Relaxation and persistent oscillations of the order parameter in non-stationary BCS theory

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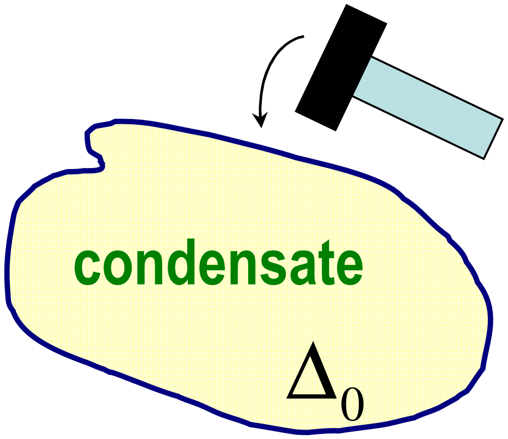
To appear on cond-mat in few days.



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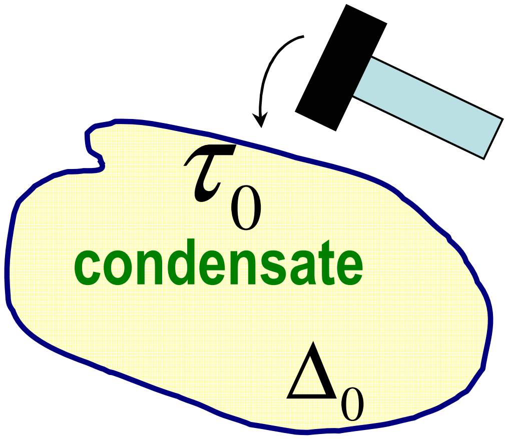
In particular, $\Delta(t) = ?$

Non-adiabatic regime: $\text{perturbation time} \ll 1/\Delta_0$



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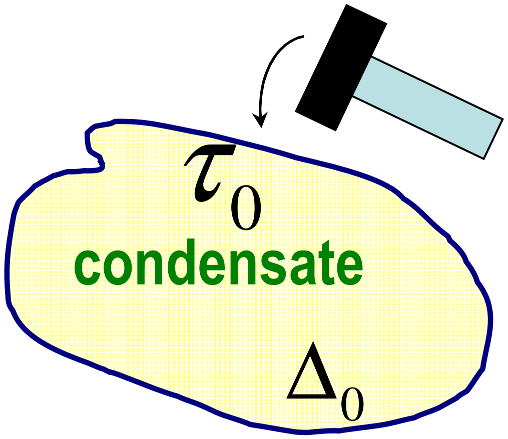
Time - dependent Ginzburg - Landau (fast pair-breaking)

Boltzman kinetic Eq. + gap Eq. (adiabatic regime)

Energy relaxation time $\tau_\varepsilon \gg 1/\Delta_0$



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Time evolution in non-adiabatic regime?

Describe the response of a fermionic condensate to a sudden perturbation at times $t \ll \tau_\varepsilon$

P.W. Anderson, 1958
V.P. Galaiko, 1972
A. F. Volkov & Sh.M. Kogan, 1974
Yu.M. Galperin, V.I. Kozub & B.Z. Spivak, 1981
V.S. Shumeiko, Doctoral Thesis, 1990

R.A. Barankov, L.S. Levitov & B.Z. Spivak, 2004
M.H. Szumanska, B.D. Simons & K. Burnett, 2005
M.H.C. Amin, E.V. Bezuglyi, A.S. Kijko, A.N. Omelyanchouk, 2004
G.L. Warner & A.J. Leggett, 2005

Physical realizations

Feshbach resonance

1) Ultra-cold fermions (^{40}K, ^{6}Li).

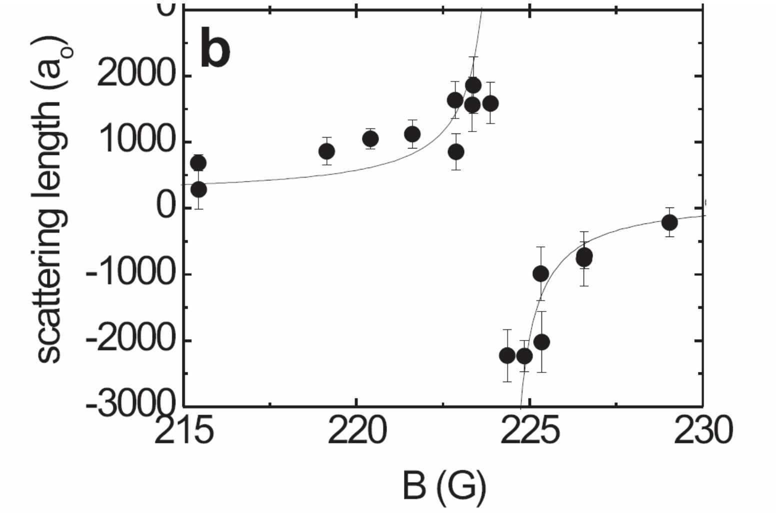
Abrupt change of the BCS coupling constant

$$g \propto a_0$$

$$\boxed{g' \rightarrow g}$$

Slow energy relaxation

$$1/\Delta_0 \simeq 0.1 \text{ ms}$$

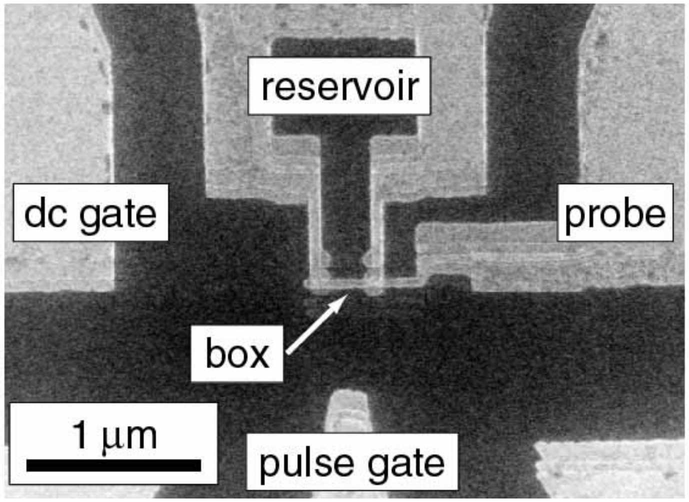


Regal et. al. (JILA, ^{40}K)

2) Superconducting qubits.

Nonequilibrium conditions can be generated by fast voltage pulses

$$1/\Delta_0 \simeq 10 \text{ ps}$$



Nakamura, Pashkin &Tsai

Time evolution in non-adiabatic regime

Problem: At $t<0$ the condensate is prepared in a nonequilibrium state by a sudden perturbation

$$\left|\Psi(t=0)\right\rangle = \left|\text{nonequilibrium state}\right\rangle$$

Determine the time evolution for $t>0$

$$\Psi(t)=?\quad \Delta(t)=?$$

Example: (cold fermions) At $t=0$ $\quad g'\to g$

$$\left|\Psi(t=0)\right\rangle = \left|\text{ground state for coupling } g'\right\rangle$$

(nonequilibrium for the new coupling g)

Time evolution in non-adiabatic regime (short answer)

Depends on the initial state $\Psi_{cond}(t=0)$!!

There are only two types of initial states

Type I: $|\Delta(t)|$ asymptotes to a constant $\Delta_\infty < \Delta_0$ as $t \to \infty$

$$\frac{|\Delta(t)|}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \varphi)}{\sqrt{\Delta_\infty t}}$$

$a \sim 1,\ \ \varphi$ depend on the details of the initial state

This happens e.g. for a sudden change of coupling $g' \to g$ in a paired ground state

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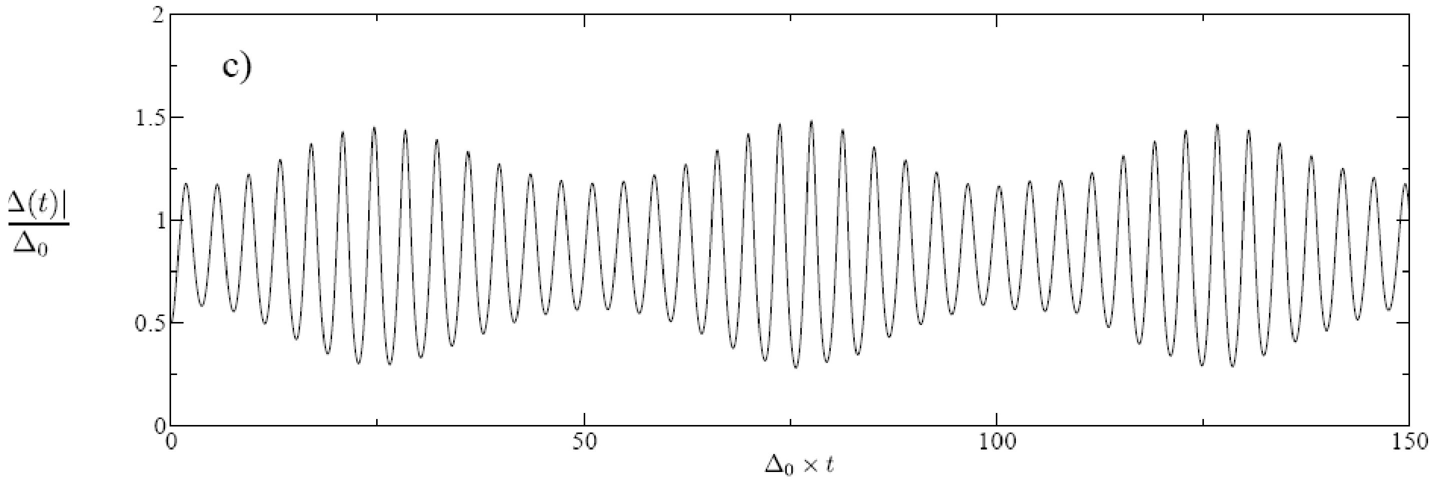
Type II: $|\Delta(t)|$ oscillates persistently with several basic frequencies



Classification of initial states, can predict dynamics from initial state

Time evolution in non-adiabatic regime (model)

- ✓ Dynamics at times $t \ll \tau_\varepsilon$ is non-dissipative
- ✓ Small system or spatially homogeneous initial state

Can use BCS model (in the presence of disorder or trapping potential)

$$H = \sum_j \varepsilon_j n_j - g \sum_{i,j} c^+_{j\uparrow} c^+_{j\downarrow} c_{i\downarrow} c_{i\uparrow}$$

single-particle levels

coupling const

Given $\Psi_{cond}(t=0)$

determine $\Psi_{cond}(t>0)$

i.e. solve time-dependent Shrodinger equation for ***H***

Quantum, many-body, far from equilibrium

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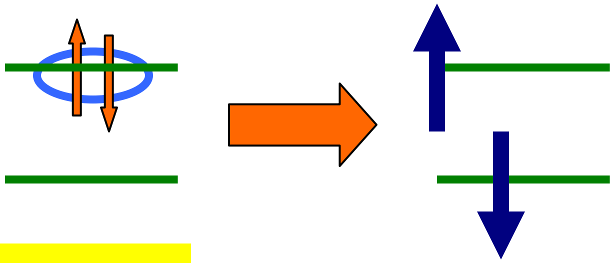
~~Quantum~~, many-body, far from equilibrium

1. Anderson’s spins

$$H = \sum_j \varepsilon_j n_j - g \sum_{i,j} c^+_{j\uparrow} c^+_{j\downarrow} c_{i\downarrow} c_{i\uparrow}$$

$$K^z_j = \frac{n_j - 1}{2}; \quad K^+_j = c^+_{j\uparrow} c^+_{j\downarrow}; \quad K^-_j = c_{j\downarrow} c_{j\uparrow}$$



$$H = \sum_j 2\varepsilon_j K^z_j - g \sum_{i,j} K^+_i K^-_j$$

Infinite range interactions – mean-field is exact in $n \to \infty$ limit

2. In mean-field spins are replaced by $\mathbf{s}_j(t) = \langle \mathbf{K}_j(t) \rangle$ their expectation values. The problem becomes classical.

$$H = \sum_j 2\varepsilon_j s^z_j - g \sum_{i,j} s^+_i s^-_j$$

$$s^-_j = u^*_j v_j \quad 2s^z_j = v^2_j - u^2_j$$

P. W. Anderson, Phys. Rev. 112, 1900 (1958)

Bogoliuobov amplitudes

Time evolution in non-adiabatic regime

Classical Hamiltonian

$$H = \sum_j 2\varepsilon_j s_j^z - g\sum_{i,j} s_i^+ s_j^-$$

$$\Delta = \Delta_x - i\Delta_y = g\sum_j s_j^-$$

$$\{s_j^x, s_j^y\} = -s_j^z$$ Angular momentum Poisson brackets

$$|\mathrm{s}_j| = \mathrm{const}$$

Eqs. of motion: $$\frac{d\mathbf{s}_j}{dt} = (-2\Delta_x, -2\Delta_y, 2\varepsilon_j)\times\mathbf{s}_j$$

Spin distribution $\mathbf{s}_j \equiv \mathbf{s}(\varepsilon_j) \equiv \mathbf{s}(\varepsilon)$ completely determines $\Psi_{cond}(t)$

Problem: Given the initial spin distribution $\mathbf{s}(\varepsilon, t=0)$ determine $\mathbf{s}(\varepsilon, t>0)$. It is sufficient to determine $\Delta(t)$

Time evolution in non-adiabatic regime

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Normally, would be intractable. However, BCS is integrable (Richardson, Gaudin). Exact solution for the dynamics: E.Y., B. Altshuler, V. Kuznetsov, V. Enolskii.

Q: What determines whether $|\Delta(t)|$ will relax or oscillate persistently? (in thermodynamic limit)

Generally, Fourier spectrum of $|\Delta(t)|$ has discrete & continuum parts

$$|\Delta(t)| = \int_{-D}^{D} A(\omega)\cos(\omega t+\varphi)\,d\omega + \sum_{i=1}^{k} B_i \cos(\omega_i t+\varphi_i) + \text{higer harmonics } (n\omega_i)$$

vanishes as $t \to \infty$

$$|\Delta(t)| \approx F_k(t) = \sum_{i=1}^{k} B_i \cos(\omega_i t+\varphi_i) + \text{higer harmonics } (n\omega_i) \text{ as } t \to \infty$$

quasiperiodic oscillations with *k* frequencies

$k = 0, \quad |\Delta(t)| \to \Delta_\infty = \text{const}$

$k = 1, \quad$ periodic oscillations

$k = 2, \quad$ two frequencies

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$k=1, \quad \text{periodic oscillations}$

$k=2, \quad \text{two frequencies}$

Frequencies of an integrable system depend only on integrals of motion (Arnold).

Can determine the frequency spectrum from the initial state

How to determine the frequency spectrum from the initial state?

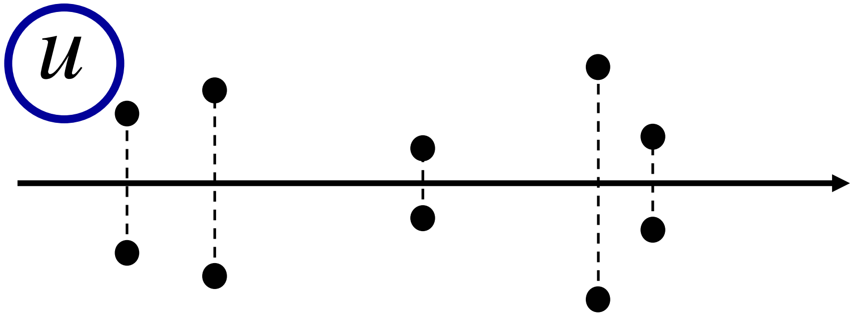
$$H = \sum_j 2\varepsilon_j s_j^z - g\sum_{i,j} s_i^+ s_j^-$$

$$\mathbf{L}(u) \equiv -\frac{\hat{\mathbf{z}}}{g} + \sum_{j=1}^{n} \frac{\mathbf{s}_j}{u-\varepsilon_j}$$

spectral parameter

$$\{H, \mathrm{L}^2(u)\} = 0 \quad \forall u$$

$\mathrm{L}^2(u)$ is conserved, generating function for integrals of motion



✓ n pairs of complex conjugate roots – branch cuts of $\sqrt{\mathrm{L}^2(u)}$ (n - # of spins)

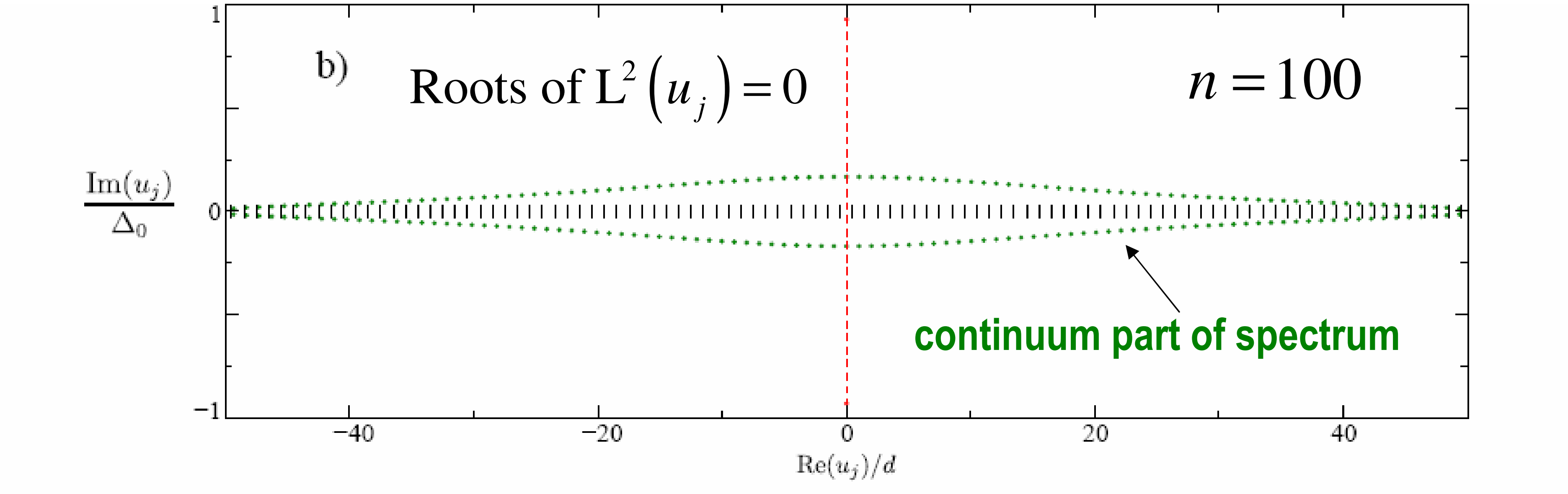
✓ one frequency per each branch cut

$$\mathrm{L}^2(u) = \frac{P_{2n}(u)}{\prod_j (u-\varepsilon_j)^2} \geq 0$$

✓ $|\Delta(t)|$ has n-1 frequencies

How to determine the frequency spectrum from the initial state?

In the thermodynamic limit $n \to \infty$ some roots merge into lines of roots, other pairs of roots (branch cuts) remain isolated



(# of isolated frequencies in $|\Delta(t)|$) = (# of isolated cuts) - 1

$L^2(u) = 0$ easy to solve in $n \to \infty$ limit

$$\mathbf{L}(u) \equiv -\frac{\hat{\mathbf{z}}}{g} + \sum_{j=1}^{n} \frac{\mathbf{s}_j}{u - \varepsilon_j}$$

can do even better!

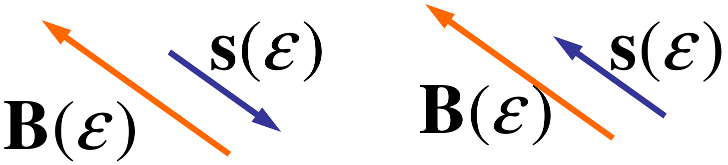
Stationary states (BCS eigenstates)

I. Anomalous states $\Delta \neq 0$ Align each spin along its field $\mathbf{s}_j \parallel \mathbf{B}_j$

$$\frac{d\mathbf{s}(\varepsilon)}{dt} = \mathbf{B}(\varepsilon)\times\mathbf{s}(\varepsilon) = 0 \quad \mathbf{B}(\varepsilon) = (-2\Delta, 0, 2\varepsilon) \quad \Delta = g\int s^x(\varepsilon)\,d\varepsilon$$

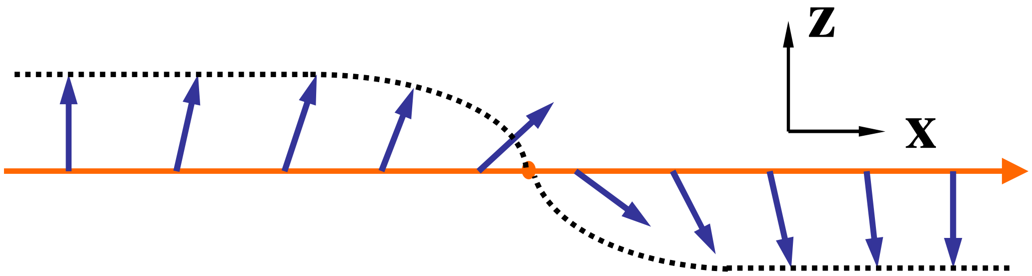
gap equation

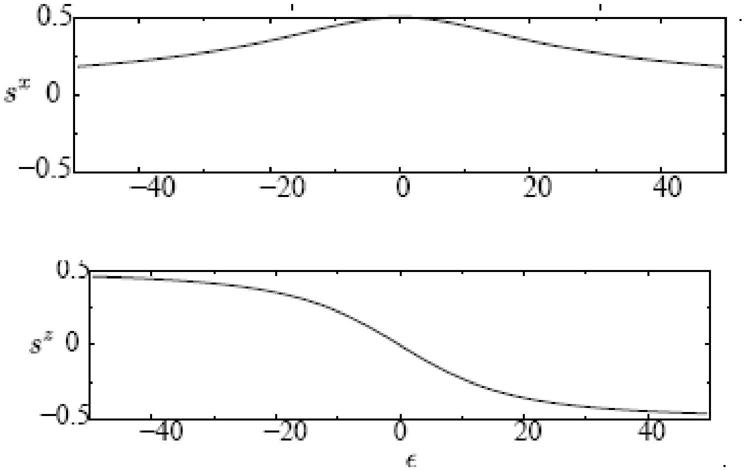
two ways to align:



Favorable alignment – BCS ground state

$\mathbf{s}(\varepsilon)$ - continuous





$$2s^z(\varepsilon) = \frac{-\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}}; \qquad 2s^x(\varepsilon) = \frac{-\Delta}{\sqrt{\varepsilon^2+\Delta^2}}$$

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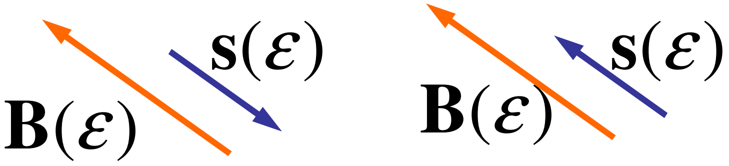
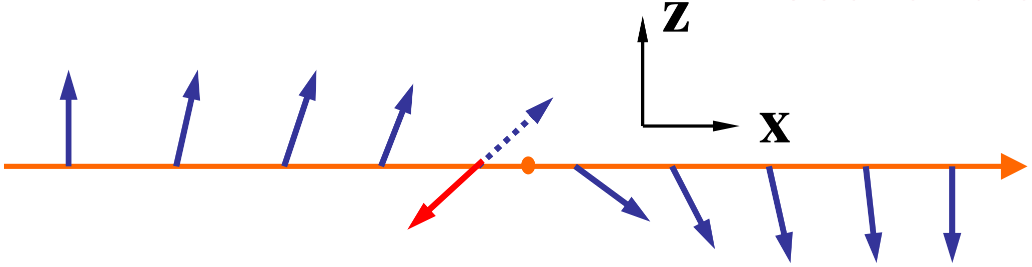


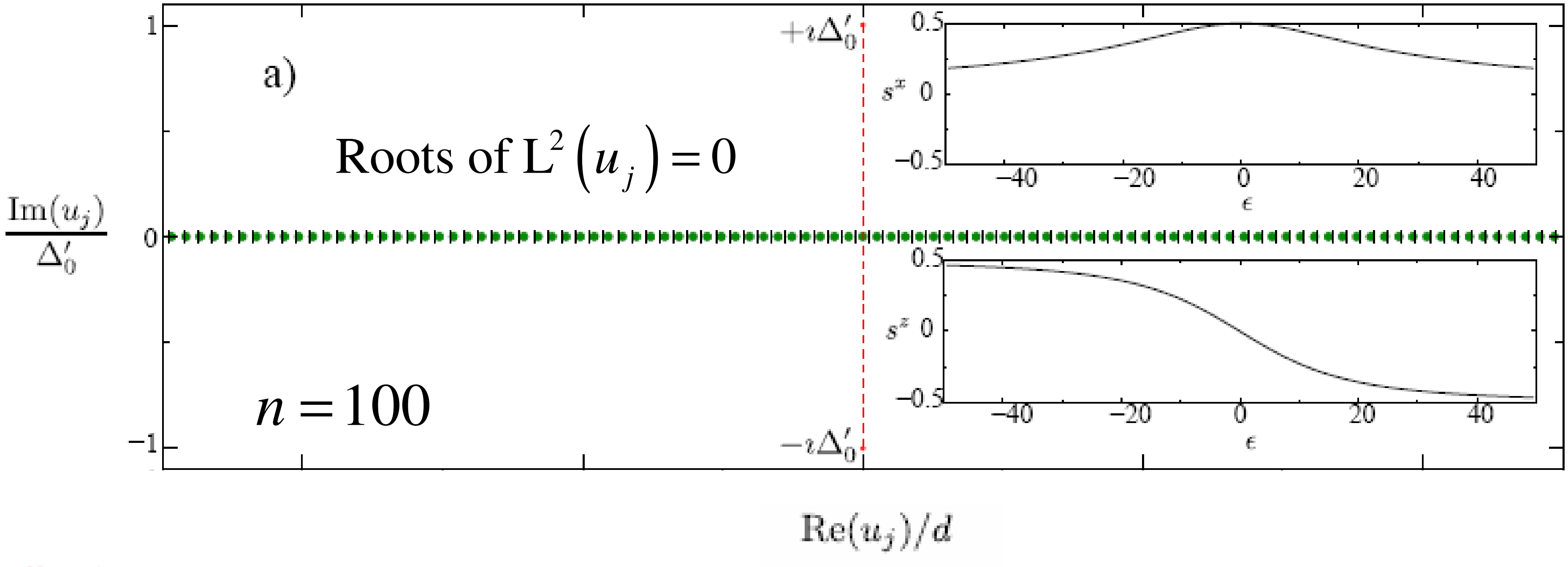
Spin flips – excited states.

Discontinuities in spin distribution $\mathbf{s}(\varepsilon)$



Excitation of energy $2\sqrt{\varepsilon^2 + \Delta^2}$

Root diagram for the BCS ground state



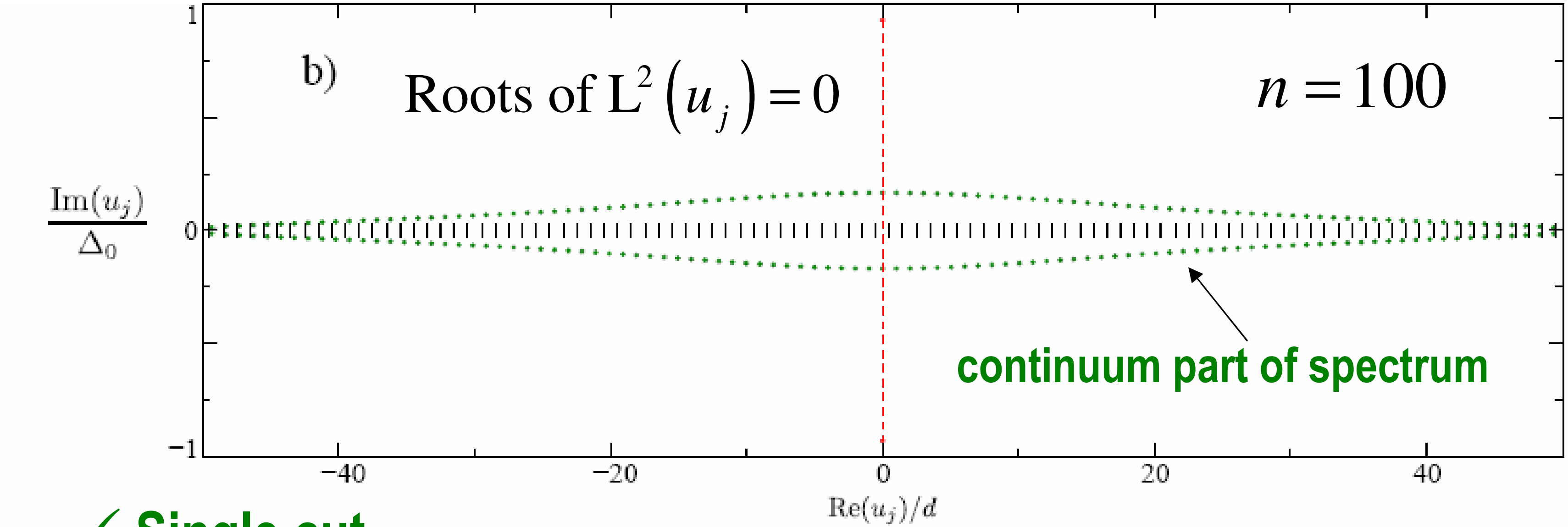
# of discontinuities in $\mathbf{s}(\varepsilon)$ **= 0**

✓ **A pair of imaginary roots at (one cut)** $\pm i\Delta_0$, Δ_0 **- ground state gap**

✓ **Other roots are doubly degenerate, real and located between consecutive** ε_j

Frequencies of small oscillations $\omega_j = 2\sqrt{u_j^2 + \Delta^2} \approx 2\sqrt{\varepsilon_j^2 + \Delta^2}$

Sudden change of coupling $g' \to g$ (not small!). Ground state gap increased 2.4 times!



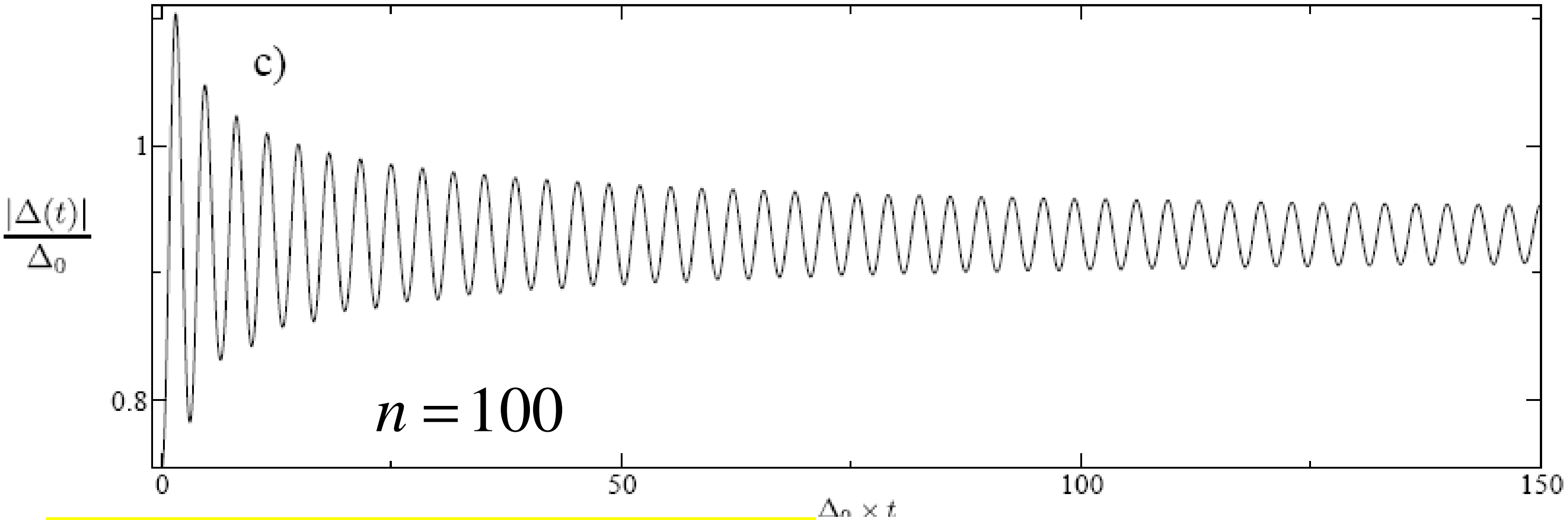
✓ Single cut

✓The line of doubly degenerate real roots splits into two complex conjugate lines

✓# of isolated frequencies = # of isolated cuts - 1 = # of discontinuities in $s(\varepsilon)$ = 0

Therefore, $|\Delta(t)|$ has continuum freq. spectrum and $|\Delta(t)| \to \Delta_\infty < \Delta_0$

Sudden change of coupling $g' \to g$ (not small!). Ground state gap increased 2.4 times!



$$\frac{|\Delta(t)|}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \varphi)}{\sqrt{\Delta_\infty t}}$$

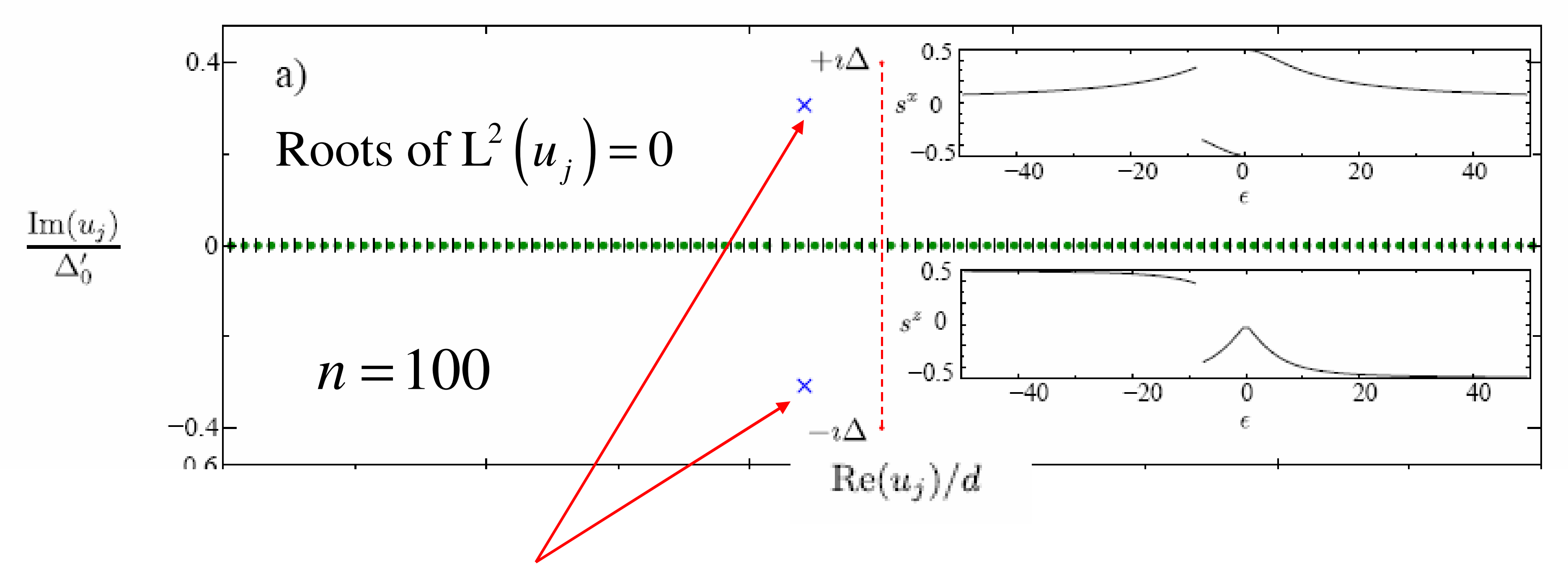
$a \sim 1, \quad \varphi$ - constants

Δ_∞ and the full final state are known

- ✓ $1/\sqrt{t}$ decay law is universal and set by non-stationary analog of square root singularity
- ✓ Similar to inhomogeneous line broadening in NMR

Root diagram for an excited (anomalous) stationary state

spins in energy interval $(-0.4\Delta_0, 0)$ are flipped



Roots of $\mathrm{L}^2\left(u_j\right) = 0$

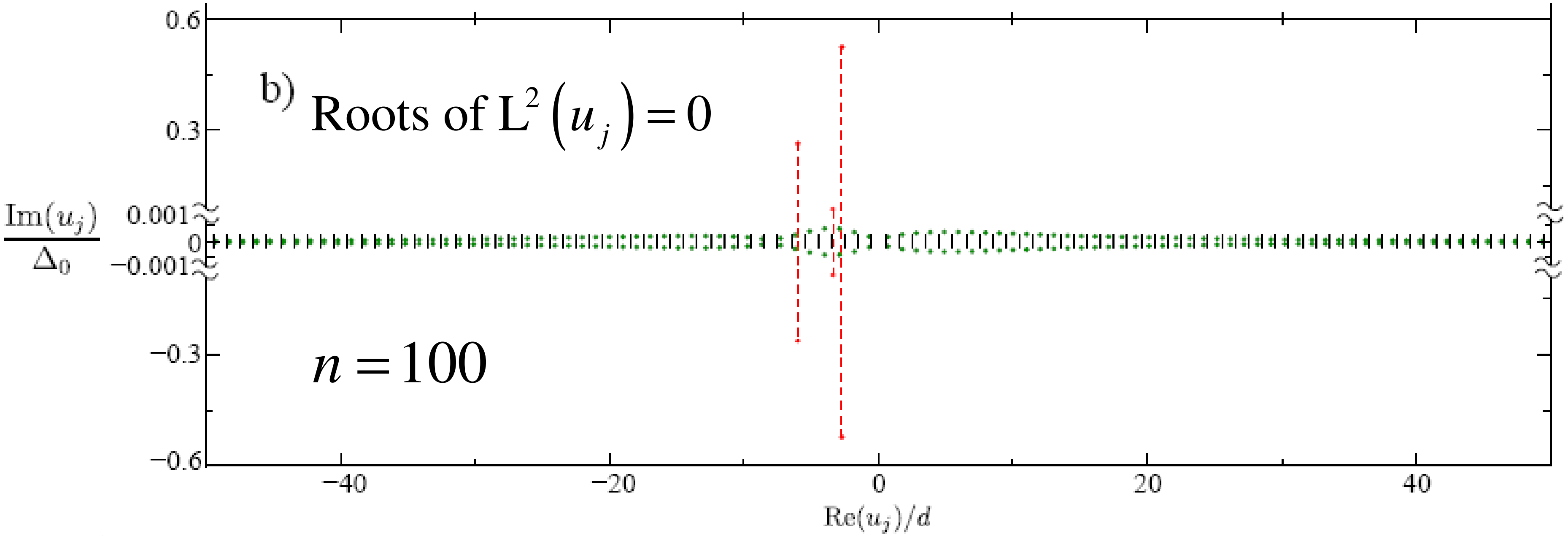
$n = 100$

Spin flips result in a pair of double complex roots

# of discontinuities in $\mathrm{s}(\varepsilon)$ = 2

Frequencies of small oscillations $\omega_j = 2\sqrt{u_j{}^2 + \Delta^2} \approx 2\sqrt{\varepsilon_j{}^2 + \Delta^2}$

Sudden change of coupling $g' \to g$ (not small!) in an excited state



✓ Double complex roots split into two additional cuts

✓The line of doubly degenerate real roots splits into two complex conjugate lines

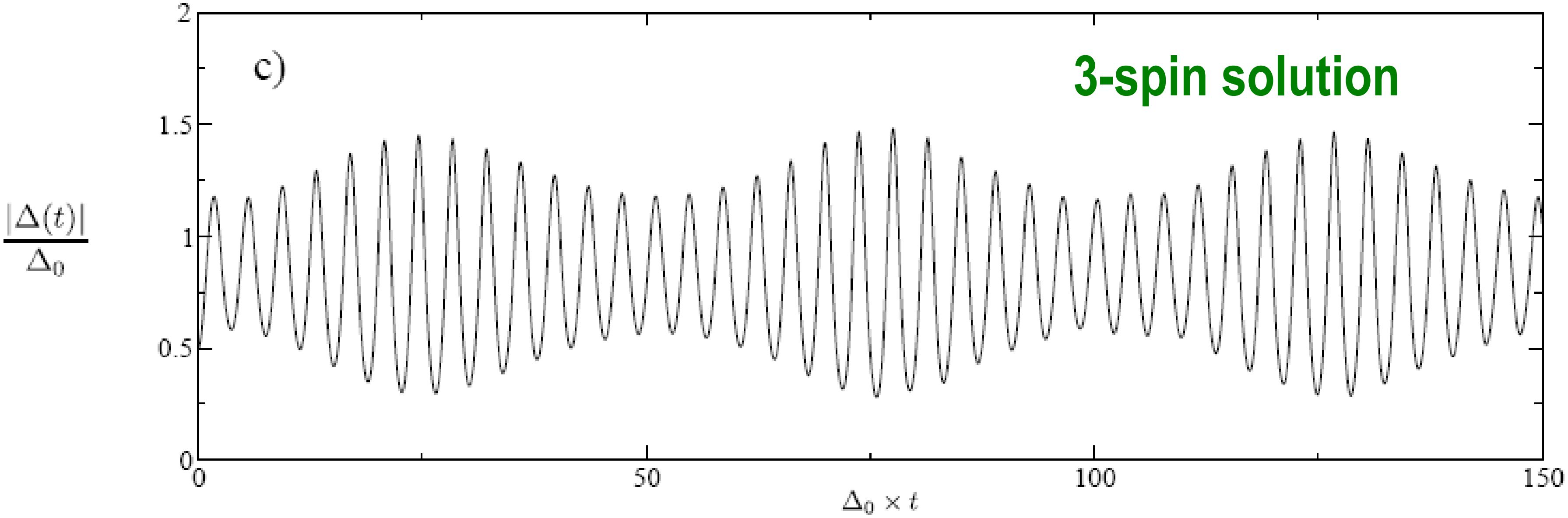
✓*# of isolated frequencies = # of isolated cuts - 1 = # of discontinuities in* $\mathrm{s}(\varepsilon)$ = 2

Therefore, $|\Delta(t)|$ oscillates persistently with two basic frequencies

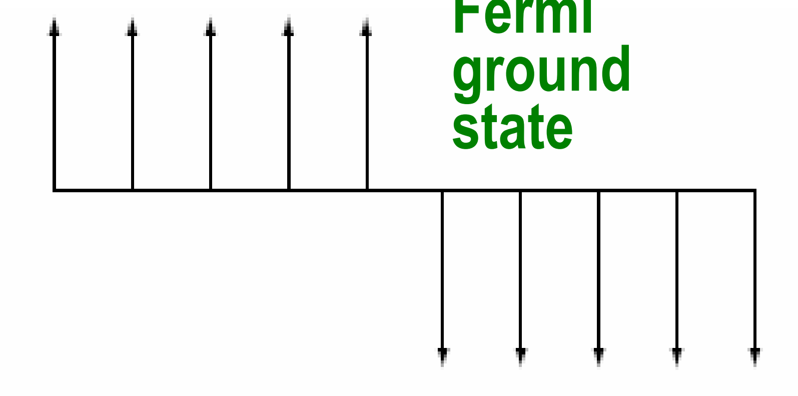
Sudden change of coupling $g' \to g$ (not small!) in excited state

$|\Delta(t)|$ oscillates persistently with two basic frequencies

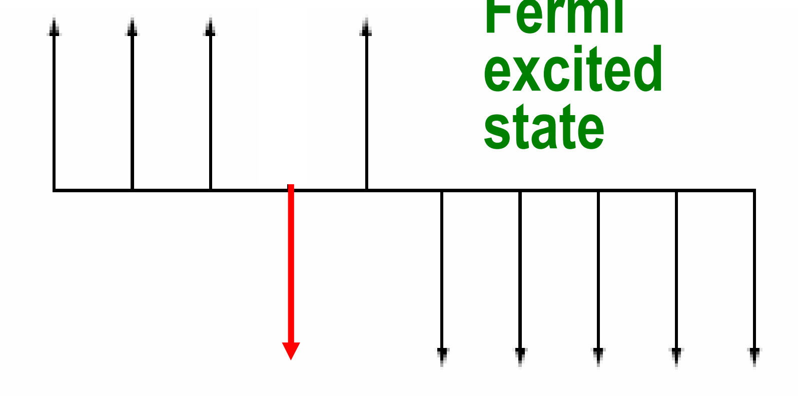
# of isolated frequencies = # of discontinuities in $s(\varepsilon)$ = 2



2nd type of stationary states – normal states



Fermi ground state

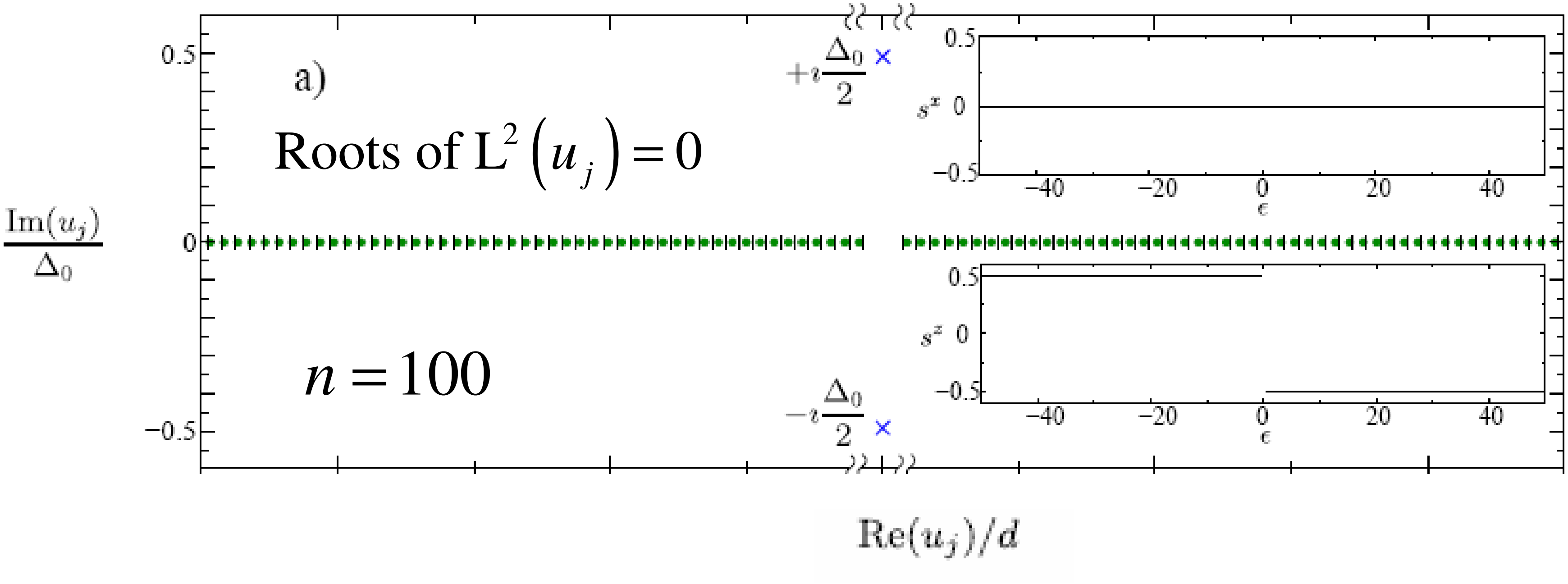


Fermi excited state

Also stationary in mean-field

$$\frac{d\mathbf{s}(\varepsilon)}{dt} = \mathbf{B}(\varepsilon)\times\mathbf{s}(\varepsilon) = 0 \qquad \text{since} \qquad \mathbf{s}(\varepsilon) \,||\, \mathbf{B}(\varepsilon) \,||\, \hat{\mathbf{z}}$$

Root diagram for the *Fermi* ground state



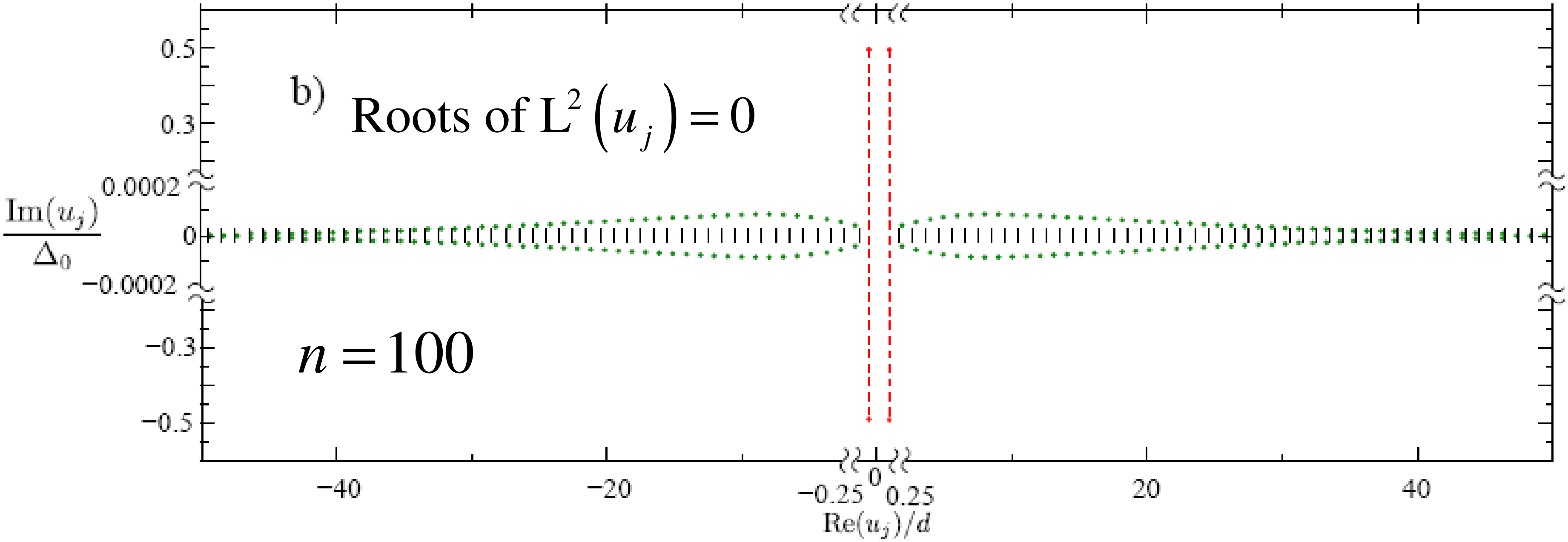
# of discontinuities in $\mathbf{s}(\varepsilon)$ **= 1 (jump at Fermi level)**

A pair of double imaginary roots at $\pm i\Delta_0/2, \;\; \Delta_0$**- ground state gap**

Normal frequencies $\omega_j = 2u_j$

One unstable mode that corresponds to $\omega_j = 2u_j = \pm i\Delta_0, \;\; |\Delta(t)| \propto e^{\Delta_0 t}$

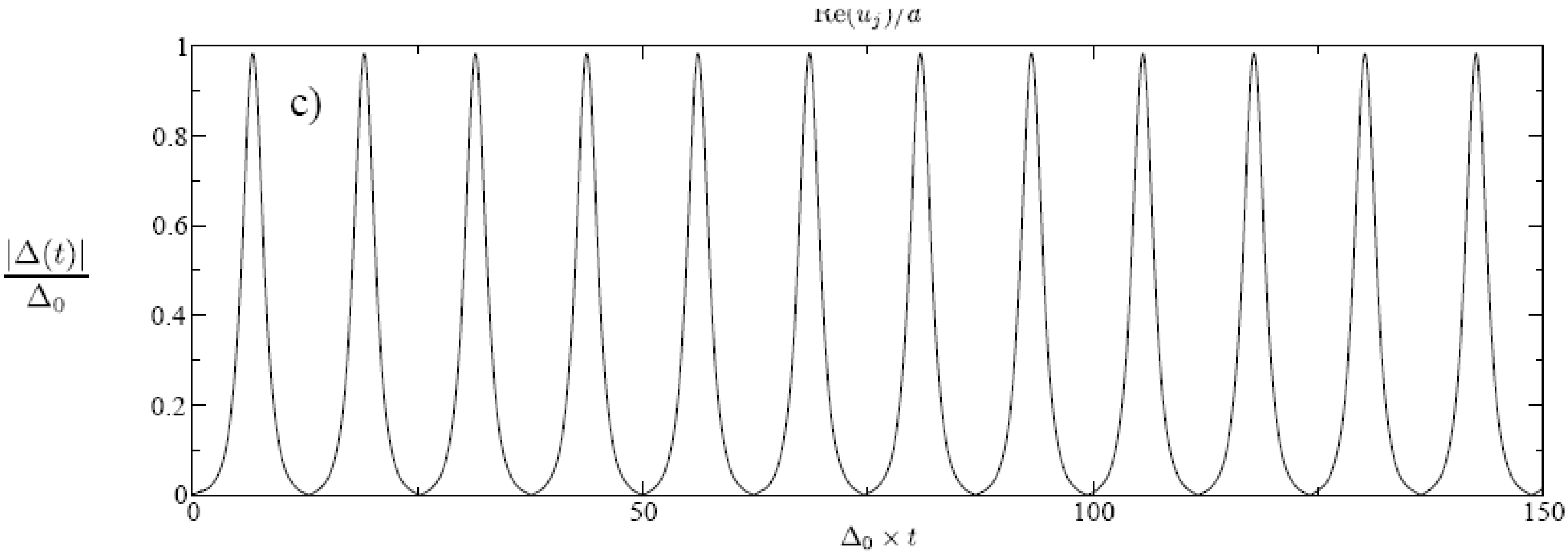
A “quantum fluctuation” splits double roots



# of isolated frequencies = # of isolated cuts - 1 = # of discontinuities in $s(\varepsilon)$ **= 1**

Therefore, $|\Delta(t)|$ **oscillates periodically (one basic frequency)**

Time evolution starting from the Fermi ground state



2-spin solution

Barankov, Levitov, Spivak

Summary

Classification of states: there are only two types of initial states

Type I: No discontinuities in the spin distribution

$|\Delta(t)|$ asymptotes to a constant $\Delta_\infty < \Delta_0$

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This happens e.g. for a sudden change of coupling in a paired ground state, $g' \to g$.

Type II: $|\Delta(t)|$ oscillates persistently with several basic frequencies

# of isolated frequencies = # of isolated cuts - 1 = # of discontinuities in $\mathbf{s}(\varepsilon)$