Migdal-Eliashberg theory as a classical spin chain

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We formulate the Migdal-Eliashberg theory of electron-phonon interactions in terms of classical spins by mapping the free energy to a Heisenberg spin chain in a Zeeman magnetic field. Spin components are energy-integrated normal and anomalous Green's functions and sites of the chain are fermionic Matsubara frequencies. The Zeeman field grows linearly with the spin coordinate and competes with ferromagnetic spin-spin interaction that falls off as the square of the inverse distance. The spin-chain representation makes a range of previously unknown properties plain to see. In particular, infinitely many new solutions of the Eliashberg equations both in the normal and superconducting states emerge at strong coupling. These saddle points of the free-energy functional correspond to spin flips. We argue that they are also fixed points of kinetic equations and play an essential role in far from equilibrium dynamics of strongly coupled superconductors. Up to an overall phase, the frequency-dependent gap function that minimizes the free energy *must* be non-negative. There are strong parallels between our *Eliashberg spins* and Anderson pseudospins, though the two sets of spins never coincide.

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I. INTRODUCTION

Electron-phonon interactions determine many properties of quantum metals, such as the charge and heat transport, thermodynamics, and superconductivity. A well-established approach to these phenomena is the Migdal-Eliashberg theory [1,2]. This theory operates with two functions of Matsubara frequency, $\Sigma_n \equiv \Sigma(\omega_n)$ and $\Phi_n \equiv \Phi(\omega_n)$, which have the meaning of the normal and anomalous self-energies. The functions Σ_n and Φ_n must satisfy two nonlinear algebraic equations known as the *Eliashberg equations*, whose kernel is the phonon-mediated electron-electron interaction $\lambda_l \equiv \lambda(\omega_l)$. The order parameter is the frequency-dependent gap function $\Delta_n = \Phi_n/Z_n$, where $Z_n = 1 + \Sigma_n/\omega_n$.

The main dimensionless parameter in the Migdal-Eliashberg theory is the electron-phonon coupling defined as the electron-electron interaction evaluated at zero Matsubara frequency, $\lambda = \lambda(\omega_l = 0)$. The coupling constant λ is inversely proportional to the square of the characteristic phonon frequency $\omega_{\rm ch}$. In conventional electron-phonon models, such as the Holstein or Frölich Hamiltonian, interactions with electrons strongly renormalize phonon frequencies [1–3]. Because of this it is crucial to distinguish *bare* (λ_0) and *renormalized* (λ) coupling constants [4].

The Migdal-Eliashberg theory enjoyed a great deal of success over many decades as a quantitative theory of conventional superconductivity. With phonon spectrum extracted from experiment, the theory makes accurate predictions for the superconducting transition temperature T_c, zero-temperature energy gap, jump in the specific heat at T_c, density of states, etc., for a broad range of superconductors, such as Al, V, Ta, Sn, Tl, In, Nb, Pb, their various alloys, Hg, and MgB_2 [5–10]. The coupling constant in these materials ranges between $\lambda \approx 0.4$ for Al and $\lambda \approx 1.6$ for Hg in simple elements and up to $\lambda \approx 3.0$ ($Pb_{0.5}Bi_{0.5}$) in alloys.

Despite the success and maturity of the Migdal-Eliashberg theory, there is also a great deal of controversy surrounding it. A number of publications [11–16] rediscover Migdal and Eliashberg's observation [1,2] that the theory is inapplicable to $\lambda_0 \gtrsim 0.5$, Migdal theorem notwithstanding [17]. Others see no upper limit on λ [4] and even hypothesize that the strong coupling, $\lambda \to \infty$, limit of the Migdal-Eliashberg theory is attainable [18]. A recent study claims Eliashberg equations acquire a one-parameter family of solutions in the strong coupling limit and as a consequence the superconducting transition temperature vanishes [19]. Even the $\lambda \to 0$ limit is not free from controversy. While it is generally accepted that the weak coupling limit of the Migdal-Eliashberg theory is the more famous Bardeen-Cooper-Schrieffer (BCS) theory [20], some argue that this is in fact untrue [21,22].

Our goal is to resolve these issues in a series of papers [23–25]. Here we establish several important properties of the free energy of the electron-phonon system. We show that the gap function Δ_n must be non-negative at the global minimum point and argue that the minimum is unique by symmetry at all temperatures and coupling strengths, up to an overall phase $e^{i\phi}$. Aside from the global minimum, we find an infinite set of new saddle points at strong coupling. As usual, stationary points other than the global minimum do not affect equilibrium properties in the thermodynamic limit. However, we reason that these saddle points play a major role in the

far from equilibrium dynamics of the electronic subsystem. Besides, their emergence and proliferation seem to be tied to the subsequent breakdown of the Migdal-Eliashberg theory discussed in the next paragraph. For the sake of completeness, we also provide an educational Appendix where we explain within the path-integral framework that the BCS theory undoubtedly *is* the weak coupling limit of the Migdal-Eliashberg theory when this limit is properly taken.

In the next paper [23], we will show that the Migdal-Eliashberg theory loses its validity at a *finite* value of the *renormalized* electron-phonon coupling λ_c due to a phase transition which breaks the lattice translational symmetry. We will then construct an extended theory (theory of lattice-fermionic superfluidity) that works past λ_c. Prior to the transition it reduces to the Migdal-Eliashberg theory. Afterwards, it describes the new state of the system.

Our main tool to achieve these goals is a representation of the Migdal-Eliashberg theory as a classical Heisenberg spin chain, which we describe below. Positions of the *Eliashberg spins* are the fermionic Matsubara frequencies $\omega_n = \pi T(2n + 1)$. Spin components are momentum-integrated normal and anomalous Green's functions. The interaction between them is ferromagnetic and falls off as $(\omega_n - \omega_m)^{-2}$ at large separation. In addition, the spins are subject to a position-dependent Zeeman field $2\pi\omega_n$ along the z axis. The energy of the spin chain is proportional to the free-energy density of the electron-phonon system. Classical spins provide a simple and intuitive picture of the normal and superconducting states and of the transition between them. In the normal state, the spins are parallel to the z axis, $\boldsymbol{S}_n = \mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is a unit vector along the z axis. Below the superconducting transition temperature, they acquire xy components softening the sharp domain wall between ω_{-1} and ω_0 in the normal state as shown in Fig. 1.

Not surprisingly, Eliashberg equations emerge as stationary point equations for the free-energy functional. Their solutions therefore correspond to either minima or maxima or saddle points of the free energy. The spin-chain formulation makes it straightforward to demonstrate several fundamental properties of these stationary points. In particular, we use it to prove that the Eliashberg gap function at the global minimum is of the form $\Delta_n = e^{i\phi}|\Delta_n|$, where $|\Delta_n|$ is even in ω_n. Furthermore, an infinite discrete set of saddle points emerges as we increase λ. In terms of the spin chain, they are equilibria with a certain number of spins flipped against effective magnetic fields acting on them. The minimum is the stable, lowest-energy spin configuration. The saddle points begin to proliferate just before the Migdal-Eliashberg theory breaks down. We argue that these saddle points play a special role in the far from equilibrium collisionless dynamics of strongly coupled conventional superconductors. Namely, they are the fixed points of the corresponding kinetic equations, which are Hamilton's equations of motion for Eliashberg spins. When sufficiently many spins are flipped, these stationary points are unstable and give rise to rich solitonlike dynamics.

The content of this paper is as follows. In Sec. II we map the Migdal-Eliashberg theory to a classical spin chain building on the path-integral formulation we develop in Appendixes A and B. We interpret the superconducting transition in terms of spins in Sec. III, construct a divergence-free form of

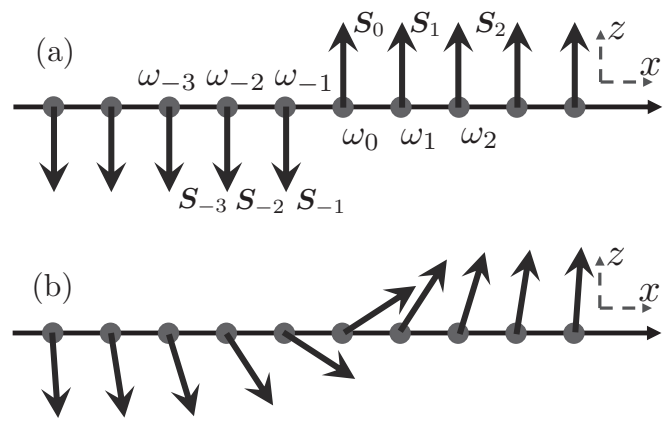

FIG. 1. Superconducting transition in terms of classical spins. As shown in the text, Migdal-Eliashberg theory maps to a classical Heisenberg spin chain. The positions of the spins are fermionic Matsubara frequencies ω_n. The spin-spin interactions are purely ferromagnetic and the spins are subject to a Zeeman magnetic field $2\pi\omega_n$ along the z axis. The figure shows (a) the normal state and (b) a superconducting state. In the superconducting state, spins acquire x components, which implies nonzero anomalous Green's function. The sharp domain wall in the normal state is smeared in the superconducting state. The normal-superconductor transition therefore translates into softening of the domain wall in the spin language.

Eliashberg equations as well as new (spin-flip) solutions for them in Sec. IV, and relate spins and the Eliashberg gap function Δ_n in Sec. V. These three sections lay the groundwork for the analysis of stationary points of the free energy in Sec. VI. Section VII focuses on the strong coupling limit $\lambda = \infty$ and in Sec. VIII we compare the two notions of spins in the theory of superconductivity (Eliashberg spins introduced in this paper and Anderson pseudospins [26]) and discuss the role of the spin-flip solutions of the Eliashberg equations in the collisionless dynamics of strongly coupled superconductors.

II. MAPPING TO A HEISENBERG SPIN CHAIN

We work with two models in this paper. One is the Holstein model (dispersionless phonons) with arbitrary hopping matrix and onsite potential. The other is a rather general electron-phonon Hamiltonian with arbitrary phonon dispersion and momentum-dependent electron-phonon coupling. The Holstein Hamiltonian reads as

$$H = \sum_{ij\sigma} h_{ij} c^\dagger_{i\sigma} c_{j\sigma} + \sum_i \left[\frac{p_i^2}{2M} + \frac{K_0 x_i^2}{2}\right] + \alpha \sum_i n_i x_i, \quad (1)$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ label lattice sites, h_{ij} are the matrix elements of an *arbitrary* single-electron Hamiltonian $\hat{h}$ [27], $c_{i\sigma}$ annihilates an electron on site $\boldsymbol{i}$ with spin projection σ, $n_i = \sum_\sigma c^\dagger_{i\sigma} c_{i\sigma}$ is the fermion occupation operator, and p_i and x_i are ion momentum and position operators. The *bare* phonon frequency is $\Omega_0 = \sqrt{K_0/M}$. The lattice and its dimensionality are at this point arbitrary.

The Hamiltonian for electrons interacting with dispersing phonons is

$$H = \sum_{p\sigma} \xi_p c^\dagger_{p\sigma} c_{p\sigma} + \sum_q \omega_0(q) b^\dagger_q b_q + \frac{1}{\sqrt{N}} \sum_{pq\sigma} \frac{\alpha_q}{\sqrt{2M\omega_0(q)}} c^\dagger_{p+q\sigma} c_{p\sigma} [b^\dagger_{-q} + b_q], \quad (2)$$

where M is the ion mass and N is the number of lattice sites.

We proceed with a path-integral formulation of the Migdal-Eliashberg theory. The first step is to integrate out phonons, which leaves us with an effective fermion-fermion interaction quartic in fermionic fields. The second step is to decouple the quartic term with three Hubbard-Stratonovich fields Φ, $\Sigma_\uparrow$, and $\Sigma_\downarrow$ that are functions of two imaginary times and, in general, two space points. In the next step, we integrate out the fermions to obtain an effective action solely in terms of the *Eliashberg fields* Φ, $\Sigma_\uparrow$, and $\Sigma_\downarrow$. The fourth and final step is to obtain the stationary point of this effective action, where the Eliashberg fields depend on the time difference only. We detailed the above procedure in Appendix A for the Holstein model and in Appendix B for dispersing phonons (2). The end result is an expression for the free-energy functional [28] of the system per site,

$$f = \nu_0 T^2 \sum_{nl} [\Phi^*_{n+l} \Lambda_l \Phi_n + \Sigma_{n+l} \Lambda_l \Sigma_n] - 2\pi \nu_0 T \sum_n \sqrt{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2}, \quad (3)$$

where ν_0 is the density of states at the Fermi energy per site per spin projection. The field $\Phi_n \equiv \Phi(\omega_n)$ is complex and $\Sigma_n \equiv \Sigma(\omega_n)$ is real. Both fields are functions of the fermionic Matsubara frequency ω_n. At the stationary point, these fields equal the anomalous and normal self-energies.

The quantity Λ_l in Eq. (3) is the Fourier transform of $1/\lambda(\tau)$ at bosonic Matsubara frequency $\omega_l = 2\pi T l$. Here $\lambda(\tau)$ is the effective fermion-fermion interaction in the imaginary-time domain. In the Matsubara frequency domain this interaction reads as, for the Holstein model,

$$\lambda(\omega_l) = \frac{g^2}{\omega_l^2 + \Omega^2}, \quad g^2 = \nu_0 \alpha^2 M^{-1}. \quad (4)$$

We also define the dimensionless electron-phonon coupling constant as

$$\lambda = \lambda(\omega_l = 0) = \frac{g^2}{\Omega^2} = \frac{\nu_0 \alpha^2}{K}, \quad (5)$$

where Ω and K are the *renormalized* phonon frequency and spring constant. See Appendix B for $\lambda(\omega_l)$ and $\lambda = \lambda(\omega_l = 0)$ for dispersing phonons. At times we will consider the strong coupling limit $\lambda \to \infty$, which is equivalent to $\Omega \to 0$ or $K \to 0$ for the Holstein model.

The stationary point equations for the free energy (3) are the *Eliashberg equations* [2]

$$\Phi_n = \pi T \sum_m \lambda_{nm} \frac{\Phi_m}{\sqrt{(\omega_m + \Sigma_m)^2 + |\Phi_m|^2}}, \quad (6a)$$

$$\Sigma_n = \pi T \sum_m \lambda_{nm} \frac{\omega_m + \Sigma_m}{\sqrt{(\omega_m + \Sigma_m)^2 + |\Phi_m|^2}}, \quad (6b)$$

where

$$\lambda_{nm} = \lambda(\omega_n - \omega_m). \quad (7)$$

Note also that

$$\lambda_{nn} = \lambda \quad (8)$$

diverges in the strong coupling limit.

It is convenient to introduce new variables, complex $F(\tau)$ and real $G(\tau)$, such that

$$\Phi(\tau) = \pi \lambda(\tau) F(\tau), \quad \Sigma(\tau) = \pi \lambda(\tau) G(\tau). \quad (9)$$

In frequency representation we have

$$\Phi_n = \pi T \sum_m \lambda_{nm} F_m, \quad \Sigma_n = \pi T \sum_m \lambda_{nm} G_m, \quad (10)$$

and Eq. (6) becomes

$$F_n = \frac{\Phi_n}{\sqrt{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2}}, \quad G_n = \frac{\omega_n + \Sigma_n}{\sqrt{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2}}. \quad (11)$$

On the stationary point, the fields F_n and G_n correspond to the anomalous and normal Green's functions integrated over the single-particle energy (see Appendix A 4). We also show in Appendix A 4 that assuming time-reversal symmetry, Σ_n is real and odd in ω_n and $|\Phi_n|$ is even. Therefore, G_n is real and odd and $|F_n|$ is even.

Observe that Eq. (11) implies a constraint on the variables G_n and F_n:

$$G_n^2 + |F_n|^2 = 1. \quad (12)$$

This means that we can trade these variables for three components of a *classical spin* $\boldsymbol{S}_n$ of unit length $\boldsymbol{S}_n^2 = 1$:

$$S_n^z = G_n, \quad S_n^x = \mathrm{Re}(F_n), \quad S_n^y = \mathrm{Im}(F_n). \quad (13)$$

It follows from Eq. (11) that

$$F_n \Phi_n^* + G_n(\omega_n + \Sigma_n) = \sqrt{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2}. \quad (14)$$

This allows us to rewrite the free energy (3) as

$$H_s \equiv \frac{f}{\nu_0 T} = -2\pi \sum_n \omega_n S_n^z - \pi^2 T \sum_{nm} \lambda_{nm} \boldsymbol{S}_n \cdot \boldsymbol{S}_m. \quad (15)$$

We interpret the free energy (15) as a Hamiltonian H_s of an open classical Heisenberg spin chain in an inhomogeneous Zeeman magnetic field. The positions of the spins are fermionic Matsubara frequencies ω_n. Spin-spin interactions are ferromagnetic and fall off at large "distance" as $\lambda_{nm} \propto (\omega_n - \omega_m)^{-2}$. The "magnetic field" is linear in the position of the spin and goes to $\pm\infty$ as $\omega_n \to \pm\infty$. Note that the Boltzmann weight is $e^{-fN/T} = e^{-\nu_0 N H_s} = e^{-H_s/\delta}$, where $\delta = (\nu_0 N)^{-1}$ is the single-particle (electron) level spacing in the original electron-phonon problem. Therefore, the spin chain is at an effective temperature

$$T_s = \delta. \quad (16)$$

Let us also write the classical spin Hamiltonian for the Holstein model as a visual [substitute Eq. (4) into Eq. (15)]:

$$H_s = -2\pi \sum_n \omega_n S_n^z - \pi^2 T g^2 \sum_{nm} \frac{\boldsymbol{S}_n \cdot \boldsymbol{S}_m}{(\omega_n - \omega_m)^2 + \Omega^2}. \quad (17)$$

See the list below Eq. (45) for more properties of the spin-chain representation of the free energy.

Sometimes an onsite Hubbard repulsion is added [6] to the Eliashberg equations by replacing λ_{nm} in Eq. (6a) with $\lambda_{nm} - u$. This adds a long-range xy term to the spin Hamiltonian (15),

$$H_C = \pi^2 T u \sum_{nm} \left(S_n^x S_m^x + S_n^y S_m^y\right), \quad (18)$$

and the classical spin Hamiltonian becomes $H_s + H_C$.

III. SUPERCONDUCTIVITY IN TERMS OF SPINS

In the spin language, the superconducting transition translates into softening of the domain wall as shown in Fig. 1. This is similar to the Anderson pseudospin description of the BCS superconductivity [26] even though Eliashberg and Anderson spins are not the same (see Sec. VIII).

To understand what happens as we lower the temperature, let us analyze the spin texture minimizing the energy H_s as a function of T. The Zeeman magnetic field and ferromagnetic interaction compete in H_s. The spin configuration minimizing the Zeeman term is $\boldsymbol{S}_n = \mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$, and the Zeeman field inevitably prevails far from the origin, so that $\boldsymbol{S}_n \to \pm\hat{\boldsymbol{z}}$ for $\omega_n \to \pm\infty$. These fixed values at infinity serve as boundary conditions for the interaction term.

Above the superconducting T_c, the anomalous averages vanish, $F_n = 0$. According to the definition (13) of the classical spin, this means that all spins are parallel to the z axis. From the behavior of $\boldsymbol{S}_n$ at large ω_n and by symmetry, it is then clear that the minimum energy spin texture is $\boldsymbol{S}_n = \mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$. This is the *normal state* in the spin language. The characteristic feature of the normal state is a sharp domain wall between ω_{-1} and ω_0 with an abrupt maximal jump of the z component of spin from $S_{-1}^z = -1$ to $S_0^z = +1$ (see Fig. 1).

Below T_c the anomalous averages are nonzero, i.e., the spins acquire x components (F_n can be made real in the spin-chain ground state). This means softening of the domain wall. Now the change in S_n^z from -1 at $\omega_n \to -\infty$ to $+1$ at $\omega_n \to +\infty$ is gradual and the jump $S_0^z - S_{-1}^z < 2$. Spin configurations with nonzero xy components are superconducting states.

IV. NEW SOLUTIONS OF ELIASHBERG EQUATIONS

We begin this section by verifying that solutions of the Eliashberg equations are equilibrium points of the spin chain. This correspondence allows us to identify new solutions of the Eliashberg equations both with zero and nonzero anomalous self-energy Φ_n, i.e., both normal and superconducting. These solutions correspond to equilibria of the spin chain with a number of spins flipped against their effective magnetic fields, as opposed to all spins being along their fields as in the ground state.

A. Eliashberg equations as equilibrium condition for spins

Consider our spin Hamiltonian in its most general form

$$H_s = -2\pi \sum_n \omega_n S_n^z - \pi^2 T \sum_{nm} \lambda_{nm} \boldsymbol{S}_n \cdot \boldsymbol{S}_m. \quad (19)$$

Terms in the Hamiltonian that contain spin $\boldsymbol{S}_n$ are

$$h_n = -\boldsymbol{b}_n \cdot \boldsymbol{S}_n, \quad \boldsymbol{b}_n = 2\pi\omega_n\hat{\boldsymbol{z}} + 2\pi^2 T \sum_{m\neq n} \lambda_{nm}\boldsymbol{S}_m, \quad (20)$$

where $\boldsymbol{b}_n$ is the effective magnetic spin acting on spin $\boldsymbol{S}_n$, which is the Zeeman field plus the field from other spins.

Equilibrium is when each spin is collinear with its field (parallel or antiparallel), i.e.,

$$\boldsymbol{S}_n = e_n \frac{\boldsymbol{b}_n}{|\boldsymbol{b}_n|}, \quad e_n = \pm 1, \quad (21)$$

where $e_n = +1$ indicates that the spin $\boldsymbol{S}_n$ is parallel to its effective field $\boldsymbol{b}_n$, while $e_n = -1$, that the spin is flipped (antiparallel to the field). However, not all spin equilibria correspond to stationary points of the Eliashberg free energy (3). The reason is that we extracted the square root in Eq. (14) to obtain the spin chain. Only such spin equilibria are stationary points of the free energy for which the left-hand side of Eq. (14) is non-negative.

To interpret this condition in terms of spins, let us introduce an additional magnetic field, which includes the action of the spin on itself,

$$\boldsymbol{B}_n = \boldsymbol{b}_n + 2\pi^2 T \lambda \boldsymbol{S}_n. \quad (22)$$

Recall that $\lambda = \lambda(\omega_l = 0) = \lambda_{nn}$. The definition of $\boldsymbol{b}_n$ in Eq. (20), the definition of spins (13), and Eq. (10) imply

$$B_n^+ = 2\pi\,\Phi_n, \quad B_n^z = 2\pi\,(\omega_n + \Sigma_n). \quad (23)$$

Here and below we use the notation $V^+ \equiv V^x + iV^y$, where V^x and V^y are the x and y components of a vector $\boldsymbol{V}$. Since in equilibrium $\boldsymbol{b}_n$ is collinear with $\boldsymbol{S}_n$, so is $\boldsymbol{B}_n$. Then, the requirement that the expression $F_n\Phi_n^* + G_n(\omega_n + \Sigma_n)$ on the right-hand side of Eq. (3) be non-negative is equivalent to

$$\boldsymbol{B}_n \cdot \boldsymbol{S}_n = \boldsymbol{b}_n \cdot \boldsymbol{S}_n + 2\pi^2\lambda T \geqslant 0. \quad (24)$$

This condition always holds when $e_n = +1$ for all n (no spin flips).

We saw that in equilibrium $\boldsymbol{S}_n$ is either parallel or antiparallel to $\boldsymbol{B}_n$. Equilibria of the spin chain with each $\boldsymbol{S}_n$ parallel to $\boldsymbol{B}_n$ correspond to solutions of the Eliashberg equations and vice versa. We consider only this type of equilibria in this paper. However, it is important to keep in mind that spin flips are nevertheless allowed as $\boldsymbol{S}_n$ can be parallel to $\boldsymbol{B}_n$, but antiparallel to $\boldsymbol{b}_n$. Later in this section we will point out equilibrium spin configurations of this type. To derive Eliashberg equations from $\boldsymbol{S}_n \parallel \boldsymbol{B}_n$, note that this condition is equivalent to

$$S_n^z = \frac{B_n^z}{|\boldsymbol{B}_n|}, \quad S_n^+ = \frac{B_n^+}{|\boldsymbol{B}_n|}. \quad (25)$$

Equation (22) now implies the following two self-consistency conditions:

$$B_n^z = \sum_m \lambda_{nm} \frac{B_m^z}{|\boldsymbol{B}_m|}, \quad B_n^+ = \sum_m \lambda_{nm} \frac{B_m^+}{|\boldsymbol{B}_m|}. \quad (26)$$

Using Eq. (23), we see that these are precisely the Eliashberg equations (6).

The spin-chain approach naturally leads to an alternative set of Eliashberg equations that are free of divergences in the strong coupling limit $\lambda \to \infty$ both in the superconducting and normal states [18]. Indeed, we see from Eqs. (6) and (8) that this limit presents a problem for the Eliashberg equations as they contain a diverging $\lambda_{nn} = \lambda$ term. As a result, both self-energies, Σ_n and Φ_n, are proportional to λ and diverge in this limit. Equation (10) shows that the divergent parts of Σ_n and Φ_n are the $m = n$ terms, $\pi\lambda T G_n$ and $\pi\lambda T F_n$, respectively. Since G_n and $|F_n|$ are both of order 1 [see Eq. (12)], these terms diverge in the strong coupling limit.

All we have to do to avoid these divergences is to use Eq. (21) instead of Eq. (25). Let us introduce reduced self-energies [cf. Eq. (10)]

$$\omega_n + \Sigma'_n = (\omega_n + \Sigma_n) - \pi T\lambda G_n = \frac{b_n^z}{2\pi},$$
$$\Phi'_n = \Phi_n - \pi T\lambda F_n = \frac{b_n^+}{2\pi}. \tag{27}$$

Σ'_n and Φ'_n are the same as Σ_n and Φ_n in Eq. (10) but without the $n = m$ terms. It follows from Eqs. (21) and the definition (20) of $\boldsymbol{b}_n$ that

$$\Phi'_n = \pi T \sum_{m\neq n} \lambda_{nm} \frac{\Phi'_m}{\sqrt{(\omega_m + \Sigma'_m)^2 + |\Phi'_m|^2}},$$
$$\Sigma'_n = \pi T \sum_{m\neq n} \lambda_{nm} \frac{\omega_m + \Sigma'_m}{\sqrt{(\omega_m + \Sigma'_m)^2 + |\Phi'_m|^2}}, \tag{28}$$

where for simplicity we took $e_n = +1$ (no spin flips). As we discuss below, this is always the case at not too strong coupling and only these $e_n = +1$ solutions have been considered in the literature until now. Then, Eqs. (28) are equivalent to the original Eliashberg equations (6), but do not contain $n = m$ terms. Note also that Eqs. (21) and (25) imply $S_n^z/S_n^+ = b_n^z/b_n^+ = B_n^z/B_n^+$ and, therefore,

$$\frac{\Phi_n}{\omega_n + \Sigma_n} = \frac{\Phi'_n}{\omega_n + \Sigma'_n}. \tag{29}$$

The z component of spin $S^z(\omega_n) \equiv S_n^z = G_n$ is odd in ω_n because G_n is odd. Now consider an equilibrium configuration such that $\mathrm{sgn}(S_n^z) = \mathrm{sgn}(\omega_n)$. Taking into account that S_n^z is odd, the expression (20) for $\boldsymbol{b}_n$ and Eq. (22) imply $\mathrm{sgn}(b_n^z) = \mathrm{sgn}(B_n^z) = \mathrm{sgn}(\omega_n)$. Since b_n^z and S_n^z have the same sign, this must be a spin configuration with no spin flips. Although less obvious, the converse is also true, i.e., $\mathrm{sgn}(b_n^z) = \mathrm{sgn}(B_n^z) = \mathrm{sgn}(\omega_n)$ holds in any configuration with no spin flips.

It is also not difficult to show that $\mathrm{sgn}(b_n^z) = \mathrm{sgn}(\omega_n)$ continues to hold when only a small number of spins are flipped out of this configuration (which we will assume). On the other hand, we saw above that $\boldsymbol{S}_n$ must be parallel to $\boldsymbol{B}_n$ in any spin equilibrium that corresponds to a solution of Eliashberg equations. Therefore, for such equilibria with few spin flips we have

$$\mathrm{sgn}\big(B_n^z\big) = \mathrm{sgn}\big(S_n^z\big) = e_n \mathrm{sgn}(\omega_n). \tag{30}$$

B. Spin-flip solutions

Let us determine when spin-flip solutions of Eliashberg equations exist. The spin chain has equilibria with any number of spins at arbitrary positions being antiparallel to their fields. However, to be also solutions of the Eliashberg equations these configurations must satisfy the inequality (24). With the help of Eq. (21) we rewrite this inequality as

$$e_n|\boldsymbol{b}_n| + 2\pi^2\lambda T \geqslant 0. \tag{31}$$

First, it is clear that $e_n = +1$ always works, i.e., stationary configurations of spins (equilibria) with no spin flips are always solutions of the Eliashberg equations.

Now suppose $\boldsymbol{S}_n$ is antiparallel to its effective field $\boldsymbol{b}_n$. Then, $e_n = -1$ and Eq. (31) becomes

$$2\pi^2\lambda T \geqslant |\boldsymbol{b}_n|. \tag{32}$$

We see that no spin flips are allowed at weak coupling, $\lambda \to 0$, and at $T = 0$ for any finite λ because $2\pi^2\lambda T$ in this inequality is either negligible or zero, while the right-hand side is positive. Conversely, arbitrary spin flips are permitted in the strong coupling limit $\lambda \to \infty$ at any finite T because this term is infinite and positive. Therefore, we expect that at finite temperature, spin-flip solutions first appear at $\lambda \sim 1$.

The existence of spin-flip solutions in the double limit $\lambda \to \infty$ and $T \to 0$ depends on whether the product λT goes to zero (spin-flip solutions do not exist), infinity (solutions with any set of $\boldsymbol{S}_n$ antiparallel to $\boldsymbol{b}_n$ exist), or a finite value (spin-flip solutions may or may not exist). Recall that $\lambda = g^2/\Omega^2$ and that we are increasing λ by decreasing Ω. Then, arbitrary spin flips are allowed as long as $Tg/\Omega^2 \to \infty$ in the double limit $T \to 0$ and $\Omega \to 0$ and there are numerous such solutions in the regime $T \gg \Omega^2/g = g/\lambda = \Omega/\sqrt{\lambda}$.

We mentioned in the previous subsection that $m = n$ contributions to the self-energies Σ_n and Φ_n in Eqs. (10) diverge in the strong coupling limit. It is precisely these divergent terms that make spin-flip solutions possible. Indeed, Eqs. (22)–(24) and the definition of $\boldsymbol{b}_n$ in Eq. (20) show that the second term in the inequality (31), without which the spin-flip solutions would not exist, arises from just these two terms.

Spin-flip solutions occur already in the normal state where $\boldsymbol{S}_n = \mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$. Suppose we flip the spin at Matsubara frequency $\omega_k > 0$. Since S_n^z is odd in ω_n, we also have to symmetrically flip the spin at $-\omega_k$. The new spin configuration is $\boldsymbol{S}_n = \mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$ for $|\omega_n| \neq \omega_k$ and $\boldsymbol{S}_n = -\mathrm{sgn}(\omega_n)\hat{\boldsymbol{z}}$ for $|\omega_n| = \omega_k$. Let us use the Holstein model (17) for simplicity. The inequality (31) becomes

$$\lambda \geqslant 2k + 1 + \sum_{m=1}^{k+1} \frac{2g^2}{4\pi^2T^2m^2 + \Omega^2}. \tag{33}$$

We see that the first spin-flip solution of the Eliashberg equations appears at $\lambda \geqslant 1$ for $k = 0$ and $T \to \infty$, the next one at $\lambda \geqslant 3$, etc. A similar analysis in the normal and superconducting states at lower T again shows that these solutions appear at $\lambda \gtrsim 1$ and proliferate at larger λ.

Let us also determine the lowest temperature at which normal ($\Delta_n \equiv 0$) solutions with spins antiparallel to their fields exist. It costs least to flip the spins at the first two Matsubara

frequencies ω_{-1} and ω_0. Setting $k = 0$ in Eq. (33) and keeping in mind that $\lambda = g^2/\Omega^2$, we find

$$T \geqslant \frac{\Omega}{2\pi}\sqrt{\frac{\lambda+1}{\lambda-1}}. \tag{34}$$

At strong coupling, this becomes $T \geqslant \Omega/(2\pi)$, which implies the condition $T \gg \Omega/\sqrt{\lambda}$ discussed above. Note also that in the strong coupling limit $\Omega \to 0$, the quantity λT diverges as g^2/Ω or stronger when T satisfies the inequality (34).

We argue in Sec. VIII B that spin-flip solutions are important for understanding the short-time nonlinear dynamics of strongly coupled superconductors: most of them are unstable fixed points of kinetic equations, similar to an inverted pendulum, that generate solitonlike waves with a rich structure. As mentioned in the Introduction, the Migdal-Eliashberg theory breaks down at a certain finite λ_c. Interestingly, both the emergence of the spin-flip saddle points at $\lambda < \lambda_c$ and the breakdown of the theory originate from the same source. We saw in this section that it is the divergent $n = m$ terms in the self-energies that bring about the spin-flip solutions. In a subsequent paper [23] we will see that these terms are also among the main culprits responsible for the breakdown of the Migdal-Eliashberg theory. In this connection, let us mention the mountain pass theorem [29] according to which there must be a saddle point between two minima. Therefore, it may be that the emergence of these saddle points indicates that the free energy develops an additional minimum aside from the one described by the Eliashberg equations.

V. SPINS AND SUPERCONDUCTING GAP FUNCTION

In this section, we introduce the superconducting gap function Δ_n, write a generalized gap equation that accounts for spin flips, and relate spin components to the gap function.

We introduce the Eliashberg gap function $\Delta(\omega_n) \equiv \Delta_n$ through equations

$$\omega_n + \Sigma_n = \omega_n Z_n, \quad \Phi_n = \Delta_n Z_n. \tag{35}$$

Equations (23) and (30) imply

$$\mathrm{sgn}(Z_n) = e_n. \tag{36}$$

In other words, Z_n is negative if spin $\boldsymbol{S}_n$ is flipped and positive otherwise. As far as we are aware, only solutions with positive Z_n, i.e., with no spin flips, have been considered in the literature until now.

Since Σ_n is odd and $|\Phi_n|$ is even in frequency, Z_n and $|\Delta_n|$ are both even. With the substitution (35), the Eliashberg equation for Σ_n in (6) determines Z_n for a given Δ_n,

$$Z_n = 1 + \frac{\pi T}{\omega_n}\sum_m \lambda_{nm}\frac{e_m\omega_m}{\sqrt{\omega_m^2 + |\Delta_m|^2}}, \tag{37}$$

while the equation for Φ_n becomes the gap equation

$$\omega_n\Delta_n = \pi T\sum_m \lambda_{nm}\frac{\omega_n e_m\Delta_m - \Delta_n e_m\omega_m}{\sqrt{\omega_m^2 + |\Delta_m|^2}}, \tag{38}$$

where we used Eq. (37) to eliminate Z_n. As soon as the sign of Z_n is fixed, the gap equation decouples from Z_n and the Eliashberg equations reduce to a single equation: the gap equation (38). The gap equation and the gap function determine various thermodynamic properties of the system, for example, the specific heat, superconducting T_c, and the condensation energy. Continued to real frequencies, the gap function determines the density of states and various response functions, such as the optical conductivity.

The $m = n$ term vanishes in the gap equation (38). This equation is therefore divergence free in the limit $\lambda_{nn} = \lambda \to \infty$. The same is not true for Z_n which diverges in this limit because the $m = n$ term does not cancel from Eq. (37). To get rid of this divergence, we use the reduced self-energies defined in Sec. IV and introduce Z_n' and Δ_n through

$$\omega_n + \Sigma_n' = \omega_n Z_n', \quad \Phi_n' = \Delta_n Z_n'. \tag{39}$$

Equation (29) guarantees that Δ_n here is the same as in Eq. (35). Therefore, the gap equation remains the same, while the expression for Z_n' is

$$Z_n' = 1 + \frac{\pi T}{\omega_n}\sum_{m\neq n} \lambda_{nm}\frac{e_m\omega_m}{\sqrt{\omega_m^2 + |\Delta_m|^2}}. \tag{40}$$

The only difference between Z_n' and Z_n is that the $m = n$ term is absent in Z_n'.

Spin components in terms of the gap function are

$$S_n^z = \frac{e_n\omega_n}{\sqrt{\omega_n^2 + |\Delta_n|^2}}, \quad S_n^+ = \frac{e_n\Delta_n}{\sqrt{\omega_n^2 + |\Delta_n|^2}}, \tag{41}$$

as follows from Eqs. (11) and (13). Since the interaction is purely ferromagnetic, it is clear that for fixed S_n^z the free energy is minimal when the xy projections of all spins $\boldsymbol{S}_n^\perp$ point in the same direction. Without loss of generality we take this direction to be the x axis. The spin Hamiltonian (15) becomes

$$H_s = -2\pi\sum_n \omega_n\cos\theta_n - \pi^2 T\sum_{nm}\lambda_{nm}\cos(\theta_n - \theta_m), \tag{42}$$

where θ_n is the angle the spin $\boldsymbol{S}_n$ makes with the z axis, $S_n^z = \cos\theta_n$, and $S_n^x = \sin\theta_n$. The stationary point of H_s with respect to θ_n is

$$\omega_n\sin\theta_n = \pi T\sum_m \lambda_{nm}\sin(\theta_m - \theta_n). \tag{43}$$

With the help of Eq. (41), we see that this is nothing but the gap equation (38) written in terms of the angular variable θ_n and that $\Delta_n = \omega_n\tan\theta_n$. Since $|\sin(\theta_m - \theta_n)| \leqslant 1$ and $\lambda_{nm} \leqslant g^2/(\omega_n - \omega_m)^2$ [see Eq. (B15)], the right hand side of Eq. (43) is bounded in absolute value. It follows that $\sin\theta_n \to 0$ when $\omega_n \to \pm\infty$. Feeding this information into the right hand side, we see that moreover $\omega_n\sin\theta_n \to 0$ and therefore $\Delta_n \to 0$ as $\omega_n \to \pm\infty$.

Let us also mention here various expressions for the free energy of boson-mediated superconductors that have been proposed over the years. Eliashberg derived one such expression in 1962 [30] (see also Ref. [31]). Notably, Bardeen and Stephen used Eliashberg's result to evaluate the corrections to the BCS condensation energy due to finite ratio of the BCS gap to the Debye energy [32]. We discovered the spin-chain representation of the free energy (17) in 2002 announcing it afterwards at various venues [33,34]. To reconcile our result with Eliashberg's, Haslinger and Chubukov obtained a somewhat less general (with all $e_n = +1$) answer for the free

energy of the Holstein model in terms of Δ_n by integrating Eliashberg's expression over the electron momentum [35]. This answer follows from Eq. (17) if we substitute Eq. (41) with $e_n = +1$ into it. Subsequently, this answer has been used in a number of publications.

It is important to emphasize that our expression (3) for the free energy is more general than Eq. (17) and therefore Eliashberg's expression. Equation (3) takes into account certain fluctuations of the fields Σ and Φ in imaginary time which Eq. (17) does not. The two expressions are equivalent only at $\lambda = \infty$, while away from this limit they coincide only at the stationary points. We will discuss this in more detail below.

VI. STATIONARY POINTS OF THE FREE ENERGY

This section contains several results regarding stationary points of the free energy. One is that at the minimum of the free energy $\Delta(\omega_n) = e^{i\phi}|\Delta_n|$, where $|\Delta_n|$ is even in ω_n and $e^{i\phi}$ is an overall phase. In other words, up to an overall phase $\Delta(\omega_n)$ is non-negative and even. Another is that any spin configuration where $\Delta(\omega_n)$ is real, but changes sign, cannot be a local or global minimum and is higher in energy than the global minimum by an amount proportional to the system size. In particular, spin flips correspond to saddle points. Fixing the overall phase, we also show that there cannot be a continuous family of stationary points connected to the global minimum and argue based on symmetry that the global minimum is unique.

We have seen in Sec. IV that stationary points of the free energy are the equilibria of the spin chain. At weak coupling only equilibria where each spin is parallel to its magnetic field ($e_n = +1$) are stationary points. As λ increases, equilibrium configurations with arbitrary number of spins antiparallel to their fields ($e_n = -1$) become stationary points as well. These spin-flip solutions of the Eliashberg equations are necessarily saddle points. Indeed, suppose $\boldsymbol{S}_n$ is antiparallel to $\boldsymbol{b}_n$. Then, its contribution $h_n = -\boldsymbol{b}_n \cdot \boldsymbol{S}_n$ to the spin Hamiltonian is positive and decreases when we rotate $\boldsymbol{S}_n$ keeping all other spins fixed. Given that there are also spins parallel to their fields, the contribution of any of them increases when it deviates from its equilibrium position. Therefore, such solutions cannot be local minima or maxima, but are saddle points.

We see that at a minimum all spins must be parallel to their fields. This condition is only necessary, but not sufficient, as there can be a collective mode involving many spins that lowers the free energy. This happens, for example, when the normal state loses stability at $T = T_c$. In the normal state all spins are parallel to their fields and yet it is a saddle point and not a minimum below T_c. Hence, for the minimum we take $e_n = +1$ for all n in Eq. (41):

$$S_n^z = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta_n|^2}}, \quad S_n^+ = \frac{\Delta_n}{\sqrt{\omega_n^2 + |\Delta_n|^2}}, \tag{44}$$

and

$$\omega_n \Delta_n = \pi T \sum_m \lambda_{nm} \frac{\omega_n \Delta_m - \Delta_n \omega_m}{\sqrt{\omega_m^2 + |\Delta_m|^2}} \tag{45}$$

is the corresponding gap equation.

Before we proceed to analyze the global minimum, it is helpful to summarize the main features of the spin-chain representation of the free energy f:

(a) The spin-chain formula $f = \nu_0 T H_s$ for the free energy holds at stationary points of f. It does not necessarily hold away from such points because we used the Eliashberg equations (6) in deriving it. However, the strong coupling limit $\lambda \to \infty$ is an exception. As we show in Sec. VII, in this limit $f = \nu_0 T H_s$ holds at every point (G_n, F_n) of the configuration space of the system, not only at the stationary points.

(b) Every stationary point of f is also a stationary point of H_s and every stationary point of H_s with $e_n = +1$ is a stationary point of f. Most importantly, the global minimum of f is the global minimum of H_s. We also saw in Sec. IV that in the $\lambda \to \infty$ limit e_n are unconstrained and, therefore, the correspondence between stationary points of f and H_s is one to one.

For easier reference, let us write the spin-chain Hamiltonian (19) here in a slightly different form,

$$H_s = -2\pi \sum_n \omega_n S_n^z - \pi^2 T \sum_{nm} \lambda_{nm} (\boldsymbol{S}_n \cdot \boldsymbol{S}_m - 1), \tag{46}$$

and remind that $\lambda_{nm} > 0$ for all n and m. In Eq. (46) we regularized the otherwise divergent interaction term by subtracting a constant from it [36]. This does not affect the stationary point (gap) equation in any way.

A. Global minimum of the free energy

Here we establish the following properties of the free energy of the electron-phonon system:

(1) At the global minimum $\Delta(\omega_n)$ is a non-negative and even function of ω_n up to an overall phase. Moreover, either $\Delta(\omega_n) = 0$ for all n (normal state) or $\Delta(\omega_n) > 0$ for all n (superconducting).

(2) For a fixed overall phase, there are no curves of stationary points that contain the global minimum.

(3) The only symmetry of the spin chain is the symmetry with respect to rotations around the z axis.

(4) Any $\Delta(\omega_n)$ that is real, but changes sign with ω_n, cannot be a minimum (local or global) of H_s.

These statements remain true at arbitrary temperature, including $T = 0$ [38]. Let us begin with the proof of property 1. Suppose S_n^z are fixed. Then, we have to minimize the xy part of the spin Hamiltonian. Clearly, since $\lambda_{nm} > 0$ for all n and m, this requires all $\boldsymbol{S}_n^\perp$ being aligned in the same direction. Any deviation from this alignment means a finite-energy cost in H_s and because the total free energy is $Nf = \nu_0 N T H_s$, the total-energy cost is proportional to the total number of sites N. As before we designate the direction in which $\boldsymbol{S}_n^\perp$ align to be the positive x direction [39]. Equation (44) then implies that Δ_n is real and, most importantly, it must be non-negative for all n at the global minimum. Since $|\Delta_n|$ is even, $\Delta_n = |\Delta_n|$ must also be an even function of frequency. Therefore, $\Delta(\omega_n)$ is a non-negative and even function. Given that Δ_n is non-negative, the gap equation (45) further shows that either all Δ_n are zero or none.

Clearly, if Δ_n is the global minimum, so is $e^{i\phi}\Delta_n$. Note that $e^{i\phi}\Delta_n$ defines a closed curve (circle) with no end points in the configuration space. Let us show that there can be no other

one-parameter families of solutions of the Eliashberg equations that contain the global minimum [property 2]. Suppose such a family exists. It then similarly defines a curve in the configuration space and both f and H_s must be constant along this curve. It follows that all their derivatives vanish on this curve as well. We saw above that at the global minimum we can always fix the global phase so that $S_n^y = 0$ and $S_n^x \geqslant 0$. Equation (44) shows that $S_n^z \geqslant 0$ for $\omega_n \geqslant 0$. Therefore, the angle θ_n that spin $\boldsymbol{S}_n$ makes with the z axis is in the range $[0, \pi/2]$. Then, the one-parameter family in question must have end points. As all derivatives of H_s vanish along the curve of solutions before the end point and at least some of the first derivatives must be nonzero after the end point, f cannot be an analytic function of all θ_n. This contradicts Eq. (42), which shows that H_s is analytic in all θ_n [40].

The model (46) is an inhomogeneous Heisenberg spin chain. Conditions under which this type of models are integrable for both quantum and classical spins have been studied in the literature [41–46]. In particular, there is a class of integrable models that describe BCS-like pairing between fermions, which become spin chains when written in terms of Anderson spins [44–46]. These models, integrable for both quantum and classical spins, are similar to the Hamiltonian (46) in that there is an inhomogeneous Zeeman field linear in the spin coordinate and interactions between spins are ferromagnetic. Models of this type are integrable only for special choices of coupling constants λ_{nm}. Integrable cases are a subset of measure zero among all Hamiltonians of this form because we need only of order N_s parameters to specify λ_{nm} in integrable cases as opposed to N_s^2 in the generic case, where N_s is the number of spins. It is straightforward to verify that our λ_{nm} do not correspond to any known integrable model. It is safe to conjecture that the Hamiltonian (46) is not integrable in the sense of Ref. [47], i.e., that it does not possess parameter-dependent integrals of motion (parameters here are ω_n and λ_{nm}). Then, any symmetry must Poisson-commute with each term in the Hamiltonian individually, i.e., with each S_n^z and $\boldsymbol{S}_n \cdot \boldsymbol{S}_m$. The only such symmetry is the z component of the total spin $J_z = \sum_n S_n^z$. Thus, we arrive at property 3 assuming nonintegrability of the spin chain (46).

Note that conservation of J_z is responsible for the degeneracy of the global minimum with respect to rotations around the z axis, i.e., with respect to the overall phase of Δ_n. Since there are no other symmetries, we expect the global minimum to be generally unique (barring accidental degeneracy) apart from the arbitrary overall phase of Δ_n.

Now consider a spin configuration with real sign alternating $\Delta(\omega_n)$. We already know from property 1 that such a configuration cannot be a global minimum and is associated with a macroscopic energy cost. Let us investigate if it can at least be a local minimum. Equation (44) implies that the projection of spin $\boldsymbol{S}_n$ onto the xy plane is $\boldsymbol{S}_n^\perp = \mathrm{sgn}(\Delta_n)|\boldsymbol{S}_n^\perp|\hat{\boldsymbol{x}}$, where $\hat{\boldsymbol{x}}$ is the unit vector along the x axis. Let us uniformly rotate all spins with $\mathrm{sgn}(\Delta_n) < 0$ around the z axis towards the positive x axis by a small angle $\delta\phi$, i.e., for these spins $S_n^+ \to S_n^+ e^{i\delta\phi}$. This does not change the Zeeman energy and the interaction energies of spins with $\mathrm{sgn}(\Delta_n) < 0$ among themselves and of spins with $\mathrm{sgn}(\Delta_n) \geqslant 0$ among themselves, while the ferromagnetic interaction energy of $\mathrm{sgn}(\Delta_n) < 0$ spins with $\mathrm{sgn}(\Delta_n) > 0$ ones decreases as they become more aligned with each other. Since the energy decreases for an infinitesimal deviation from this configuration, it cannot be a local minimum of H_s. At $\lambda = \infty$, when $f = \nu_0 T H_s$ holds in the entire configuration space, spin configurations with sign alternating $\Delta(\omega_n)$ cannot be local minima of f as well. This proves property (4).

In connection with the above, let us mention papers by Wu *et al.* that claim that at $T = 0$ and $\lambda = \infty$ there is a one-parameter family of sign-alternating solutions of the Eliashberg equations [19,48,49]. One end point of this purported family of solutions is said to be the global minimum and the other, the normal state, i.e., the free energy must *increase* along this continuous family of stationary points (see, e.g., Fig. 4 of Ref. [19] and Fig. 1 of Ref. [48]). This is impossible because any curve of stationary points must also be a curve of constant free energy, such as, e.g., the curve traced out by changing the global phase ϕ. It also contradicts property (2) above.

Moreover, we saw above that the energy cost of configurations with sign alternating Δ_n is proportional to the system volume. Their Boltzmann weight relative to the global minimum is $e^{-N\Delta f/T}$, where N is the number of sites and Δf is the free-energy density difference (which is finite in $N \to \infty$ limit). Naturally, such states cannot contribute to thermal equilibrium properties of a bulk system. In the present case not only $N \to \infty$, but also $T \to 0$, so the global minimum becomes the ground state and entirely determines all equilibrium physics.

VII. STRONG COUPLING LIMIT

There are two universal limits of the Migdal-Eliashberg theory: the weak and strong coupling limits $\lambda \to 0$ and $\lambda \to \infty$. "Universal" here means that the theory becomes independent of the phonon dispersion and momentum dependence of the electron-phonon coupling. In both limits there is only one low-energy scale. The weak coupling limit of the Migdal-Eliashberg theory is the BCS theory (see Appendix B 2).

We review the strong coupling limit [6,8,50–53] in this section for two related reasons. First, the derivation of the spin-chain representation of the free energy does not rely on Eliashberg equations in this limit and therefore holds everywhere in the configuration space. Second, we will need it in a later study. However, it is important to emphasize that the strong coupling limit of the Migdal-Eliashberg theory is *unphysical* because the theory breaks down at finite λ before reaching $\lambda = \infty$ as discussed in the Introduction. We will use the strong coupling limit as a technical tool only, e.g., to show that the breakdown occurs universally, irrespective of the model electron-phonon Hamiltonian.

Consider the Holstein model first. By definition,

$$\lambda = \frac{g^2}{\Omega^2} = \frac{\nu_0 \alpha^2 M^{-1}}{\Omega^2}, \tag{47}$$

where α is the dimensionful electron-phonon coupling constant. Note that α does not have units of energy, while g does. We define the strong coupling limit as $\alpha \to \infty$. This implies $\lambda \to \infty$ and $\alpha, g \propto \lambda^{1/2}$. As Ω is negligible compared to g, the latter remains the only intrinsic low-energy scale in the problem [54]. Other energies characterizing the superconductor

are proportional to g, for example, $T_c \approx 0.18g$ and the spectral gap is $1.16g$ [6,53]. Measured in units of g these quantities are finite, while the phonon frequency goes to zero, $\Omega \propto \lambda^{-1/2}$. Therefore, an equivalent, but more convenient, way to obtain the strong coupling limit is to keep g fixed and send Ω to zero.

We already saw that Z_n and therefore Φ_n and Σ_n diverge when $\lambda \to \infty$, while F_n and G_n stay finite. Separating the divergent part in the variable change (10), $\Sigma_n = \lambda\pi T G_n + \cdots$, $\Phi_n = \lambda\pi T F_n + \cdots$ and expanding the square root in Eq. (3) in Ω^2, we directly obtain the spin Hamiltonian (15) up to terms of order Ω^2 that vanish in the strong coupling limit. Moreover, it turns out that the mass of fluctuations violating the constraint (12) is infinite [25] and hence $\boldsymbol{S}_n^2 = 1$. Note that here we cast the free energy into the spin-chain form without ever using the stationary point (Eliashberg) equations. Therefore, in the strong coupling limit the spin-chain representation is guaranteed at any point (G_n, F_n) in the configuration space with no stationary point constraints on G_n and F_n. The spin-chain Hamiltonian (17) in the strong coupling limit $\Omega \to 0$ becomes

$$H_s = -2\pi \sum_n \omega_n S_n^z - \pi^2 T g^2 \sum_{nm} \frac{\boldsymbol{S}_n \cdot \boldsymbol{S}_m - 1}{(\omega_n - \omega_m)^2}, \tag{48}$$

where we subtracted a constant from the Hamiltonian as discussed below Eq. (19).

The procedure for dispersing phonons is the same. Now,

$$\lambda(\omega_l) = \frac{1}{2p_F^2} \int_0^{2p_F} \frac{g_q^2 q\, dq}{\omega_l^2 + \omega_q^2},$$

$$\lambda = \lambda(\omega_l = 0) = \frac{1}{2p_F^2} \int_0^{2p_F} \frac{g_q^2 q\, dq}{\omega_q^2} \tag{49}$$

(see Appendix B). Suppose $g_q \to \infty$ for some range of q. Then, the energy g defined as

$$g^2 \equiv \frac{1}{2p_F^2} \int_0^{2p_F} g_q^2 q\, dq \tag{50}$$

diverges and all other low energies are either negligible or proportional to it. This is equivalent to keeping g fixed and sending the phonon frequencies ω_q to zero. In this limit, the spin-chain Hamiltonian (15) for dispersing phonons turns into the Hamiltonian (48), same as for the Holstein model. We see that the spin chain (48) provides a complete and universal description of the the thermodynamics of the strong coupling limit: it determines the Boltzmann weight of any field configuration and is independent of the underlying electron-phonon model except for a single energy g.

VIII. COMPARISON TO ANDERSON PSEUDOSPINS

We already mentioned in Sec. III that the description of the normal and superconducting states and of the transition between them in terms of spins introduced in this paper is very similar to the Anderson pseudospin description of the BCS theory of superconductivity [26]. In particular, our Fig. 1 is identical to Fig. 1 of Ref. [26], except that in the latter figure the cites of the chain are single-fermion energies $\xi_{\boldsymbol{p}}$ instead of fermionic Matsubara frequencies ω_n. Indeed, we will see in this section that equilibrium configurations of Eliashberg and Anderson spins map into each other under the interchange of $\xi_{\boldsymbol{p}}$ with ω_n and of BCS gap Δ with the Eliashberg gap function Δ_n.

An overall principle common to both spin formulations of superconductivity is that there are three real functions of energy or frequency (the normal average and real and imaginary parts of the anomalous average) that admit an interpretation as three components of a classical spin. Nevertheless, Eliashberg and Anderson spins are distinct. Eliashberg spins are energy-integrated normal and anomalous thermal Green's functions. They exist at any temperature and coupling λ. Anderson spins are frequency-integrated normal and anomalous Keldysh Green's functions (see below). They are well defined both in and out of equilibrium, but only at $T = 0$. The two sets of spins do not coincide even where their domains of definition overlap, i.e., at $T = 0$ in the weak coupling (BCS) limit of the Migdal-Eliashberg theory.

A. In and out of equilibrium BCS superconductivity in terms of classical Anderson pseudospins: Brief review

Soon after the publication of the BCS theory, Anderson realized [26] that the BCS Hamiltonian

$$H = \sum_{p\sigma} \xi_{\boldsymbol{p}} c_{p\sigma}^\dagger c_{p\sigma} - \lambda\delta \sum_{pp'} c_{-p\uparrow}^\dagger c_{p\downarrow}^\dagger c_{p'\downarrow} c_{-p'\uparrow}, \tag{51}$$

where $\delta = (\nu_0 N)^{-1}$, maps to a spin-$\frac{1}{2}$ model. We will need quantum averages of pseudospin-$\frac{1}{2}$ operators defined by Anderson [55]

$$s_{\boldsymbol{p}}^z = \frac{\langle c_{p\uparrow}^\dagger c_{p\uparrow} + c_{-p\downarrow}^\dagger c_{-p\downarrow}\rangle - 1}{2}, \quad s_{\boldsymbol{p}}^- = \langle c_{p\downarrow} c_{-p\uparrow}\rangle, \tag{52}$$

where $s_{\boldsymbol{p}}^\pm = s_{\boldsymbol{p}}^x \pm i s_{\boldsymbol{p}}^y$. Note that $\boldsymbol{s}_{\boldsymbol{p}}$ depend on time out of equilibrium, when the state of the system in which we evaluate the averages (52) is itself time-dependent.

In the mean field treatment, which is exact for the BCS model in the thermodynamic limit [56–58], the Hamiltonian (51) becomes [59]

$$H = \sum_{\boldsymbol{p}} 2\xi_{\boldsymbol{p}} s_{\boldsymbol{p}}^z - \lambda\delta \sum_{pp'} s_{\boldsymbol{p}}^+ s_{\boldsymbol{p}'}^-. \tag{53}$$

The eigenstates of the BCS Hamiltonian are product states of the form $\prod_{\boldsymbol{p}} (u_{\boldsymbol{p}} + v_{\boldsymbol{p}} c_{p\uparrow}^\dagger c_{-p\downarrow}^\dagger)|0\rangle$. For any state of this form,

$$s_{\boldsymbol{p}}^z = \frac{|v_{\boldsymbol{p}}|^2 - |u_{\boldsymbol{p}}|^2}{2}, \quad s_{\boldsymbol{p}}^- = u_{\boldsymbol{p}}^* v_{\boldsymbol{p}}. \tag{54}$$

Therefore, the normalization condition $|u_{\boldsymbol{p}}|^2 + |v_{\boldsymbol{p}}|^2 = 1$ implies that $\boldsymbol{s}_{\boldsymbol{p}} = (s_{\boldsymbol{p}}^x, s_{\boldsymbol{p}}^y, s_{\boldsymbol{p}}^z)$ is a vector of length $\frac{1}{2}$. Given the spin configuration, we determine the state of the system using Eq. (54).

Equilibria of the spin Hamiltonian (53) correspond to the eigenstates of the BCS Hamiltonian [26,60–62]. Spin $\boldsymbol{s}_{\boldsymbol{p}}$ experiences an effective magnetic field $\boldsymbol{B}_{\boldsymbol{p}} = (-2\Delta_x, -2\Delta_y, 2\xi_{\boldsymbol{p}})$, where Δ_x and $-\Delta_y$ are the real and imaginary parts of the BCS order parameter defined by

$$\Delta = \lambda\delta \sum_{\boldsymbol{p}} s_{\boldsymbol{p}}^-. \tag{55}$$

In equilibrium the spin is either antiparallel ($e_{\boldsymbol{p}} = +1$) or parallel ($e_{\boldsymbol{p}} = -1$) to its effective field. It follows that equilibrium spin configurations are

$$2s^z_{\boldsymbol{p}} = -\frac{e_{\boldsymbol{p}}\xi_{\boldsymbol{p}}}{\sqrt{\xi^2_{\boldsymbol{p}} + |\Delta|^2}}, \quad 2s^-_{\boldsymbol{p}} = \frac{e_{\boldsymbol{p}}\Delta}{\sqrt{\xi^2_{\boldsymbol{p}} + |\Delta|^2}}. \tag{56}$$

The self-consistency condition (55) becomes

$$2\Delta = \lambda\delta \sum_{\boldsymbol{p}} \frac{e_{\boldsymbol{p}}\Delta}{\sqrt{\xi^2_{\boldsymbol{p}} + |\Delta|^2}}. \tag{57}$$

The BCS ground state is obtained by aligning all spins antiparallel to their fields, i.e., $e_{\boldsymbol{p}} = +1$ for all $\boldsymbol{p}$. States where one of the spins is flipped opposite to its ground-state orientation correspond to excited or "real" pairs in the terminology of Bardeen, Cooper, and Schrieffer [20,26]. Note that Eqs. (56) and (57) describe two types of equilibrium configurations: configurations with $\Delta = 0$ [which is always a solution of Eq. (57)] and configurations where $\Delta \neq 0$. In the terminology of Ref. [62], these are the *normal* and *anomalous* states, respectively.

Classical Anderson pseudospins play a central role in understanding collisionless dynamics of BCS condensates in response to fast perturbations [59,61–67] when the sample size is of the order of the superconducting coherence length or smaller [64]. Hamilton's equations of motion for the classical spin Hamiltonian (53) with the usual angular momentum Poisson brackets for components of $\boldsymbol{s}_{\boldsymbol{p}}$ are equivalent to the time-dependent Bogoliubov–de Gennes equations [61]. These classical spin equations of motion have been used by Anderson to analyze the collective modes of a BCS superconductor and more recently to study its time evolution after a quantum quench [59,62,63,65–67]. Normal and anomalous states mentioned in the preceding paragraph play a special role in the dynamics [61]. Normal states are dynamically unstable and anomalous states are unstable when sufficiently many spins are flipped. These unstable equilibria give rise to normal and anomalous multisolitons that start in an unstable equilibrium at $t = -\infty$ and return into it at $t = +\infty$, similar to an inverted pendulum. Moreover, dynamics for many physical initial conditions (e.g., for an interaction quench) can be described in terms of multisolitons.

Let us also mention the relationship between Anderson pseudospins and equal time, zero-temperature Keldysh Green's functions defined by [68–70]

$$\begin{aligned} \mathsf{G}_{\boldsymbol{p}}(t,t') &= -i\langle[c_{\boldsymbol{p}\sigma}(t), c^\dagger_{\boldsymbol{p}\sigma}(t')]\rangle, \\ \mathsf{F}_{\boldsymbol{p}}(t,t') &= -i\langle[c_{\boldsymbol{p}\uparrow}(t), c_{-\boldsymbol{p}\downarrow}(t')]\rangle, \end{aligned} \tag{58}$$

where the square brackets denote the commutator and we assumed time-reversal symmetry, so that $\mathsf{G}_{\boldsymbol{p}}(t,t')$ does not depend on σ. We see that $G_{\boldsymbol{p}}(t,t) = 2is^z_{\boldsymbol{p}}$ and $F_{\boldsymbol{p}}(t,t) = 2is^-_{\boldsymbol{p}}$. Fourier transforming the Green's functions with respect to $t' - t$, we find

$$2is^z_{\boldsymbol{p}} = \int_{-\infty}^{\infty} d\omega\, \mathsf{G}_{\boldsymbol{p}}(\omega), \quad 2is^-_{\boldsymbol{p}} = \int_{-\infty}^{\infty} d\omega\, \mathsf{F}_{\boldsymbol{p}}(\omega). \tag{59}$$

We suppressed the dependence on t in spin components $\mathsf{G}_{\boldsymbol{p}}(\omega)$ and $\mathsf{F}_{\boldsymbol{p}}(\omega)$. Thus, we see that Anderson spins are frequency-integrated normal and anomalous Keldysh Green's functions (divided by $2i$).

B. Strongly coupled superconductors in and out of equilibrium in terms of Eliashberg spins

Let us first compare equilibrium configurations of Eliashberg and Anderson spins. In equilibrium, Eliashberg spins are [Eq. (41)]

$$S^z_n = \frac{e_n\omega_n}{\sqrt{\omega^2_n + |\Delta_n|^2}}, \quad S^+_n = \frac{e_n\Delta_n}{\sqrt{\omega^2_n + |\Delta_n|^2}}. \tag{60}$$

We see that they turn into Anderson spins (56), up to a factor of 2 and opposite sign of y and z components, if we replace the Matsubara frequency ω_n with the single-particle energy $\xi_{\boldsymbol{p}}$ and the Eliashberg gap function Δ_n with the BCS gap Δ. The factor of 2 difference comes from normalization, the arbitrary choice of spin length $\frac{1}{2}$ for Anderson spins and 1 for Eliashberg spins. Opposite signs of y and z components of spins are a similar "gauge" degree of freedom that does not affect the Poisson brackets between spin components. It is straightforward to redefine Anderson or Eliashberg spins to eliminate these differences. However, the gap equations (57) and (38) are rather different due to the retarded nature of interactions and the presence of zz interactions in the spin formulation of the Migdal-Eliashberg theory.

There are also similarities between the two approaches in the overall physical picture they provide. For example, in both cases the normal state is a state where all spins are down (up) below a certain energy and up (down) above it as shown in Fig. 1 and spins acquire xy components in the superconducting state thus softening the domain wall at zero frequency (energy).

Nevertheless, the two sets of spins never coincide. Anderson spins are ill defined away from $\lambda \to 0$, $T = 0$ limit, i.e., in the Migdal-Eliashberg theory at finite λ or finite T, in the sense that the model cannot be formulated entirely in terms of them and the length of the vector $\boldsymbol{s}_{\boldsymbol{p}}$ depends on $\boldsymbol{p}$ and is not conserved by the dynamics. In contrast, Eliashberg spins remain well defined in the BCS ($\lambda \to 0$) limit and, in particular, the classical spin Hamiltonian (46) minus its value in the normal state gives the BCS condensation energy as a function of T. Even in the BCS limit the two sets of spins are distinct and provide alternative descriptions of the system: Eliashberg spins determine the free energy as a function of T, while Anderson spins work at $T = 0$ and determine the ground state and excited states of the Hamiltonian.

While Anderson spins are frequency-integrated Keldysh Green's functions [see Eq. (59)], Eliashberg spins correspond to energy-integrated normal and anomalous Matsubara Green's functions

$$\begin{aligned} \mathcal{G}_{\sigma\boldsymbol{p}}(\tau - \tau') &= -\langle T_\tau c_{\boldsymbol{p}\sigma}(\tau) c^\dagger_{\boldsymbol{p}\sigma}(\tau')\rangle, \\ \mathcal{F}_{\boldsymbol{p}}(\tau - \tau') &= \langle T_\tau c_{-\boldsymbol{p}\downarrow}(\tau) c_{\boldsymbol{p}\uparrow}(\tau')\rangle. \end{aligned} \tag{61}$$

We have evaluated these Green's functions in the Matsubara frequency domain in Appendix A 4:

$$\begin{aligned} \mathcal{G}_{\boldsymbol{p}n} &= -\frac{i(\omega_n + \Sigma_n) + \xi_{\boldsymbol{p}} + \chi_n}{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2 + (\chi_n + \xi_{\boldsymbol{p}})^2}, \\ \mathcal{F}_{\boldsymbol{p}n} &= -\frac{\Phi_n}{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2 + (\chi_n + \xi_{\boldsymbol{p}})^2}. \end{aligned} \tag{62}$$

Integrating $\mathcal{G}_{\boldsymbol{p}n}$ and $\mathcal{F}_{\boldsymbol{p}n}$ and comparing with the expressions (11) for Eliashberg spins on the stationary point, we see that

$$S_n^z = -i\int_{-\infty}^{\infty} d\xi_{\boldsymbol{p}}\mathcal{G}_{\boldsymbol{p}n}, \quad S_n^+ = -\int_{-\infty}^{\infty} d\xi_{\boldsymbol{p}}\mathcal{F}_{\boldsymbol{p}n}. \tag{63}$$

Note that the single-particle energy $\xi_{\boldsymbol{p}}$ plays the same role for Eliashberg spins as the frequency ω does for Anderson spins.

We explained in the previous subsection how Anderson spins describe the far from equilibrium dynamics of a BCS condensate. Likewise, we expect the time evolution of Eliashberg spins with the Hamiltonian (46) to describe the dynamics of electronic degrees of freedom of a strongly coupled (Eliashberg) superconductor in a similar regime. Hamiltonian (Bloch) equations of motion for Eliashberg spins read as

$$\frac{d\boldsymbol{S}_n}{dt} = -\boldsymbol{b}_n \times \boldsymbol{S}_n,$$
$$\boldsymbol{b}_n = 2\pi\omega_n\hat{\boldsymbol{z}} + 2\pi^2 T\sum_{m\neq n}\lambda_{nm}\boldsymbol{S}_m. \tag{64}$$

These equations or a certain version of them should be valid for the short time far from equilibrium dynamics, at least when the sample size is of the order of the coherence length or smaller. In this case, dynamics are spatially uniform [64], phonons are in thermal equilibrium with the outside environment [71], and collision integrals are negligible [72]. Indeed, kinetic equations very similar to Eq. (64) have already been derived by Eliashberg [71] (see also Ref. [72]). In the weak coupling limit $\lambda_{nm} \to \lambda$ at relevant frequencies and, with the replacement of ω_n with $\xi_{\boldsymbol{p}}$, Eq. (64) is equivalent to the Bloch equation for Anderson spins $\boldsymbol{s}_{\boldsymbol{p}}$ defined by Eq. (52).

The spin-flip solutions we identified in Sec. IV B play a special role in the dynamics. These saddle points of free energy are by construction stationary solutions of the Bloch equation (64). Just as with flipped Anderson spins discussed below Eq. (57), these equilibria are dynamically unstable when sufficiently many spins are flipped and there are two types of them, normal and anomalous, depending on whether or not the xy components of spins are zero. Therefore, we anticipate that these saddle points should give rise to dynamics analogous to normal and anomalous solitons in the far from equilibrium BCS superconductivity.

IX. CONCLUSION

In this paper, we mapped the Migdal-Eliashberg theory of superconductivity to a classical Heisenberg spin chain in Zeeman magnetic field. Lattice sites are fermionic Matsubara frequencies ω_n. The interaction between spins is purely ferromagnetic. It depends on the phonon spectrum and falls off as $(\omega_n - \omega_m)^{-2}$ at large separation between spins. The Zeeman field is proportional to ω_n, the spin coordinate along the chain. As an example, the spin Hamiltonian for the Holstein model is

$$H_s = -2\pi\sum_n \omega_n S_n^z - \pi^2 T g^2 \sum_{nm}\frac{\boldsymbol{S}_n\cdot\boldsymbol{S}_m - 1}{(\omega_n-\omega_m)^2+\Omega^2}, \tag{65}$$

where T is the temperature, g is the electron-phonon coupling, Ω is the renormalized Einstein phonon frequency, and $\boldsymbol{S}_n^2 = 1$. The free-energy density of the system of fermions interacting through phonons is $f = \nu_0 T H_s$.

The spin-chain formulation made the analysis of the free-energy functional simple. In particular, we saw that, up to an overall phase $e^{i\phi}$, the Eliashberg order parameter $\Delta(\omega_n)$ is non-negative and even at the global minimum and there can be no continuous families of stationary points that include the global minimum. We also discussed symmetry and non-integrability arguments that further support our claim that the global minimum is unique.

It became apparent that the free energy acquires infinitely many new saddle points (which are additional solutions of the Eliashberg equations) at strong coupling, which correspond to spin flips. We saw that these saddle points most likely play a significant role in the collisionless far from equilibrium dynamics of strongly coupled superconductors. Apparently, they are the fixed points of the corresponding kinetic equations and give rise to rich multisoliton dynamics. These saddle points emerge and proliferate just before the breakdown of the Migdal-Eliashberg theory and, moreover, these two phenomena have similar origin as we discussed in Sec. IV B. It is evident that the spin chain has unstable equilibria where a number of spins are flipped against their magnetic fields. However, without the spin-chain representation these solutions are hard to notice and indeed they have not been seen before.

Our spin-flip solutions of Eliashberg equations are unrelated to the continuous family of sign-alternating solutions for Δ_n conjectured by Wu *et al.* [19] at $\lambda = \infty$. We believe that this conjecture is internally inconsistent in several ways. Therefore, the claims by the aforementioned reference of vanishing of the superconducting T_c, destruction of the superconducting phase coherence, etc., are unsubstantiated. Thinking in terms of the spin chain also made it clear that any sign alternating Δ_n costs a macroscopic amount of energy to create at all λ and therefore cannot contribute to thermal equilibrium properties of a macroscopic system. Most importantly, because as we discussed above the Migdal-Eliashberg theory breaks down at finite λ, its strong coupling limit is *unphysical* altogether.

Classical spins in the Migdal-Eliashberg theory (Eliashberg spins) are in many ways similar to Anderson spins in the BCS theory of superconductivity, though the two sets of spins never coincide. Common to both spin notions is that three real numbers, the normal Green's function and real and imaginary parts of the anomalous Green's function integrated over single-fermion energy (Eliashberg) or frequency (Anderson), become the three components of a classical spin vector. Eliashberg spins exist at any temperature and any λ, while Anderson spins are well defined in and out of equilibrium, but only at $T = 0$ and in the BCS limit of the Migdal-Eliashberg theory. Equilibrium configurations of Anderson and Eliashberg spins map into each other under the interchange of single-fermion energy $\xi_{\boldsymbol{p}}$ with fermionic Matsubara frequency ω_n and of the BCS gap Δ with Eliashberg gap function Δ_n. An interesting open question is to study the short-time far from equilibrium dynamics of strongly coupled conventional superconductors with the help of Eliashberg spins.

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APPENDIX A: ELIASHBERG FREE ENERGY FOR THE HOLSTEIN MODEL WITH ARBITRARY SINGLE-PARTICLE POTENTIAL

In this Appendix, we derive the Eliashberg free energy from the path integral for the Holstein model (see also Refs. [73,74]). For the most part the derivation follows a standard sequence of steps. One notable difference is that we keep the single-electron potential arbitrary throughout the entire calculation and rewrite the action in its eigenbasis. This will be especially useful later, in our study of the breakdown of the Eliashberg theory at finite λ [23].

1. Effective action

The Lagrangian corresponding to the Holstein Hamiltonian (1) is

$$L=\sum_{ij,\sigma}\psi^*_{i\sigma}G^{-1}_{0ij}\psi_{j\sigma}+\sum_i\left[\frac{M\Omega_0^2\varphi_i^2}{2}+\frac{M(\partial_\tau\varphi_i)^2}{2}\right]+\alpha\sum_{i\sigma}\psi^*_{i\sigma}\psi_{i\sigma}\varphi_i+\sum_{i\sigma}(J^*_{i\sigma}\psi_{i\sigma}+\psi^*_{i\sigma}J_{i\sigma}), \tag{A1}$$

where $G^{-1}_{0ij}=\partial_\tau\delta_{ij}+h_{ij}-\mu\delta_{ij}$, $\psi_{i\sigma}$ and $\psi^*_{i\sigma}$ are Grassmann variables, and φ_i is a real field that corresponds to the ion displacement operator x_i. We also introduced a chemical potential μ and Grassmann source fields $J^*_{i\sigma}$ and $J_{i\sigma}$, which we will use to evaluate Green's functions. All fields in the Lagrangian depend on the imaginary time τ.

Integrating out phonons, we obtain the following effective action:

$$S_{\text{eff}}=\sum_{ij\tau\sigma}\psi^*_{i\tau\sigma}G^{-1}_{0ij}\psi_{j\tau\sigma}-\frac{1}{2}\sum_{i\tau\tau'\sigma\sigma'}\psi^*_{i\tau'\sigma'}\psi_{i\tau'\sigma'}D_{\tau'\tau}\psi^*_{i\tau\sigma}\psi_{i\tau\sigma}+\sum_{i\tau\sigma}(J^*_{i\tau\sigma}\psi_{i\tau\sigma}+\psi^*_{i\tau\sigma}J_{i\tau\sigma}). \tag{A2}$$

Summations indicate summation over $\boldsymbol{i}$ and integration over τ and τ',

$$D_{\tau'\tau}=T\sum_l\frac{\alpha^2M^{-1}}{\omega_l^2+\Omega^2}e^{i\omega_l(\tau'-\tau)}, \tag{A3}$$

is the effective electron-electron interaction, and $\omega_l=2\pi Tl$ are bosonic Matsubara frequencies. The interaction is proportional to the phonon propagator as usual [3].

We replaced the bare phonon frequency Ω_0 with the renormalized frequency Ω. To obtain an equation for Ω within the path-integral framework, one needs to introduce an additional Hubbard-Stratonovich field $\Pi_i(\tau',\tau)$ for phonons [73]. On the stationary point this leads to the usual phonon renormalization procedure (see Ref. [3], p. 181). We reject it here because Holstein and other standard electron-phonon Hamiltonians do not renormalize phonons correctly [75,76]. Instead, we treat Ω as a parameter of the model. Renormalized phonon frequencies are generally momentum dependent even when the bare spectrum is dispersionless. Then, the effective action is that for dispersing phonons, which we treat in Appendix B. Here we disregard the momentum dependence of Ω for simplicity: Eq. (A2) is a legitimate model of the electron-phonon system in its own right, no less legitimate than the Holstein model itself.

We decouple the four-fermion term in Eq. (A2) with a Hubbard-Stratonovich transformation in the particle-particle and particle-hole channels with three fields $\Phi_i(\tau',\tau)$, $\Sigma_{i\uparrow}(\tau',\tau)$, and $\Sigma_{i\downarrow}(\tau',\tau)$:

$$S_{\text{eff}}=\sum_i\iint d\tau' d\tau\left[\frac{\Phi^*_i(\tau',\tau)\Phi_i(\tau',\tau)}{D(\tau'-\tau)}+\sum_\sigma\frac{\Sigma_{i\sigma}(\tau',\tau)\Sigma_{i\sigma}(\tau,\tau')}{2D(\tau'-\tau)}\right]+\sum_{x'x}\Psi^\dagger_{x'}M_{x'x}\Psi_x+\sum_x(K^\dagger_x\Psi_x+\Psi^\dagger_xK_x), \tag{A4}$$

where Ψ_x and K_x are two-component Nambu fields, $M_{x'x}$ is a 2×2 matrix in the Nambu space,

$$\Psi_x=\begin{bmatrix}\psi_{\uparrow x}\\ \psi^*_{\downarrow x}\end{bmatrix},\quad K_x=\begin{bmatrix}J_{i\uparrow}(\tau)\\ -J^*_{i\downarrow}(\tau)\end{bmatrix},\quad M_{x'x}=\begin{bmatrix}G^{-1}_{0ij}\delta(\tau'-\tau)-i\Sigma_{i\uparrow}(\tau',\tau)\delta_{ij} & \Phi_i(\tau',\tau)\delta_{ij}\\ \Phi^*_i(\tau,\tau')\delta_{ij} & \bar{G}^{-1}_{0ij}\delta(\tau'-\tau)+i\Sigma_{i\downarrow}(\tau,\tau')\delta_{ij}\end{bmatrix}, \tag{A5}$$

and $\bar{G}^{-1}_{0ij}=\partial_\tau\delta_{ij}-h_{ji}+\mu\delta_{ij}$. By definition the fields $\Sigma_{i\sigma}(\tau',\tau)$ are Hermitian

$$\Sigma_{i\sigma}(\tau',\tau)=\Sigma^*_{i\sigma}(\tau,\tau'). \tag{A6}$$

The statement here is that integrating $e^{-S_{\text{eff}}}$ with S_{eff} from Eq. (A4) over the fields $\Phi_i(\tau',\tau)$ and $\Sigma_{i\sigma}(\tau',\tau)$, we obtain Eq. (A2). On the stationary point, the fields $\Phi_i(\tau',\tau)$ and $\Sigma_{i\sigma}(\tau',\tau)$ depend only on the difference $\tau'-\tau$ and represent the anomalous and normal self-energies (see below).

The action (A4) is quadratic in fermion fields. Performing the Gaussian integral over these fields, we obtain the effective action for the fields $\Phi_i(\tau',\tau)$ and $\Sigma_{i\sigma}(\tau',\tau)$:

$$S_{\text{eff}}=\sum_i\iint d\tau' d\tau\left[\frac{\Phi^*_i(\tau',\tau)\Phi_i(\tau',\tau)}{D(\tau'-\tau)}+\sum_\sigma\frac{\Sigma_{i\sigma}(\tau',\tau)\Sigma_{i\sigma}(\tau,\tau')}{2D(\tau'-\tau)}\right]-\operatorname{Tr}\ln M-\sum_{x'x}K^\dagger_{x'}M^{-1}_{x'x}K_x. \tag{A7}$$

The action is a scalar matrix trace. It is convenient to transform the matrices and vectors in S_{eff} to fermionic Matsubara frequencies ω_n and the eigenbasis of the single-electron

Hamiltonian $\hat{h}$ with a unitary matrix

$$U_{\tau n,i\alpha} = \sqrt{T}\begin{bmatrix} \pi_{i\alpha} e^{-i\tau\omega_n} & 0 \\ 0 & \pi_{i\alpha} e^{-i\tau\omega_n} \end{bmatrix}, \tag{A8}$$

which leaves S_{eff} invariant. Here π_α are the eigenstates of $\hat{h}$,

$$\sum_j h_{ij}\pi_{j\alpha} = \epsilon_\alpha \pi_{i\alpha}. \tag{A9}$$

For example, the matrix M transforms into $(U^\dagger M U)_{nm,\alpha\beta} = \sum_{ij}\int\int d\tau' d\tau\, U^\dagger_{n\tau,\alpha i} M_{i\tau,i'\tau'} U_{\tau' m,i'\beta}$ and similarly $K \to U^\dagger K$ and $\Psi \to U^\dagger \Psi$. Carrying out this transformation in Eq. (A7), we find

$$S_{\text{eff}} = T\sum_{nml\alpha\beta}\left[\left(\Phi^{\alpha\beta}_{n+l,m+l}\right)^* D_l^{-1}\Phi^{\alpha\beta}_{nm} + \frac{1}{2}\sum_\sigma \Sigma^{\alpha\beta}_{\sigma,m+l,n+l} D_l^{-1}\Sigma^{\beta\alpha}_{\sigma nm}\right] - \text{Tr ln}\, M - TK^\dagger M^{-1} K. \tag{A10}$$

Here D_l^{-1} is the bosonic Matsubara frequency ω_l component of $1/D(\tau)$,

$$\Phi^{\alpha\beta}_{nm} = T\sum_i \iint d\tau' d\tau\, e^{i\omega_n\tau'}\pi^*_{i\alpha}\Phi_i(\tau',\tau)\pi_{i\beta}e^{-i\omega_m\tau}, \tag{A11}$$

and similarly for $\Sigma_{\sigma nm}$. Hermiticity of $\Sigma_{i\sigma}(\tau',\tau)$ implies Hermiticity of $\Sigma^{\alpha\beta}_{\sigma nm}$ since the transformation is unitary. The matrix M in this basis reads as

$$M = \begin{bmatrix} (-i\omega_n + \xi_\alpha)\delta_{nm} - i\Sigma^{\alpha\beta}_{\uparrow nm} & \Phi^{\alpha\beta}_{nm} \\ \left(\Phi^{\beta\alpha}_{mn}\right)^* & (-i\omega_n - \xi_\alpha)\delta_{nm} + i\Sigma^{\alpha\beta}_{\downarrow,-m,-n} \end{bmatrix}, \tag{A12}$$

and the source fields are

$$K_{\alpha m} = \begin{bmatrix} J_{\uparrow\alpha m} \\ -J^*_{\downarrow,\alpha',-m} \end{bmatrix}, \quad J_{\uparrow\alpha m} = \sum_i \int d\tau\, e^{i\omega_m\tau}\pi^*_{i\alpha}J_{i\uparrow}(\tau),$$
$$J_{\downarrow\alpha' m} = \sum_i \int d\tau\, e^{i\omega_m\tau}(\pi^*_{i\alpha})^* J_{i\downarrow}(\tau). \tag{A13}$$

Note that α' labels the state π^*_α, which is related to π_α by time-reversal operation and $A_{-m,-n}$ stands for $A(-\omega_m,-\omega_n)$.

2. Stationary point

The Eliashberg equations are a stationary point of the effective action, such that the fields $\Phi_i(\tau',\tau)$ and $\Sigma_{i\sigma}(\tau',\tau)$ are spatially uniform and depend only on the time difference,

$$\Phi_i(\tau',\tau) = \Phi(\tau'-\tau), \quad \Sigma_{i\sigma}(\tau'-\tau) = \Sigma_\sigma(\tau'-\tau). \tag{A14}$$

Equation (A11) then implies

$$\Phi^{\alpha\beta}_{nm} = \Phi_n\delta_{nm}\delta_{\alpha\beta}, \quad \Sigma^{\alpha\beta}_{\sigma nm} = \Sigma_{\sigma n}\delta_{nm}\delta_{\alpha\beta}. \tag{A15}$$

In words, off-diagonal matrix elements are zero and diagonal ones depend only on the Matsubara frequency.

The derivative of the action (A10) with respect to any off-diagonal matrix element is a sum of terms each of which is proportional to one of the other off-diagonal matrix elements. This means that setting these matrix elements to zero automatically solves all stationary point equations for them. To determine stationary point equations for the diagonal matrix elements, we substitute Eq. (A15) into Eq. (A10). Setting also the sources to zero, we obtain

$$S_{\text{eff}} = TN\sum_{nl}\left[\Phi^*_{n+l}D_l^{-1}\Phi_n + \Sigma_{n+l}D_l^{-1}\Sigma_n - \chi_{n+l}D_l^{-1}\chi_n\right] - \sum_{n\alpha}\ln[(\omega_n+\Sigma_n)^2 + |\Phi_n|^2 + (\chi_n+\xi_\alpha)^2], \tag{A16}$$

where N is the number of sites (which is equal to the number of the single-electron states π_α) and

$$\Sigma_n = \frac{\Sigma_{\uparrow n} - \Sigma_{\downarrow,-n}}{2}, \quad i\chi_n = \frac{\Sigma_{\uparrow n} + \Sigma_{\downarrow,-n}}{2}, \tag{A17}$$

where A_{-n} means $A(-\omega_n)$. Minimizing the effective action with respect to Φ^*_n, Σ_n, and χ_n, we derive three Eliashberg equations

$$\Phi_n = \frac{T}{N}\sum_{m\alpha} D_{n-m}\frac{\Phi_m}{(\omega_m+\Sigma_m)^2 + |\Phi_m|^2 + (\chi_m+\xi_\alpha)^2},$$
$$\Sigma_n = \frac{T}{N}\sum_{m\alpha} D_{n-m}\frac{\omega_m+\Sigma_m}{(\omega_m+\Sigma_m)^2 + |\Phi_m|^2 + (\chi_m+\xi_\alpha)^2},$$
$$\chi_n = -\frac{T}{N}\sum_{m\alpha} D_{n-m}\frac{\xi_\alpha+\chi_m}{(\omega_m+\Sigma_m)^2 + |\Phi_m|^2 + (\chi_m+\xi_\alpha)^2}, \tag{A18}$$

where

$$D_{n-m} = \frac{\alpha^2 M^{-1}}{(\omega_n-\omega_m)^2 + \Omega^2}. \tag{A19}$$

When ξ_α are symmetric with respect to zero, $\chi_n \equiv 0$ solves the last equation in (A18).

As is normally done in the Migdal-Eliashberg theory, we now send the Fermi energy to infinity and take the density of

states per site per spin orientation ν_0 to be constant. We implement this limit in Eq. (A16) by integrating the logarithm over ξ_α from $-\mathbb{\Lambda}$ to $\mathbb{\Lambda}$, discarding the constant term that depends only on $\mathbb{\Lambda}$, and then taking the limit $\mathbb{\Lambda}\to\infty$. The result is

$$S_{\rm eff} = TN\sum_{nl}\left[\Phi^*_{n+l}D_l^{-1}\Phi_n + \Sigma_{n+l}D_l^{-1}\Sigma_n\right] - 2\pi\nu_0 N\sum_n\sqrt{(\omega_n+\Sigma_n)^2+|\Phi_n|^2}. \tag{A20}$$

Eliashberg equations become

$$\Phi_n = \pi T\sum_m \lambda(\omega_n-\omega_m)\frac{\Phi_m}{\sqrt{(\omega_m+\Sigma_m)^2+|\Phi_m|^2}},$$
$$\Sigma_n = \pi T\sum_m \lambda(\omega_n-\omega_m)\frac{\omega_m+\Sigma_m}{\sqrt{(\omega_m+\Sigma_m)^2+|\Phi_m|^2}}. \tag{A21}$$

with

$$\lambda(\omega_n-\omega_m) = \frac{g^2}{(\omega_n-\omega_m)^2+\Omega^2}, \quad g^2 = \nu_0\alpha^2 M^{-1}. \tag{A22}$$

The fermionic part of the free-energy functional per site is $f = TS_{\rm eff}/N$,

$$f = \nu_0 T^2\sum_{nl}[\Phi^*_{n+l}\Lambda_l\Phi_n + \Sigma_{n+l}\Lambda_l\Sigma_n] - 2\pi\nu_0 T\sum_n\sqrt{(\omega_n+\Sigma_n)^2+|\Phi_n|^2}, \tag{A23}$$

where Λ_l is the discrete Fourier transform of $1/\lambda(\tau)$ at bosonic Matsubara frequency ω_l.

3. Spatially nonuniform stationary point

For future reference let us also consider spatially nonuniform solutions

$$\Phi_i(\tau',\tau) = \Phi_i(\tau'-\tau), \quad \Sigma_{i\sigma}(\tau'-\tau) = \Sigma_{i\sigma}(\tau'-\tau). \tag{A24}$$

Let Φ_{in} be the Fourier transform of $\Phi_i(\tau'-\tau)$ with respect to $\tau'-\tau$. Its matrix elements in the eigenbasis of the single-particle Hamiltonian h_{ij} are $\Phi_n^{\alpha\beta} = \sum_i \pi^*_{i\alpha}\Phi_{in}\pi_{i\beta}$. If the eigenstates $\pi_{i\alpha}$ are localized or delocalized but highly oscillatory, the off-diagonal, $\alpha\neq\beta$, matrix elements are negligible. In this case, a suitable ansatz for the stationary point is

$$\Phi^{\alpha\beta}_{nm} = \Phi^\alpha_n\delta_{nm}\delta_{\alpha\beta}, \quad \Sigma^{\alpha\beta}_{\sigma nm} = \Sigma^\alpha_{\sigma n}\delta_{nm}\delta_{\alpha\beta}. \tag{A25}$$

The derivation of the stationary point equations is similar to the uniform case. The effective action now reads as

$$S_{\rm eff} = T\sum_{nl\alpha}\left[\left(\Phi^\alpha_{n+l}\right)^* D_l^{-1}\Phi^\alpha_n + \Sigma^\alpha_{n+l}D_l^{-1}\Sigma^\alpha_n - \chi^\alpha_{n+l}D_l^{-1}\chi^\alpha_n\right] - \sum_{n\alpha}\ln\left[\left(\omega_n+\Sigma^\alpha_n\right)^2+|\Phi^\alpha_n|^2+\left(\chi^\alpha_n+\xi_\alpha\right)^2\right], \tag{A26}$$

where

$$\Sigma^\alpha_n = \frac{\Sigma^\alpha_{\uparrow n}-\Sigma^\alpha_{\downarrow,-n}}{2}, \quad i\chi^\alpha_n = \frac{\Sigma^\alpha_{\uparrow n}+\Sigma^\alpha_{\downarrow,-n}}{2}. \tag{A27}$$

It is incorrect to minimize $S_{\rm eff}$ with respect to Φ^α_n because $\Phi^\alpha_n = \sum_i|\pi_{i\alpha}|^2\Phi_{in}$ are not independent. The original independent variables are Φ_{in} and the matrix $R_{i\alpha} = |\pi_{i\alpha}|^2$ that relates Φ^α_n to Φ_{in} is degenerate. For example, the completeness relation $\sum_\alpha \pi_{i\alpha}\pi_{j\alpha} = \delta_{ij}$ implies $\sum_\alpha R_{i\alpha} = 1$ meaning that the columns of the matrix R are linearly dependent. Similarly, the normalization condition $\sum_i|\pi_{i\alpha}|^2 = 1$ says that the rows of R are linearly dependent too. The same applies to Σ^α_n and χ^α_n.

We should instead minimize Eq. (A26) with respect to Φ_{in}, Σ_{in}, and χ_{in} using

$$\frac{\partial\Phi^\alpha_n}{\partial\Phi_{in}} = |\pi_{i\alpha}|^2, \text{ etc.} \tag{A28}$$

We obtain

$$\sum_\alpha\Phi^\alpha_n|\pi_{i\alpha}|^2 = T\sum_{m\alpha}D_{n-m}\frac{\Phi^\alpha_m|\pi_{i\alpha}|^2}{\left(\omega_m+\Sigma^\alpha_m\right)^2+\left|\Phi^\alpha_m\right|^2+\left(\chi^\alpha_m+\xi_\alpha\right)^2},$$
$$\sum_\alpha\Sigma^\alpha_n|\pi_{i\alpha}|^2 = T\sum_{m\alpha}D_{n-m}\frac{\left(\omega_m+\Sigma^\alpha_m\right)|\pi_{i\alpha}|^2}{\left(\omega_m+\Sigma^\alpha_m\right)^2+\left|\Phi^\alpha_m\right|^2+\left(\chi^\alpha_m+\xi_\alpha\right)^2},$$
$$\sum_\alpha\chi^\alpha_n|\pi_{i\alpha}|^2 = -T\sum_{m\alpha}D_{n-m}\frac{\left(\xi_\alpha+\chi^\alpha_m\right)|\pi_{i\alpha}|^2}{\left(\omega_m+\Sigma^\alpha_m\right)^2+\left|\Phi^\alpha_m\right|^2+\left(\chi^\alpha_m+\xi_\alpha\right)^2}. \tag{A29}$$

If the single-particle Hamiltonian h_{ij} has the periodicity of the lattice, $\pi_{i\alpha}$ are plane waves and $|\pi_{i\alpha}|^2 = N^{-1}$. Then, $\Phi^\alpha_n = \Phi_n$, $\Sigma^\alpha_n = \Sigma_n$, $\chi^\alpha_n = \chi_n$, and we recover the usual Eliashberg equations (A18).

4. Green's functions

Let us evaluate the Fourier transforms $\mathcal{G}_{\sigma\alpha n}$ and $\mathcal{F}_{\alpha n}$ of the normal and anomalous thermal Green's functions at the stationary point. The latter are defined as

$$\mathcal{G}_{\sigma\alpha}(\tau-\tau') = -\langle T_\tau c_{\sigma\alpha}(\tau)c^\dagger_{\sigma\alpha}(\tau')\rangle,$$
$$\mathcal{F}_\alpha(\tau-\tau') = \langle T_\tau c_{\downarrow\alpha'}(\tau)c_{\uparrow\alpha}(\tau')\rangle. \tag{A30}$$

We obtain $\mathcal{G}_{\sigma\alpha n}$ and $\mathcal{F}_{\alpha n}$ by differentiating the partition function $\mathcal{Z}$ with respect to the source fields

$$T\mathcal{G}_{\sigma\alpha n} = \frac{1}{\mathcal{Z}}\frac{\partial^2\mathcal{Z}}{\partial J_{\sigma\alpha n}\partial J^*_{\sigma\alpha n}}, \quad T\mathcal{F}_{\alpha n} = \frac{1}{\mathcal{Z}}\frac{\partial^2\mathcal{Z}}{\partial J^*_{\downarrow\alpha' n}\partial J^*_{\uparrow\alpha,-n}}. \tag{A31}$$

The source fields enter through the $K^\dagger M^{-1} K$ term in the effective action (A10). At the stationary point this term becomes

$$K^\dagger M^{-1} K = \sum_{\alpha n} K^\dagger_{n\alpha} M^{-1}_{\alpha n} K_{n\alpha} = \sum_{\alpha n} \frac{1}{\Theta_{\alpha n}} [a^+_{\alpha n} J^*_A J_A - \Phi_n J^*_A J^*_{A'} - (\Phi_n)^* J_{A'} J_A + a^-_{\alpha n} J_{A'} J^*_{A'}], \tag{A32}$$

where $a^\pm_{\alpha n} = i(\omega_n + \Sigma_n) \pm (\xi_\alpha + \chi_n)$, the labels A and A' stand for $\uparrow \alpha n$ and $\downarrow \alpha', -n$, respectively, and $\Theta_{\alpha n} = (\omega_n + \Sigma^\alpha_n)^2 + |\Phi^\alpha_n|^2 + (\chi^\alpha_n + \xi_\alpha)^2 = -\det M_{\alpha n}$.

The source-dependent part of the partition function at the stationary point is $\mathcal{Z}_s = e^{-TK^\dagger M^{-1} K}$. Using Eq. (A31), we find

$$\mathcal{G}_{\uparrow\alpha n} = -\frac{i(\omega_n + \Sigma_n) + \xi_\alpha + \chi_n}{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2 + (\chi_n + \xi_\alpha)^2},$$
$$\mathcal{G}_{\downarrow\alpha n} = -\frac{i(\omega_n - \Sigma_{-n}) + \xi_\alpha + \chi_{-n}}{(\omega_n - \Sigma_{-n})^2 + |\Phi_{-n}|^2 + (\chi_{-n} + \xi_\alpha)^2},$$
$$\mathcal{F}_{\alpha n} = -\frac{\Phi_{-n}}{(\omega_n - \Sigma_{-n})^2 + |\Phi_{-n}|^2 + (\chi_{-n} + \xi_\alpha)^2}. \tag{A33}$$

Suppose there is a time-reversal symmetry so that $\Sigma_{\uparrow n} = \Sigma_{\downarrow n}$ and $\mathcal{G}_{\uparrow\alpha n} = \mathcal{G}_{\downarrow\alpha n}$. Equation (A17) then implies that Σ_n is odd in frequency and χ_n is even. It further follows from the above formulas for $\mathcal{G}_{\uparrow\alpha n}$ and $\mathcal{G}_{\downarrow\alpha n}$ that $|\Phi_n|$ is even and we arrive at the usual expressions for the thermal Green's functions in the Migdal-Eliashberg theory:

$$\mathcal{G}_{\uparrow\alpha n} = \mathcal{G}_{\downarrow\alpha n} \equiv \mathcal{G}_{\alpha n} = -\frac{i(\omega_n + \Sigma_n) + \xi_\alpha + \chi_n}{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2 + (\chi_n + \xi_\alpha)^2},$$
$$\mathcal{F}_{\alpha n} = -\frac{\Phi_n}{(\omega_n + \Sigma_n)^2 + |\Phi_n|^2 + (\chi_n + \xi_\alpha)^2}. \tag{A34}$$

APPENDIX B: PATH-INTEGRAL FORMULATION OF THE MIGDAL-ELIASHBERG THEORY FOR DISPERSING PHONONS

Here we derive Eliashberg equations and free energy for arbitrary phonon spectrum and momentum-dependent electron-phonon coupling. We will see that these quantities have the same form as for the Holstein model, but with a more general kernel which now involves an integral over the phonon spectrum. Our starting point is the standard electron-phonon Hamiltonian

$$H = \sum_{\boldsymbol{p}\sigma} \xi_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma} c_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{q}} \omega_0(\boldsymbol{q}) b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} + \frac{1}{\sqrt{N}} \sum_{\boldsymbol{p}\boldsymbol{q}\sigma} \frac{\alpha_{\boldsymbol{q}}}{\sqrt{2M\omega_0(\boldsymbol{q})}} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\sigma} c_{\boldsymbol{p}\sigma} [b^\dagger_{-\boldsymbol{q}} + b_{\boldsymbol{q}}]. \tag{B1}$$

The Hermitian property requires $\alpha_{\boldsymbol{q}} = \alpha^*_{-\boldsymbol{q}}$. For simplicity, we take $\alpha_{\boldsymbol{q}}$ and $\omega_0(\boldsymbol{q})$ to depend only on the magnitude of the momentum $\boldsymbol{q}$. Integrating out phonons, we obtain an effective action for the fermionic fields

$$S_{\text{eff}} = T \sum_{\mathsf{p}\sigma} \psi^*_{\mathsf{p}\sigma} G^{-1}_{0\mathsf{p}} \psi_{\mathsf{p}\sigma} - \frac{1}{2} \frac{T^3}{N} \sum_{\mathsf{p}\mathsf{p}'\mathsf{q}\sigma\sigma'} \frac{|\alpha_q|^2 M^{-1}}{\omega_m^2 + \omega_q^2} \times \psi^*_{\mathsf{p}+\mathsf{q}\sigma} \psi_{\mathsf{p}\sigma} \psi^*_{\mathsf{p}'-\mathsf{q}\sigma'} \psi_{\mathsf{p}'\sigma'}, \tag{B2}$$

where $G^{-1}_{0\mathsf{p}} = -i\omega_n + \xi_{\boldsymbol{p}}$, and $\mathsf{p} = (\omega_n, \boldsymbol{p})$; $\mathsf{q} = (\omega_m, \boldsymbol{q})$ are the frequency-momentum 4-vectors. As in Eq. (A3) we replace the bare phonon frequencies $\omega_0(q)$ with the renormalized frequencies ω_q in the denominator of the effective electron-electron interaction.

It is helpful to rewrite the interaction as

$$\frac{1}{2} \sum_{\sigma\sigma'} \frac{T^2}{N} \frac{|\alpha_q|^2 M^{-1}}{\omega_n^2 + \omega_q^2} \psi^*_{\mathsf{p}+\mathsf{q}\sigma} \psi_{\mathsf{p}\sigma} \psi^*_{\mathsf{p}'-\mathsf{q}\sigma'} \psi_{\mathsf{p}'\sigma'} = \psi^*_{\mathsf{p}'_1\uparrow} \psi^*_{-\mathsf{p}'_2\downarrow} D^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \psi_{-\mathsf{p}_2\downarrow} \psi_{\mathsf{p}_1\uparrow} - \frac{1}{2} \psi^*_{\mathsf{p}'_1\uparrow} \psi_{\mathsf{p}'_2\uparrow} D^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \psi^*_{\mathsf{p}_2\uparrow} \psi_{\mathsf{p}_1\uparrow} - \frac{1}{2} \psi^*_{\mathsf{p}'_1\downarrow} \psi_{\mathsf{p}'_2\downarrow} D^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \psi^*_{\mathsf{p}_2\downarrow} \psi_{\mathsf{p}_1\downarrow}, \tag{B3}$$

where we treat D as a matrix with matrix elements

$$D^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} = D_{\mathsf{q}} \delta_{\mathsf{p}'_1, \mathsf{p}_1+q} \delta_{\mathsf{p}'_2, \mathsf{p}_2+\mathsf{q}}, \quad D_{\mathsf{q}} = \frac{T^2}{N} \frac{|\alpha_q|^2 M^{-1}}{\omega_n^2 + \omega_q^2}. \tag{B4}$$

As before, we decouple the interaction with fields Φ and Σ_σ,

$$S_{\text{eff}} = T \sum_{\mathsf{p}'_1\mathsf{p}'_2\mathsf{p}_1\mathsf{p}_2} \left\{ \Phi^*_{\mathsf{p}'_1\mathsf{p}'_2} [D^{-1}]^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \Phi_{\mathsf{p}_1\mathsf{p}_2} + \frac{1}{2} \sum_\sigma \Sigma^*_{\sigma\mathsf{p}'_1\mathsf{p}'_2} [D^{-1}]^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \Sigma_{\sigma p_1 p_2} \right\} + T \sum_{\mathsf{p}_1\mathsf{p}_2} \Psi^\dagger_{\mathsf{p}_1} M_{\mathsf{p}_1\mathsf{p}_2} \Psi_{\mathsf{p}_2}, \tag{B5}$$

where

$$M_{\mathsf{p}_1\mathsf{p}_2} = \begin{pmatrix} [-i\omega_n + \xi_{\boldsymbol{p}_1}]\delta_{\mathsf{p}_1\mathsf{p}_2} - i\Sigma_{\uparrow\mathsf{p}_1\mathsf{p}_2} & \Phi_{\mathsf{p}_1\mathsf{p}_2} \\ \Phi^*_{\mathsf{p}_2\mathsf{p}_1} & [-i\omega_n - \xi_{\boldsymbol{p}_1}]\delta_{\mathsf{p}_1\mathsf{p}_2} + i\Sigma_{\downarrow,-\mathsf{p}_2,-\mathsf{p}_1} \end{pmatrix}, \quad \Psi_p = \begin{pmatrix} \psi_{\mathsf{p}\uparrow} \\ \psi^*_{-\mathsf{p}\downarrow} \end{pmatrix}. \tag{B6}$$

Integrating out the fermions, we find

$$S_{\text{eff}} = T \sum_{\mathsf{p}'_1\mathsf{p}'_2\mathsf{p}_1\mathsf{p}_2} \left\{ \Phi^*_{\mathsf{p}'_1\mathsf{p}'_2} [D^{-1}]^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \Phi_{\mathsf{p}_1\mathsf{p}_2} + \frac{1}{2} \sum_\sigma \Sigma^*_{\sigma\mathsf{p}'_1\mathsf{p}'_2} [D^{-1}]^{\mathsf{p}'_1\mathsf{p}'_2}_{\mathsf{p}_1\mathsf{p}_2} \Sigma_{\sigma p_1 p_2} \right\} - \text{Tr} \ln M. \tag{B7}$$

1. Stationary point

It is reasonable to expect that in a translationally invariant system, the stationary point is also translationally invariant. Translational invariance means that matrices $\Phi_{\mathsf{p}_1\mathsf{p}_2}$ and $\Sigma_{\sigma\mathsf{p}_1\mathsf{p}_2}$ are diagonal,

$$\Phi_{\mathsf{p}_1\mathsf{p}_2} = \Phi_{\mathsf{p}_1} \delta_{\mathsf{p}_1\mathsf{p}_2}, \quad \Sigma_{\sigma\mathsf{p}_1\mathsf{p}_2} = \Sigma_{\sigma\mathsf{p}_1} \delta_{\mathsf{p}_1\mathsf{p}_2}, \tag{B8}$$

since $\mathsf{p}_1 = \mathsf{p}_2$ ensures that $e^{ip_1x_1 - ip_2x_2} = e^{ip_1(x_1 - x_2)}$ depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\tau_1 - \tau_2$.

Setting the off-diagonal matrix elements to zero in the effective action (B7), we obtain

$$S_{\rm eff} = T\sum_{\mathsf{p}'\mathsf{p}}\left\{\Phi^*_{\mathsf{p}'}[D^{-1}]^{\mathsf{p}'}_{\mathsf{p}}\Phi_{\mathsf{p}} + \Sigma_{\mathsf{p}'}[D^{-1}]^{\mathsf{p}'}_{\mathsf{p}}\Sigma_{\mathsf{p}} - \chi_{\mathsf{p}'}[D^{-1}]^{\mathsf{p}'}_{\mathsf{p}}\chi_{\mathsf{p}}\right\} - \sum_{\mathsf{p}}\ln[(\omega_n+\Sigma_{\mathsf{p}})^2 + |\Phi_p|^2 + (\chi_{\mathsf{p}}+\xi_{\boldsymbol{p}})^2], \tag{B9}$$

where

$$\Sigma_{\mathsf{p}} = \frac{\Sigma_{\uparrow\mathsf{p}} - \Sigma_{\downarrow,-\mathsf{p}}}{2}, \quad i\chi_{\mathsf{p}} = \frac{\Sigma_{\uparrow\mathsf{p}} + \Sigma_{\downarrow,-\mathsf{p}}}{2}. \tag{B10}$$

Now we evaluate the stationary point of this effective action with respect to Φ^*_{p}, Σ_{p}, and χ_{p} and multiply the resulting expression by matrix D on both sides. The result is

$$\begin{aligned}
\Phi_{n\boldsymbol{p}} &= \frac{T}{N}\sum_{m\boldsymbol{q}}\frac{|\alpha_q|^2M^{-1}}{(\omega_n-\omega_m)^2+\omega_q^2}\frac{\Phi_{m\boldsymbol{p}'}}{(\omega_m+\Sigma_{m\boldsymbol{p}'})^2+|\Phi_{m\boldsymbol{p}'}|^2+(\chi_{m\boldsymbol{p}'}+\xi_{\boldsymbol{p}'})^2},\\
\Sigma_{n\boldsymbol{p}} &= \frac{T}{N}\sum_{m\boldsymbol{q}}\frac{|\alpha_q|^2M^{-1}}{(\omega_n-\omega_m)^2+\omega_q^2}\frac{\omega_m+\Sigma_{m\boldsymbol{p}'}}{(\omega_m+\Sigma_{m\boldsymbol{p}'})^2+|\Phi_{m\boldsymbol{p}'}|^2+(\chi_{m\boldsymbol{p}'}+\xi_{\boldsymbol{p}'})^2},\\
\chi_{n\boldsymbol{p}} &= -\frac{T}{N}\sum_{m\boldsymbol{q}}\frac{|\alpha_q|^2M^{-1}}{(\omega_n-\omega_m)^2+\omega_q^2}\frac{\chi_{m\boldsymbol{p}'}+\xi_{\boldsymbol{p}'}}{(\omega_m+\Sigma_{m\boldsymbol{p}'})^2+|\Phi_{m\boldsymbol{p}'}|^2+(\chi_{m\boldsymbol{p}'}+\xi_{\boldsymbol{p}'})^2},
\end{aligned} \tag{B11}$$

where $\boldsymbol{p}' = \boldsymbol{p}+\boldsymbol{q}$. These are the momentum-dependent Eliashberg equations for the Hamiltonian (B1) [cf. Eq. (A18) for the Holstein model].

We look for an isotropic solution of Eq. (B11), so that $\Phi_{n\boldsymbol{p}}$, $\Sigma_{n\boldsymbol{p}}$, and $\chi_{n\boldsymbol{p}}$ are independent of the direction of $\boldsymbol{p}$. Then, it is useful to convert the summation over $\boldsymbol{q}$ in Eq. (B11) into an integral over q and p', where q and p' are the magnitudes of vectors $\boldsymbol{q}$ and $\boldsymbol{p}'$. Using $p'^2 = p^2+q^2+2pq\cos\theta$, we evaluate the Jacobian for the change of variables $(q,\cos\theta)\to(q,p')$ and find

$$\sum_{\boldsymbol{q}} = \frac{V}{4\pi^2}\iint q^2dq\,d\cos\theta = \frac{V}{4\pi^2}\frac{1}{p}\int q\,dq\int p'dp'. \tag{B12}$$

Since the right-hand side of Eq. (B11) is a product of a function of q only and a function of p' only, the integration over $\boldsymbol{q}$ factorizes into a product of an integral over q and an integral over p'.

In the Migdal-Eliashberg theory, the fluctuations around the stationary point are negligible only when the Fermi energy is much larger than any other characteristic energy [1,2]. This means that the integration is confined to the vicinity of the Fermi surface, which we take to be spherical, and $p\approx p'\approx p_F$, where p_F is the Fermi momentum. Since $\boldsymbol{q}$ connects $\boldsymbol{p}$ and $\boldsymbol{p}'$, which are both on the Fermi surface, its magnitude varies between 0 and $2p_F$. This allows as to rewrite Eq. (B12) as

$$\sum_{\boldsymbol{q}} = \frac{V}{4\pi^2}\frac{1}{p_F^2}\int_0^{2p_F} q\,dq\int p'^2dp' = \frac{1}{2p_F^2}\int_0^{2p_F} q\,dq\sum_{\boldsymbol{p}'}. \tag{B13}$$

Using this in Eq. (B11), we observe $\Phi_{n\boldsymbol{p}}$, $\Sigma_{n\boldsymbol{p}}$, and $\chi_{n\boldsymbol{p}}$ are independent of the momentum $\boldsymbol{p}$ and performing the summation over $\boldsymbol{p}'$ we arrive at the following two equations:

$$\begin{aligned}
\Phi_n &= \pi T\sum_m \lambda(\omega_n-\omega_m)\frac{\Phi_m}{\sqrt{(\omega_m+\Sigma_m)^2+|\Phi_m|^2}},\\
\Sigma_n &= \pi T\sum_m \lambda(\omega_n-\omega_m)\frac{\omega_m+\Sigma_m}{\sqrt{(\omega_m+\Sigma_m)^2+|\Phi_m|^2}},
\end{aligned} \tag{B14}$$

where

$$\lambda(\omega_l) = \frac{1}{2p_F^2}\int_0^{2p_F}\frac{g_q^2 q\,dq}{\omega_l^2+\omega_q^2}, \quad g_q^2 = |\alpha_q|^2M^{-1}. \tag{B15}$$

This expression is for a spherical Fermi surface in $d=3$ dimensions, but it is straightforward to extend it to any $d\geqslant 2$. Equations (B14) are of the same form as Eqs. (A21), but with a more general kernel $\lambda(\omega_n-\omega_m)$. For dispersionless phonons, $\omega_q=\Omega$ and $g_q=g$, this kernel too coincides with that in Eq. (A21). By definition the dimensionless coupling constant for dispersing phonons is

$$\lambda = \lambda(\omega_l=0) = \frac{1}{2p_F^2}\int_0^{2p_F}\frac{g_q^2 q\,dq}{\omega_q^2}. \tag{B16}$$

The effective action evaluated at the stationary point is [cf. Eq. (A20)]

$$S_{\rm eff} = T\nu_0V\sum_{nl}[\Phi^*_{n+l}\Lambda_l\Phi_n + \Sigma_{n+l}\Lambda_l\Sigma_n] - 2\pi\nu_0V\sum_n\sqrt{(\omega_n+\Sigma_n)^2+|\Phi_n|^2}. \tag{B17}$$

Here Λ_l is the Fourier transform of $1/\lambda(\tau)$ at frequency ω_l. The corresponding free energy $f = TS_{\rm eff}/N$ per lattice site is

$$f = \nu_0T^2\sum_{nl}\left[\Phi^*_{n+l}\lambda_l^{-1}\Phi_n + \Sigma_{n+l}\lambda_l^{-1}\Sigma_n\right] - 2\pi\nu_0T\sum_n\sqrt{(\omega_n+\Sigma_n)^2+|\Phi_n|^2}. \tag{B18}$$

2. BCS theory as the weak coupling limit of the Migdal-Eliashberg theory

Confusion exists in the literature to this day regarding whether or not the BCS theory is the weak coupling limit of the Migdal-Eliashberg theory [21,22]. This confusion arises from attempts to compare cutoff-dependent quantities which are illegitimate within the BCS theory. The BCS theory is only valid in the limit $\Delta_{\rm BCS}/\Xi \to 0$, where $\Delta_{\rm BCS}$ is the BCS ground-state gap and Ξ is the cutoff of the order of the Debye energy for acoustic phonons and Ω_0 for Einstein phonons (Holstein model).

In other words, one should take the limit $\Xi \to \infty$ and the coupling constant $\lambda \to 0$ while keeping $\Delta_{\rm BCS} \propto \Xi e^{-1/\lambda}$ fixed. Only quantities that survive this limit are legitimate within the BCS theory. In particular, it is meaningless to compare $\Delta_{\rm BCS}/\Xi$ or T_c/Ξ, but, for example, $\Delta_{\rm BCS}/T_c$, the condensation energy, the normalized jump in the specific heat are meaningful and their BCS values should agree with those in the weak coupling limit of the Migdal-Eliashberg theory, which they indeed do [8,32,77,78].

It is well-understood how the BCS theory emerges from the Migdal-Eliashberg theory in the weak coupling limit. Here we reproduce known arguments within our path-integral framework. Our aim in doing so is to be able to compare later the weak and strong coupling limits of the Migdal-Eliashberg theory as well as Anderson and Eliashberg spins. We define the weak coupling limit as the limit where the ratio of the energy gap to the characteristic phonon frequency $\omega_{\rm ch}$ goes to zero. This is equivalent to $\omega_{\rm ch} \to \infty$ (cf. $\omega_{\rm ch} = \Omega \to 0$) in the strong coupling limit of the Holstein model [see Eq. (5)].

Since relevant energies and frequencies are of the order of the gap, ω_n in the denominator is negligible in the effective action (B2). Then, the interaction is instantaneous and the effective action corresponds to the Hamiltonian

$$H = \sum_{p\sigma} \xi_p c^\dagger_{p\sigma} c_{p\sigma} - \frac{1}{2N} \sum_{pp'q\sigma\sigma'} \frac{\alpha_q^2 M^{-1}}{\omega_q^2} c^\dagger_{p+q\sigma} c_{p\sigma} c^\dagger_{p'-q\sigma'} c_{p'\sigma'}. \tag{B19}$$

The hierarchy of scales $\varepsilon_F \gg \omega_{\rm ch} \gg \Delta_{\rm BCS}$ implies that the pairing interaction is confined to the vicinity of the Fermi surface and that momenta of phonons mediating the interaction are of the order of the Fermi momentum. Therefore, the range of the electron-electron interaction is of order of the Fermi wavelength λ_F. The characteristic length scale (coherence length) in a weakly coupled superconductor is of the order of $v_F/\Delta_{\rm BCS}$, which is orders of magnitude larger than λ_F. We see that the dimensionless range of interactions goes to zero in the weak coupling limit. This means the interaction potential is a delta function in the position space and a constant in the momentum space. Since the pairing occurs at the Fermi surface in this limit, we determine the constant by averaging the interaction over this surface,

$$\text{const} = \frac{1}{2p_F^2} \int_0^{2p_F} \frac{\alpha_q^2 M^{-1} q\, dq}{\omega_q^2} = \frac{\lambda}{\nu_0}, \tag{B20}$$

where we used Eq. (5). This is very similar to how we obtained Eq. (B15) in the Migdal-Eliashberg theory. The interaction in the position basis takes the form $c^\dagger_\sigma(\boldsymbol{r}) c_\sigma(\boldsymbol{r}) c^\dagger_{\sigma'}(\boldsymbol{r}) c_{\sigma'}(\boldsymbol{r})$. For $\sigma = \sigma'$ this is equal to $c^\dagger_\sigma(\boldsymbol{r}) c_\sigma(\boldsymbol{r})$, which we absorb into the chemical potential term. We are left with $\sigma \neq \sigma'$ terms, i.e., with the BCS Hamiltonian

$$H = \sum_{p\sigma} \xi_p c^\dagger_{p\sigma} c_{p\sigma} - \lambda\delta \sum_{pp'q} c^\dagger_{p+q\uparrow} c^\dagger_{p'-q\downarrow} c_{p'\downarrow} c_{p\uparrow}, \tag{B21}$$

where $\delta = (\nu_0 N)^{-1}$ is the single-particle level spacing at the Fermi energy.

Both BCS and Eliashberg theories are mean-field theories. In mean-field approach, only Cooper pairs with zero total momentum determine the ground state and low-lying excitation spectrum. Keeping only such pairs, we end up with the reduced BCS Hamiltonian [20]

$$H = \sum_{p\sigma} \xi_p c^\dagger_{p\sigma} c_{p\sigma} - \lambda\delta \sum_{pp'} c^\dagger_{-p\uparrow} c^\dagger_{p\downarrow} c_{p'\downarrow} c_{-p'\uparrow}. \tag{B22}$$

We expect the above sequence of steps leading to Eq. (B22) to be exact when we take both the weak coupling and thermodynamic limits. The neglect of frequency and momentum dependencies of the electron-electron interaction is valid at energies much smaller than $\omega_{\rm ch}$. It resulted in infinite-range interaction in momentum space between Cooper pairs in Eq. (B22) [or, equivalently, spins in Eq. (53)]. Therefore, this infinite-range interaction has to be cut off at $\Xi \sim \omega_{\rm ch}$. Any quantity to which energies of order $\omega_{\rm ch}$ contribute is beyond the BCS theory.

The weak coupling limit of the Migdal-Eliashberg theory, the BCS theory, is universal in the sense that it is independent of the phonon dispersion and momentum dependence of the electron-phonon coupling. There is only one low-energy scale in this limit: the BCS gap $\Delta_{\rm BCS}$. The gap itself *cannot* be determined accurately from within the BCS theory. However, the ratio of any other energy to the gap, e.g., $T_c/\Delta_{\rm BCS}$ is well defined. The strong coupling limit of the Migdal-Eliashberg theory is similarly universal [53].

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