solutions of the following equation:

$$B = \sum_{k=1}^N \frac{s_j}{x_m^G - \varepsilon_k}. \quad (36)$$

Note that if this equation is brought to the common denominator, the numerator becomes a polynomial of order N in x_m^G . Therefore, there are N solutions for x_m^G and N eigenvalues (35) as it should be since there are N states in this sector, see equation (31). The unnormalized eigenstates (common to all $\hat{h}^i(B)$) corresponding to eigenvalues (35) are

$$|\lambda_m\rangle_G = \sum_{k=1}^N \frac{\sqrt{s_k}|k\rangle}{x_m^G - \varepsilon_k}, \quad (37)$$

where the basic states $|k\rangle$ have been introduced in equation (31). A concise derivation of equations (35), (40) and (37) can be found in [17, 21, 34].

Using the mapping (34) between basic maximal operators $h^i(u)$ and Gaudin Hamiltonians, we obtain from equation (35) the energies of $h^i(u)$

$$\lambda_m^i = \frac{\gamma_i^2}{x_m - \varepsilon_i}. \quad (38)$$

Note that we set the overall shift $\psi^i = 0$ in accordance to the discussion surrounding equation (33). The corresponding common eigenstates of all $h^k(u)$ are

$$|\lambda_m\rangle = \sum_{k=1}^N \frac{\gamma_k|k\rangle}{x_m - \varepsilon_k}, \quad (39)$$

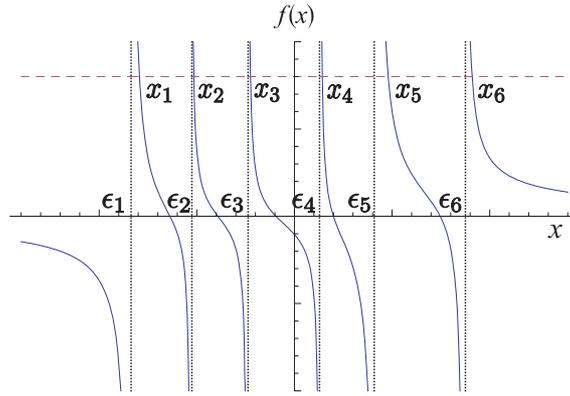


Figure 3. A plot of $f(x) = \sum_{k=1}^N \gamma_k^2 (x - \epsilon_k)^{-1}$ for $N = 6$. Solutions of the equation $f(x_m) = u$ determine the energies (41) and eigenstates (39) of a general maximal Hamiltonian $H(u)$. There are N points, x_m with $m = 1, \dots, N$, where $y = f(x)$ intersects $y = u$ (dashed horizontal line) yielding $N = 6$ eigenstates. Note that $\epsilon_m < x_m < \epsilon_{m+1}$ except in the case of x_N for which we have $\epsilon_N < x_N$ for $u > 0$ and $x_N < \epsilon_{N+1} \equiv \epsilon_1$ for $u < 0$. Furthermore, we see that $x_m \rightarrow \epsilon_m$ as $u \rightarrow \infty$ and $x_m \rightarrow \epsilon_{m+1}$ as $u \rightarrow -\infty$. This observation allows us to determine the behavior of the energies in the $u \rightarrow \pm\infty$ limits, see equation (44).

where $|k\rangle$ now stands for a basic vector for matrices $h^i(u)$, i.e. its j th component is $|k\rangle_j = \delta_{jk}$. In equations (38) and (37) x_m are solutions of the following equation:

$$u = \sum_{k=1}^N \frac{\gamma_k^2}{x_m - \epsilon_k} \equiv f(x_m), \quad m = 1, \dots, N, \tag{40}$$

which follows from equation (36). That equations (38),(39) and (40) yield the correct spectrum of $h^i(u)$ can be verified directly using the matrix form (19) of $h^i(u)$. Finally, using equation (23), we derive the energies of a general maximally commuting Hamiltonian, $H(u) = T + uV$,

$$\omega_m = \sum_{k=1}^N \frac{d_k \gamma_k^2}{x_m - \epsilon_k}. \tag{41}$$

The corresponding eigenstates are still given by equation (39).

Let us analyze the flow of eigenvalues ω_m of $H(u)$ with u and determine their behavior in the $u \rightarrow \pm\infty$ limits. Consider equation (40). Let ϵ_k be ordered as $\epsilon_1 < \epsilon_2 < \dots < \epsilon_N$. The right-hand side of equation (40) is plotted in figure 3. Note that $f(x) \rightarrow +\infty$ as $x \rightarrow \epsilon_k^+$ and $f(x) \rightarrow -\infty$ as $x \rightarrow \epsilon_{k+1}^-$. It follows that the equation $u = f(x_m)$ has a real root between ϵ_k and ϵ_{k+1} for any k . Let us number the roots x_m so that $\epsilon_m < x_m < \epsilon_{m+1}$. Note from figure 3 that for the last root x_N we have $\epsilon_N < x_N$ for $u > 0$ and $x_N < \epsilon_{N+1} \equiv \epsilon_1$ for $u < 0$, where from now on we identify indices m and $m + N$ that differ by a multiple of N . Further, observe from figure 3 that $x_m \rightarrow \epsilon_m$ as $u \rightarrow +\infty$. In this limit the $k = m$ term dominates equations (40) and (41) and we obtain $\gamma_m^2 / (x_m - \epsilon_m) \approx u$ and $\omega_m \approx d_m \gamma_m^2 / (x_m - \epsilon_m) \approx u d_m$. Similarly, for $u \rightarrow -\infty$ we have $x_m \rightarrow \epsilon_{m+1}$ and $\omega_m \approx u d_{m+1}$. Therefore,

$$\omega_m \rightarrow -|u|d_{m+1} \quad \text{as } u \rightarrow -\infty, \quad \omega_m \rightarrow |u|d_m \quad \text{as } u \rightarrow +\infty. \tag{42}$$

At this point it is convenient to rescale the Hamiltonian

$$H'(u) = \frac{H(u)}{\sqrt{u^2 + 1}} = \frac{T + uV}{\sqrt{u^2 + 1}}. \tag{43}$$

Note that this does not affect the level crossings, i.e. $H'(u)$ and $H(u)$ have crossings (if any) at the same values of u . Equation (42) implies

$$\omega'_m \rightarrow -d_{m+1} \quad \text{as } u \rightarrow -\infty, \quad \omega'_m \rightarrow d_m \quad \text{as } u \rightarrow +\infty, \quad (44)$$

where ω'_m is the eigenvalue of $H'(u)$ corresponding to the eigenstate $|\lambda_m\rangle$. Recall that d_k are the eigenvalues of V , see the text below equation (6). We see from equation (43) that the eigenvalues of $H'(u)$ indeed should tend to d_k in $u \rightarrow \pm\infty$ limits consistent with equation (44). The latter equation however provides much more detailed information—it shows to which particular d_k the eigenvalue corresponding to a given eigenvector tends in each limit. We will use equation (44) in the following section to study the crossings of energy levels of a general maximally commuting Hamiltonian $H(u)$.

6. Level crossings

In this section, we establish the presence of energy level crossings in the spectrum of an arbitrary maximally commuting Hamiltonian $H(u) = T + uV$ (6). This provides an explanation of the level crossing phenomenon in the absence of any u -independent symmetry based solely on the fact that $H(u)$ has the maximum possible number of independent commuting partners, see the text above equation (4). Further, we determine the number of level crossings as it depends on the ordering of the eigenvalues d_k of the perturbation operator V and argue that this number takes values

$$M_c = \frac{(N-1)(N-2)}{2} - 2K, \quad K = 0, 1, \dots, K_{\max}, \quad (45)$$

where N is the dimensionality of the state space of $H(u)$ and K_{\max} is the integer part (floor) of $(N-1)(N-2)/4 - 1/2$. For example, $N = 3$ maximally commuting operators have a single level crossing, while for $N = 6$ we have $M_c = 2, 4, 6, 8$ and 10 . We also develop an approach that allows us to readily predict the minimum allowed number of crossings for a given $H(u)$ from the ordering of d_k .

Consider a Hamiltonian (not necessarily belonging to any commuting family) that depends on a real parameter u . Suppose $|n_i\rangle$ are its eigenstates and $E_{n_1}^- < E_{n_2}^- < \dots$ are the corresponding energies at large negative u . There is only one way to avoid crossings—the order of eigenvalues $E_{n_i}^+$ at $u \rightarrow \infty$ must be exactly the same as that at $u \rightarrow -\infty$, i.e. $E_{n_1}^+ < E_{n_2}^+ < \dots$. This is what happens with a typical Hamiltonian in agreement with the Wigner–von Neumann non-crossing rule, figure 2. If, on the other hand, the relative order of any two energies changes, at least one level crossing must occur. For example, $E_{n_1}^- < E_{n_4}^-$ and $E_{n_1}^+ > E_{n_4}^+$ means that the difference $E_{n_1}(u) - E_{n_4}(u)$ changes sign as u evolves from $-\infty$ to ∞ . By continuity this implies a crossing of levels corresponding to eigenstates $|n_1\rangle$ and $|n_4\rangle$ at a certain value of u . This is observed in blocks of quantum integrable Hamiltonians characterized by the same u -independent symmetry, see e.g. figure 1. Numerical spectra of maximal Hamiltonians display the same behavior, figures 5 and 8(a).

Now let us turn our attention to an arbitrary maximally commuting Hamiltonian $H(u)$. In the previous section we have established the behavior of its energies in $u \rightarrow \pm\infty$ limits. It follows from equation (44) that the energy level of $H(u) = T + uV$ (with appropriate rescaling (43)) that starts from $-d_k$ at $u \rightarrow -\infty$ ends at d_{k-1} at $u \rightarrow \infty$. Symbolically, this can be represented by

$$k \mapsto k - 1, \quad (\text{mod } N). \quad (46)$$

Note that we cannot fix an ordering of d_k without loss of generality, as d_k correspond to ε_k , see e.g. equation (41), and we have already fixed the order of ε_k so that $\varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_N$.

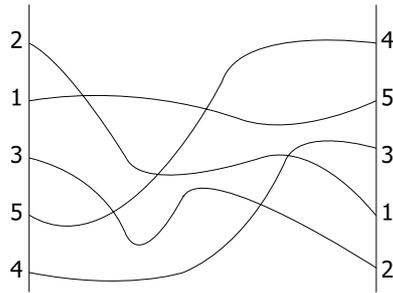


Figure 4. Schematic energy level diagram for general 5×5 maximal Hamiltonians $H(u) = T + uV$ drawn using the rules (47). The eigenvalues d_k of V are ordered such that $d_4 > d_5 > d_3 > d_1 > d_2$. The diagram shows six level crossings for this ordering and specifies which levels cross, e.g. the level $4 \rightarrow 3$ first crosses $3 \rightarrow 2$ and then $2 \rightarrow 1$. Compare to figure 5 and note that the crossings predicted by the above diagram are exactly the same as actual numerical crossings for this ordering. Note also that according to equation (45) six is the maximum allowed number of crossings for this ordering of d_k and multiple crossings of the same two levels are therefore forbidden for this ordering.

First, we assume that all d_k are distinct as is generally the case. Equation (46) implies that the flow of energy levels from $u = -\infty$ to $u = \infty$ can be schematically depicted using the following rules:

- (1) Create two columns in which $\{-d_k\}$ and $\{d_k\}$ are both in descending order and replace each d_k with its lower index k , i.e.

$$\begin{array}{c}
 -d_i \\
 -d_j \\
 \vdots \\
 -d_l \\
 -d_m
 \end{array}
 \left| \begin{array}{cc}
 d_m & i \\
 d_l & j \\
 \vdots & \vdots \\
 d_j & l \\
 d_i & m
 \end{array} \right.
 \Rightarrow
 \begin{array}{c}
 i \\
 j \\
 \vdots \\
 l \\
 m
 \end{array}
 \tag{47}$$

- (2) Draw a line connecting j in the left column to $j - 1$ in the right. These lines represent energy levels of $H(u)$. Consequently, their crossings imply crossings of the corresponding energy levels of $H(u)$.

An example of an energy level diagram generated using the above prescription for $N = 5$ is shown in figure 4. It corresponds to the ordering $d_4 > d_5 > d_3 > d_1 > d_2$ and predicts six level crossings. It also specifies which levels cross, e.g. the top level connecting 2 and 1 crosses with the next in energy level connecting 1 and 5. We see that the crossing predicted by figure 4 are exactly the same as those of actual levels of a maximally commuting operator with that ordering shown in figure 5. The latter has been obtained by numerical diagonalization of a 5×5 maximally commuting operator (19), $H(u)$, with randomly chosen γ_k, ε_k and random d_k obeying the above ordering. More examples of level diagrams are shown in figures 6, 7, and 9 and discussed in detail below. Next, we explore further consequences of equation (46).

Inevitability of level crossings for maximally commuting operators. In section 3 we have seen that the maximum allowed number of level crossings is $(N - 1)(N - 2)/2$, see equation (29). Now let us show that at least one crossing must be present in the spectrum of *any* maximally commuting $H(u) = T + uV$. Suppose the eigenvalues of V are ordered as $d_i < d_j < \dots < d_l < d_m$ as shown in the diagram (47) and assume there are no crossings.

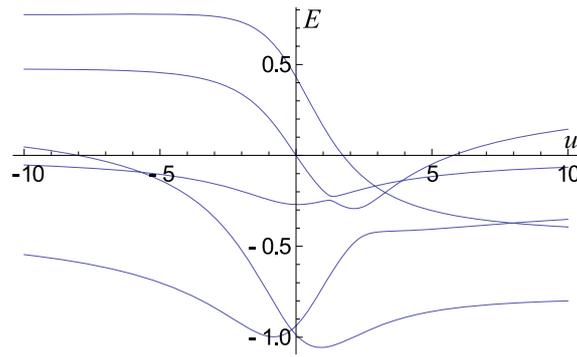


Figure 5. Numerical energy levels of a 5×5 maximal Hamiltonian $H(u) = T + uV$ with the same ordering of eigenvalues of V as that in figure 4. Energies are scaled with a factor $(u^2 + 1)^{-1/2}$ to highlight their asymptotic approach to eigenvalues of V . Matrix elements of $H(u)$ are generated using equation (19) with random γ_k, ε_k and random d_k constrained to obey the ordering of figure 4. Note that the number of crossings as well as the levels that cross are exactly the same as those predicted by figure 4.

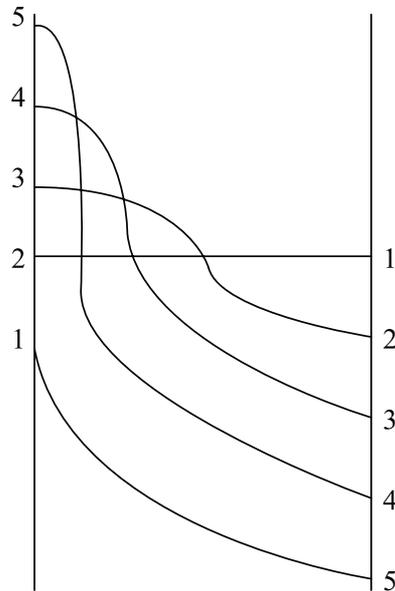


Figure 6. A schematic diagram corresponding to maximum level crossings with $N = 5$. As is evident, $N \rightarrow N - 1$ line has $N - 2$ crossings, $N - 1 \rightarrow N - 2$ adds $N - 3$ new crossings and so on, till the line $3 \rightarrow 2$ adds only 1 new crossing. Thus, the maximum number of crossings, $1 + 2 + 3 + \dots + (N - 2) = (N - 1)(N - 2)/2$, can be confirmed.

Then, the top level must go from i on the left to m on the right, i.e. $i \rightarrow m$, the next level starting at j on the left must be connected to l on the right, $j \rightarrow l$ etc. Finally, we must have $l \rightarrow j$ and $m \rightarrow i$. Consider in particular levels $i \rightarrow m$ and $m \rightarrow i$. According to equation (46), this asymptotic behavior implies $m = i - 1 \pmod{N}$ and $i = m - 1 \pmod{N}$. We obtain $0 = 2 \pmod{N}$, which does not hold for any $N \geq 3$, i.e. the above assumption that levels do not cross cannot be true. Thus, we have demonstrated that at least one level crossing is always present.

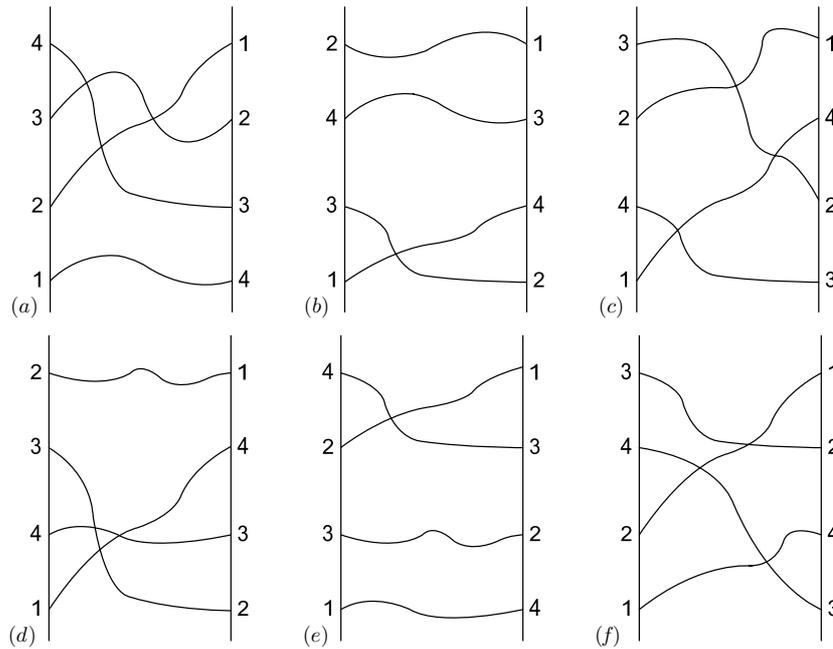


Figure 7. All distinct level diagrams of $N = 4$ maximal Hamiltonians, $H(u) = T + uV$, drawn according to (47). There are $(N - 1)! = 6$ distinct diagrams each corresponding to $N = 4$ different orderings of eigenvalues of V (see the text). For example, diagram (b) corresponds to $d_1 > d_3 > d_4 > d_2$ and three other orderings obtained with a shift of the indices by an integer mod N , e.g. $d_2 > d_4 > d_1 > d_3$ etc. The diagrams predict either one or three level crossings in agreement with equation (45) and specify which levels cross. However, when the number of crossings is less than the maximum (three), additional multiple crossings of the same two levels can occur. This can increase the number of crossings by $2K$, see figure 8. In the present case, the number of crossings for orderings (b) and (e) can increase from one to three.

The maximum number of crossings $M_c^{\max} = (N - 1)(N - 2)/2$ is realized e.g. for the ordering $d_1 > d_2 > \dots > d_N$, see figure 6. In this case equation (46) implies $N \rightarrow N - 1$ yielding $m = N - 2$ crossings, $N - 1 \rightarrow N - 2$ giving rise to another $m = N - 3$ crossings etc., so that altogether we have $\sum_{m=1}^{N-2} m = M_c^{\max}$ level crossings. Note however that the schematic level diagrams, such as those shown in figures 4, 6, 7 and 9, do not account for the possibility of two levels crossing more than once. For example, the level $2 \rightarrow 1$ in figure 7(b) can go below the level $1 \rightarrow 4$ and come back above it again generating two additional crossings, see figure 8. Therefore, multiple crossings of two levels can increase the total number of crossings M_c for a given ordering of d_k by an even number except when $M_c = M_c^{\max}$. In the latter case, since M_c cannot exceed M_c^{\max} , multiple crossings of the same two levels are prohibited. We see that multiple crossings do not modify equation (45). Interestingly, numerically we have found that for as low as $N = 8$, multiple crossings of the same two levels are very common.

Thus far, we have established that the total number of crossings M_c in the spectrum of an arbitrary maximally commuting Hamiltonian is $1 \leq M_c \leq M_c^{\max}$. By inspecting all level diagrams for $3 \leq N \leq 6$, we have also found that for a given N the total number of level crossings changes in increments of 2 from M_c^{\max} to 1 (2) for odd (even) M_c^{\max} , i.e. we verified equation (45) for these N . Moreover, this equation is also supported by the parity considerations in the end of section 3 and is consistent with all preceding observations regarding the properties of level diagrams. As such, we adopt it without a formal proof.

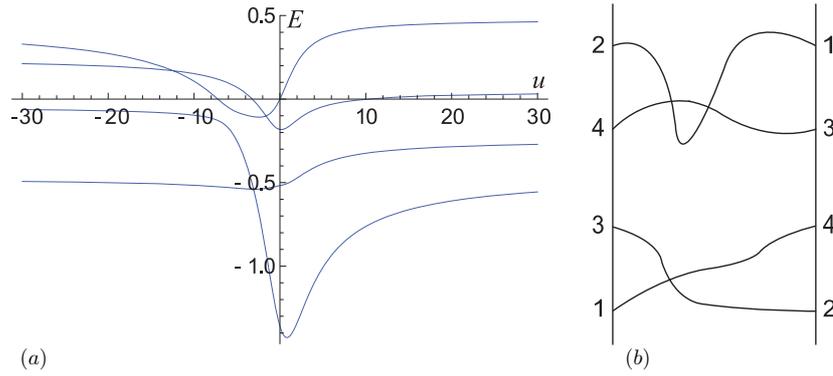


Figure 8. (a) Numerical energy levels of a 4×4 maximal Hamiltonian $H(u) = T + uV$ with the same ordering of d_k as in figure 7(b). Energies are scaled with $(u^2 + 1)^{-1/2}$ to highlight their asymptotic approach to d_k . Matrix elements of $H(u)$ are generated using equation (19) with random γ_k, ε_k and random d_k constrained to obey the above ordering. The multiple (twofold) crossing of the top two levels increases the number of crossings from one as in figure 7(b) to three. This illustrates the generic situation arising when multiple crossings increase the number of crossings by an even integer over and above the number enforced by the diagrams (47). Nevertheless, as discussed in the text, this does not affect equation (45). (b) The schematic of (a).

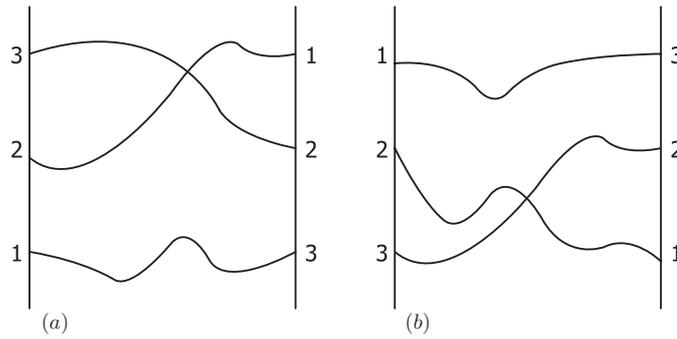


Figure 9. All distinct level diagrams for $N = 3$ maximal Hamiltonians, $H(u) = T + uV$, drawn according to (47). There are $(N - 1)! = 2$ distinct diagrams each corresponding to $N = 3$ different orderings of eigenvalues of V (see the text). For example, diagram (a) corresponds to $d_1 > d_2 > d_3$ and two other orderings obtained with a shift of the indices by an integer mod N , i.e. $d_2 > d_3 > d_1$ and $d_3 > d_1 > d_2$. The diagrams predict a single level crossing in agreement with equation (45) and specify which levels cross. Since this is also the maximum number of crossings for $N = 3$, no multiple crossings of the same two levels are allowed, cf figures 7 and 8.

Let us also comment that cases when some of the eigenvalues of V, d_k , are degenerate should be regarded as crossings at $u \rightarrow \pm\infty$. Equivalently, one can treat T and V on equal footing by defining $H(u, v) = vT + uV$. Then, degenerate d_k correspond to crossings at $v/u = 0$, while the crossings considered above occur either at finite v/u or at $v/u \rightarrow \pm\infty$, or equivalently at $u/v = 0$. For example, levels of the BCS Hamiltonian, which is a linear combination of Gaudin magnets [37], $\hat{H}_{BCS} = \frac{1}{B} \sum \varepsilon_i \hat{h}^i(B) + \text{const}$, cross at the value of the BCS coupling constant $g = 1/B = 1/u = \infty$ or, equivalently, at $u = 0$, see e.g. [17, 39].

We conclude this section with a discussion of useful properties and examples of energy level diagrams. There are $N!$ diagrams for a given N corresponding to permutations of

eigenvalues d_1, d_2, \dots, d_N . However, some of them are identical. Specifically, orderings $d_i < d_m < \dots < d_l$ and $d_{i+a} < d_{m+a} < \dots < d_{l+a}$ that differ by a shift of indices by an integer a yield identical diagrams, since equation (46) is invariant with respect to the replacement $k \rightarrow k + a \pmod{N}$. Because N different orderings can be generated using this shift, it leaves $(N - 1)!$ distinct diagrams. For example, there are two distinct diagrams for $N = 3$, see figure 9. Each corresponds to three different orderings of d_k . Both diagrams predict a single level crossing. Since this is also the maximum number of crossings for $N = 3$, repeated crossings of the same two levels are not allowed. Therefore, a single crossing of either two top or two bottom levels is the only option for $N = 3$ maximally commuting Hamiltonians. For $N = 4$ there are six distinct level diagrams shown in figure 7. Four of them—diagrams (a), (c), (d) and (f) in figure 7—exhibit the maximum number, $M_c^{\max} = 3$, of level crossings. In a manner similar to that of the $N = 3$ case this is the only option for the corresponding sixteen orderings of d_k . In contrast, in diagrams (b) and (e) showing a single crossing, multiple crossings can occur. This will increase the total number of crossings from one to three, see figure 8.

7. Submaximal Hamiltonians

The preceding sections have focused on maximally commuting Hamiltonians, where we have explicitly constructed these operators, solved them exactly, and used the solution to explain the level crossings in such systems. In this section, we explore Hamiltonians linear in a parameter u characterized by less than the maximum number of commuting partners. Most importantly, we demonstrate that some of these submaximal Hamiltonians have *no* energy level crossings, i.e. the *inevitability* of level crossings due to parameter-dependent commuting partners appears to be an exclusive property of maximal Hamiltonians.

As discussed in the introduction, a given family of maximal Hamiltonians contains $N - 1$ nontrivial independent commuting operators (see the discussion in the paragraph following equation (6)). It is reasonable to expect that there exist submaximal families with $N - 2, N - 3$ etc. Hamiltonians. Similar to equation (5), any common u -independent symmetry is assumed to be factored out by going to blocks of the same symmetry. We may adopt a convenient terminology, where families with $N - L$ nontrivial Hamiltonians are identified as being Type L (cf Type I and II of [19]). Then, the maximally commuting Hamiltonians are Type 1, those with $N - 2$ commuting operators are Type 2 and so on. Since a nontrivial family must contain at least two nontrivial commuting operators, the first nontrivial instance of Type 1 occurs for $N = 3$, Type 2 for $N = 4$ etc., where N is the dimensionality of the state space.

First, let us construct 4×4 Type 2 Hamiltonians linear in a real parameter u . Our task is therefore to identify two 4×4 commuting real symmetric matrices that do not have the third independent commuting partner other than $(a + ub)I$. We will do so by employing the construction of maximal Hamiltonians detailed in section 2. Consider 4×4 operators $H(x, y, u) = xT + yK + uV$ and $\tilde{H}(x, y, u) = x\tilde{T} + y\tilde{K} + u\tilde{V}$, linear in parameters x, y and u , such that

$$[H(x, y, u), \tilde{H}(x, y, u)] = 0. \tag{48}$$

Since this equation is to hold for all x, y , and u , the coefficients of the xy, xu, yu etc. terms must vanish individually. We obtain

$$[T, \tilde{T}] = [V, \tilde{V}] = [K, \tilde{K}] = 0, \tag{49}$$

$$[T, \tilde{V}] = [\tilde{T}, V], \quad [T, \tilde{K}] = [\tilde{T}, K], \tag{50}$$

$$[V, \tilde{K}] = [\tilde{V}, K]. \tag{51}$$

Let us choose these real symmetric matrices so that $(T + uV, \tilde{T} + u\tilde{V})$ and $(K + uV, \tilde{K} + u\tilde{V})$ belong to two different families of maximally commuting Hamiltonians parameterized by γ_m, ε_m and $\tilde{\gamma}_m = \gamma_m, \tilde{\varepsilon}_m$, respectively, see equation (19). Then, equations (49) and (50) are satisfied by construction. Consequently, it remains to solve equation (51).

Thus, from equation (21) we have in the common eigenbasis of V and \tilde{V}

$$\begin{aligned}
 [H(x, y, u)]_{mn} &= x\gamma_m\gamma_n \frac{d_m - d_n}{\varepsilon_m - \varepsilon_n} + y\gamma_m\gamma_n \frac{d_m - d_n}{\tilde{\varepsilon}_m - \tilde{\varepsilon}_n}, \quad m \neq n, \\
 [H(x, y, u)]_{mm} &= ud_m - x \sum_{j \neq m} \gamma_j^2 \frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} - y \sum_{j \neq m} \tilde{\varepsilon}_m \gamma_j^2 \frac{d_m - d_j}{\tilde{\varepsilon}_m - \tilde{\varepsilon}_j},
 \end{aligned} \tag{52}$$

where d_k are the eigenvalues of V . Matrix elements of $\tilde{H}(x, y, u)$ are obtained from equation (52) by replacing $d_k \rightarrow \tilde{d}_k$. Using these expressions for the matrix elements, one can rewrite the remaining commutation relation (51) as follows:

$$\gamma_l^2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ d_i & d_j & d_k \\ \tilde{d}_i & \tilde{d}_j & \tilde{d}_k \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ \varepsilon_i & \varepsilon_j & \varepsilon_k \\ \tilde{\varepsilon}_i & \tilde{\varepsilon}_j & \tilde{\varepsilon}_k \end{vmatrix}^2}{(\varepsilon_i - \varepsilon_j)(\varepsilon_j - \varepsilon_k)(\varepsilon_i - \varepsilon_k)(\tilde{\varepsilon}_i - \tilde{\varepsilon}_j)(\tilde{\varepsilon}_j - \tilde{\varepsilon}_k)(\tilde{\varepsilon}_i - \tilde{\varepsilon}_k)}, \quad l \neq i, j, k. \tag{53}$$

Therefore, choosing $d_r, \tilde{d}_r, \varepsilon_r$, and $\tilde{\varepsilon}_r$, we obtain γ_r from equation (53). This yields two commuting matrices $H(x, y, u)$ and $\tilde{H}(x, y, u)$. Fixing nonzero values of $x = x_0$ and $y = y_0$, we obtain a Type 2 family of Hamiltonians linear in u ,

$$H(u) = (x_0T + y_0K) + uV, \quad \tilde{H}(u) = (x_0\tilde{T} + y_0\tilde{K}) + u\tilde{V}. \tag{54}$$

There are a number of equivalent ways to verify that these operators are indeed Type 2 rather than maximally commuting. For example, one can show that their matrix elements (52) cannot be cast into the form (19). Alternatively, it can be demonstrated that conditions (A.3) necessary for any maximal operator do not hold. However, a less formal, but more fruitful verification uses the following argument. We have seen in section 6 that *any* $N = 4$ maximal Hamiltonian must have either one or three level crossings. Let us check if this holds for the Hamiltonians (54). To this end, we set $x_0 = 1$, generate random y_0, d_k, ε_k , and $\tilde{\varepsilon}_k$, and obtain γ_k from equation (53) and $H(u)$ from equation (52). Doing so repeatedly and numerically diagonalizing the resulting Hamiltonians we observe that they always have either no or two level crossings. An example with no crossings is shown in figure 10. Thus, operators (54) are Type 2.

We see that level crossings are not guaranteed when the number of commuting operators is less than the maximum—nontrivial solutions of equation (3) do not necessarily imply crossings. The converse is also false, i.e. level crossings can occur in the absence of any nontrivial commuting partner linear in u and any u -independent symmetry. For example, one can show that the 4×4 Hamiltonian given by equation (A.4) in the appendix with a single level crossing at $u = 0$ has no nontrivial commuting partners and no u -independent symmetry. Interestingly, $N = 4$ is the first dimensionality where this happens as for 3×3 real symmetric matrices linear in u a level crossing implies a nontrivial commuting partner linear in u and vice versa [18].

8. Summary and open questions

In this paper, we addressed the problem of the violation of the Wigner–von Neumann non-crossing rule in quantum integrable systems. For this purpose, we introduced and studied a

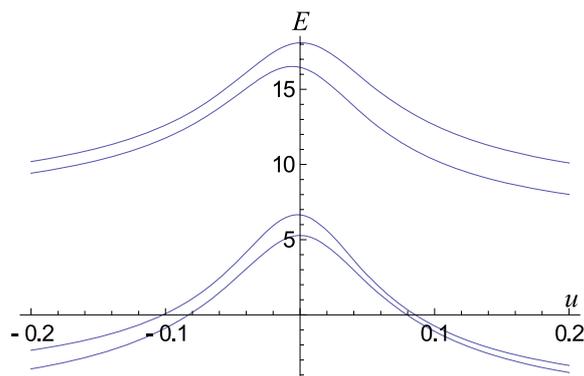


Figure 10. Numerical energy levels of a 4×4 submaximal Hamiltonian $H(u)$ obtained from equations (52) and (54) with $x_0 = 1$, random y_0, d_k, ε_k and $\tilde{\varepsilon}_k$. Energies are scaled with $(u^2+1)^{-1/2}$ as in figure 8. Unlike $N = 4$ maximal Hamiltonians, which always have two nontrivial commuting partners, this $H(u)$ has only one such partner (see the text). Note that levels of $H(u)$ do not cross at any u , i.e. the mere existence of a nontrivial commuting partner does not guarantee level crossings. This is to be contrasted with 4×4 maximally commuting Hamiltonians which always exhibit either three or one crossings, see figures 7 and 8.

general class of maximal Hamiltonians—a vector space of $N \times N$ real symmetric Hamiltonians, $H(u) = T + uV$, characterized by the existence of the maximum possible number (N) of independent mutually commuting integrals similarly linear in the coupling u . We have resolved this commutation property and explicitly constructed general maximal Hamiltonians, see equation (19). Interestingly, these operators are equivalent to the Gaudin magnets (30) in the next to highest weight sector, $J^z = J_{\max}^z - 1$, where J^z is the z projection of the total spin.

The mapping to Gaudin magnets allowed us to obtain a complete exact solution for the eigenstates and eigenvalues of $H(u)$, equations (40) and (41). Furthermore, we have demonstrated that energy level crossings are inevitable for maximal operators, i.e. there is always at least one crossing. The total number of crossings varies from 1 or 2 to $(N-1)(N-2)/2$, see section 6. Thus, the mere existence of the maximum number of commuting partners guarantees a) an exact solution and b) level crossings. This relationship between the existence of conservation laws and exact solution is a quantum analog of the famous Liouville–Arnold theorem in classical mechanics. The latter states that if a classical model with n degrees of freedom has n Poisson-commuting integrals, its equations of motion are exactly solvable [20].

At the same time, by constructing an explicit example we have demonstrated that submaximal Hamiltonians—real symmetric operators of the form $T + uV$ with less than the maximum number of linear in u commuting partners—often show no instances of level crossings at any u . Thus, the inevitability of crossings is an exclusive feature of maximally commuting operators. Similarly, we have also shown that the presence of level crossings does not necessarily imply the existence of a nontrivial commuting partner linear in u , i.e. crossings can occur even in the absence of such partners as well as u -independent symmetries.

Our understanding of properties of parameter-dependent energy spectra in integrable models is far from complete. We conclude this section with a list of open questions stemming from the results of this work.

- (i) We have shown that there are submaximal Hamiltonians with no level crossings. Nevertheless, crossings often do occur in these systems in violation of the non-crossing

rule, see section 7. This indicates that there is more to the link between crossings and the presence of commuting partners. It is interesting to understand this link for submaximal operators, what lifts level repulsion in this case, and why crossings happen only for a fraction of submaximal Hamiltonians.

- (ii) In section 7, we have also introduced a notion of Type L commuting family characterized by $N - L$ nontrivial integrals. In this classification maximally commuting operators are Type 1, while submaximal operators correspond to $L \geq 2$. A natural question is whether there is a general explicit parametrization for Type 2, 3 etc. similar to that obtained in this paper for maximal systems. For instance, one can show that $|J^z| \leq J_{\max}^z - 2$ sectors provide examples of submaximal commuting families.
- (iii) What is the role of maximal Hamiltonians in the context of general quantum integrable Hamiltonians? For the central spin Hamiltonians (Gaudin magnets) they represent the next to highest weight sector. Do other integrable models have maximally commuting sectors?
- (iv) In this paper, we focused on operators linear in the coupling u . An interesting question is how our results can be generalized to operators with a more general, e.g. polynomial, dependence on the coupling.
- (v) We have established hard bounds for the number M_c of level crossings in $N \times N$ maximally commuting operators. Can one also determine the distribution of M_c for large N , i.e. the relative prevalence of maximal Hamiltonians with a particular number of level crossings?

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Appendix A. Shastry's construction of commuting matrices

In [19] Shastry developed an approach to generate commuting real symmetric $N \times N$ matrices linear in a parameter u . Here we show that matrices obtained with this approach belong to maximally commuting set constructed in section 2, see equation (19).

First, we briefly review the results of [19]. Consider equation (3). In the common eigenbasis of V and \tilde{V} the second relation in equation (3) becomes

$$S_{ij} \equiv \frac{T_{ij}}{d_i - d_j} = \frac{\tilde{T}_{ij}}{\tilde{d}_i - \tilde{d}_j}, \quad i \neq j, \quad (\text{A.1})$$

where T_{ij} (\tilde{T}_{ij}) are the matrix elements of T (\tilde{T}) and d_i and \tilde{d}_i are the eigenvalues of V and \tilde{V} , respectively. It remains to consider the $[T, \tilde{T}] = 0$ commutation relation in equation (3). This can be cast into the following form:

$$\mu_{ijk}\tilde{d}_i + \mu_{jki}\tilde{d}_j + \mu_{kij}\tilde{d}_k + \sum_{l \neq i, j, k} \nu_{lijk}\tilde{d}_l = 0, \quad \text{distinct } i, j, k, \quad (\text{A.2})$$

where μ_{ijk} and ν_{lijk} depend only on matrix elements of $H(u)$ and not on those of $\tilde{H}(u)$. Specifically, they involve only d_r , T_{rr} and S_{rm} .

A set of particular solutions to equation (3) can be obtained by setting the coefficients in equation (A.2) at each \tilde{d}_r individually to zero, i.e.

$$\mu_{ijk} = 0, \quad \nu_{lijk} = 0, \quad \text{distinct } l, i, j, k. \quad (\text{A.3})$$

Now commuting $H(u)$ and $\tilde{H}(u)$ can be generated as follows. One first chooses $3N - 1$ parameters, e.g. $2N - 3$ variables S_{1r} and S_{2r} for $\{2, 3\} \leq r \leq N$ and $N + 2$ variables $\{d_r\}$, T_{11} , and T_{22} . Then, equation (A.3) reduce to linear equations and can be solved for the remaining variables. Once $H(u)$ is determined in this way, $\tilde{H}(u)$ can also be constructed, see [19] for details. This scheme is quite suitable for numerical implementation and, having examined several examples, Shastry observed crossings in all cases. Based on this and the results of [18] he conjectured that these matrices will always exhibit them.

To show that this construction always yields maximal Hamiltonians, we note that equation (A.3) is a sufficient condition for equation (A.2) to have N linearly independent solutions for $(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_N)$. Since \tilde{d}_r are the eigenvalues of \tilde{V} , the existence of N linearly independent solutions means that there are N Hamiltonians $\tilde{H}(u)$ with linearly independent \tilde{V} s. The absence of u -independent symmetries can also be demonstrated (it follows from $S_{ij} \neq 0$ for all $i \neq j$). Thus, we have a maximally commuting set, see the introduction. The only difference is that by construction d_r are not allowed to be degenerate, see equation (A.1), while the maximal set contains these matrices as well.

Finally, we write down an example (see the discussion at the end of section 7) of a 4×4 Hamiltonian $H(u)$ with a level crossing but no u -independent symmetry and no commuting partners linear in u other than trivial ones— $cH(u) + (a + ub)I$, where a, b , and c are real numbers and I is the identity matrix,

$$H(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + u \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & 2 \end{pmatrix}. \quad (\text{A.4})$$

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