Rutgers University
Department of Physics & Astronomy

01:750:271 Honors Physics I
Fall 2015

Lecture 4

4. Motion in two and three dimensions

Goals:

- To study position, velocity, and acceleration vectors
- To apply position, velocity, and acceleration insights to projectile motion
- To extend our linear investigations to uniform circular motion
- To investigate relative velocity

in **two and three** dimensions.

Position and Displacement

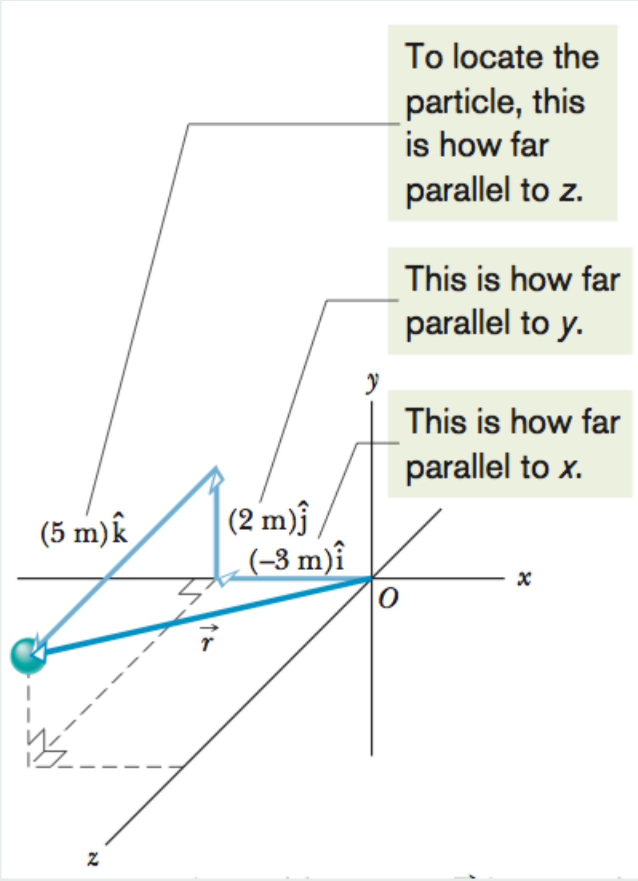
• In **3D** the position of a particle is given by a **position vector**

$$\vec{r} = x\widehat{i} + y\widehat{j} + z\widehat{k}$$

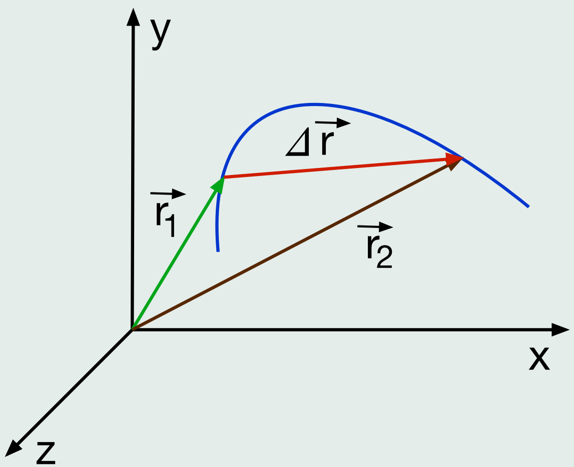
starting at the origin O.

• x, y, z are the **components** of $\vec{r}$, also called the **coordinates** of the particle.

• The path of the particle is generally a **curve**.



- **Displacement:** the **change** of the position vector $\vec{r}$ over a time interval Δt.



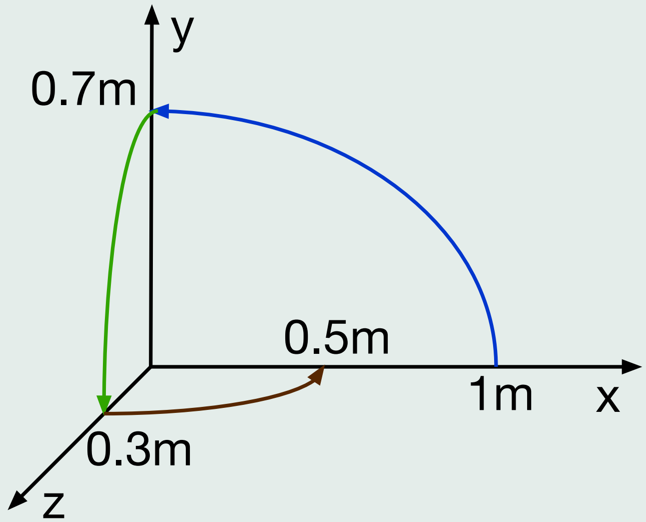
$$\vec{r}_1 = x_1\widehat{i} + y_1\widehat{j} + z_1\widehat{k} \quad \text{at time } t_1$$

$$\vec{r}_2 = x_2\widehat{i} + y_2\widehat{j} + z_2\widehat{k} \quad \text{at time } t_2$$

$$\begin{aligned}\Delta\vec{r} &= (x_2\widehat{i} + y_2\widehat{j} + z_2\widehat{k}) - (x_1\widehat{i} + y_1\widehat{j} + z_1\widehat{k}) \\ &= (x_2 - x_1)\widehat{i} + (y_2 - y_1)\widehat{j} + (z_2 - z_1)\widehat{k}. \\ &= \Delta x\,\widehat{i} + \Delta y\,\widehat{j} + \Delta z\,\widehat{k}.\end{aligned}$$

i-Clicker

A particle travels along a 3D path as shown in the figure starting from the initial position $\vec{r}_1 = (1\,\text{m})\widehat{i}$ and ending with the final position $\vec{r}_2 = (0.5\,\text{m})\widehat{i}$. What is the displacement vector?



$A)\ \Delta\vec{r} = (-1\,\text{m})\widehat{i} + (0.7\,\text{m})\widehat{j}$

$B)\ \Delta\vec{r} = (0.7\,\text{m})\widehat{j} - (0.3\,\text{m})\widehat{k}$

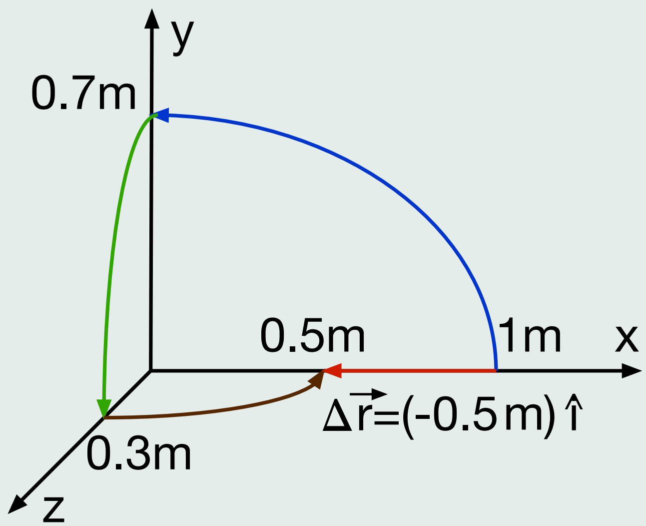
$C)\ \Delta\vec{r} = (-0.5\,\text{m})\widehat{i}$

$D)\ \Delta\vec{r} = 0$

$E)\ \Delta\vec{r} = (0.5\,\text{m})\widehat{i}$

Answer

A particle travels along a 3D path as shown in the figure starting from the initial position $\vec{r}_1 = (1\,\mathrm{m})\hat{i}$ and ending with the final position $\vec{r}_2 = (0.5\,\mathrm{m})\hat{i}$. What is the displacement vector?



$A)\ \Delta\vec{r} = (-1\,\mathrm{m})\hat{i} + (0.7\,\mathrm{m})\hat{j}$

$B)\ \Delta\vec{r} = (0.7\,\mathrm{m})\hat{j} - (0.3\,\mathrm{m})\hat{k}$

$C)\ \Delta\vec{r} = (-0.5\,\mathrm{m})\hat{i}$

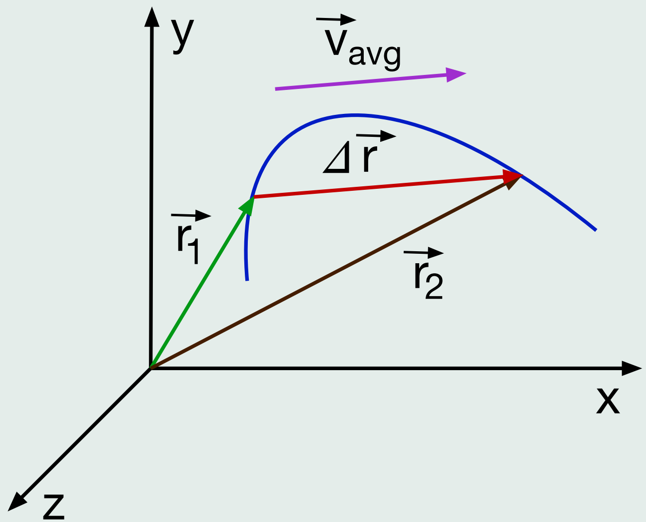
$D)\ \Delta\vec{r} = 0$

$E)\ \Delta\vec{r} = (0.5\,\mathrm{m})\hat{i}$

Average and Instantaneous Velocity

- **Average velocity** over a time interval Δt

$$\vec{v}_{\mathsf{avg}} = \frac{\text{Displacement vector}}{\text{time interval}} = \frac{\Delta\vec{r}}{\Delta t}$$

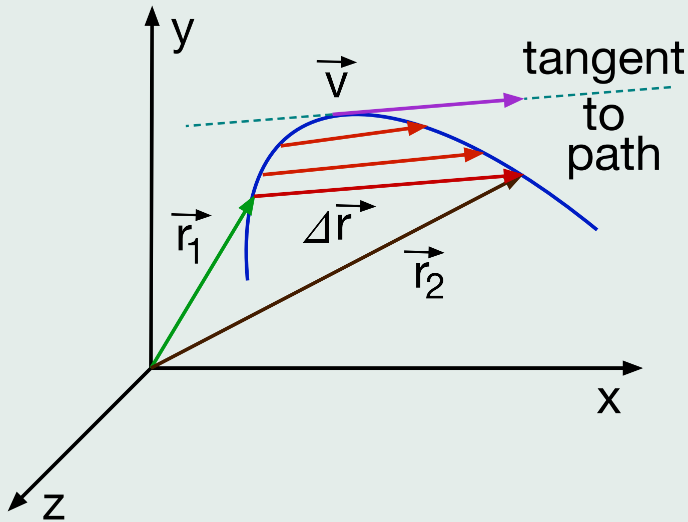


Note:

Since $\Delta t > 0 \;\Rightarrow\; \vec{v}_{\mathsf{avg}}$ is always parallel with $\Delta\vec{r}$ and points in the same direction.

- **Instantaneous velocity** at time t

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\widehat{i} + \left(\frac{dy}{dt}\right)\widehat{j} + \left(\frac{dz}{dt}\right)\widehat{k}$$

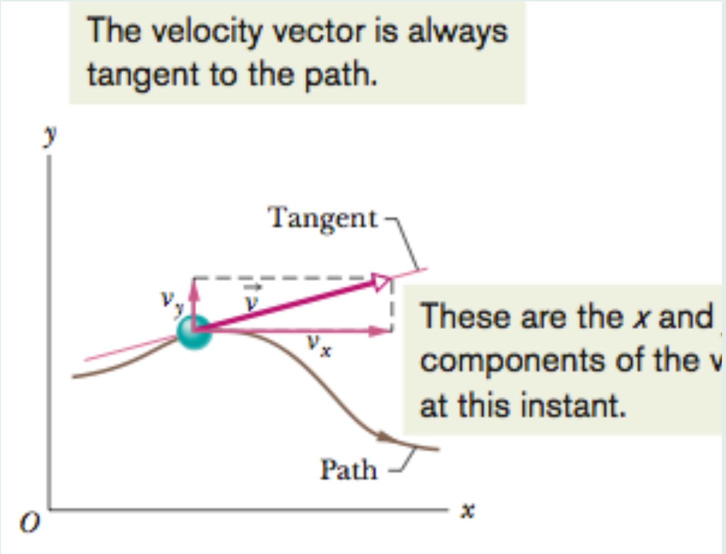


Note:

- $\vec{v}$ is always **tangent** to the path of the particle at its current position.

- **Components** of $\vec{v}$:

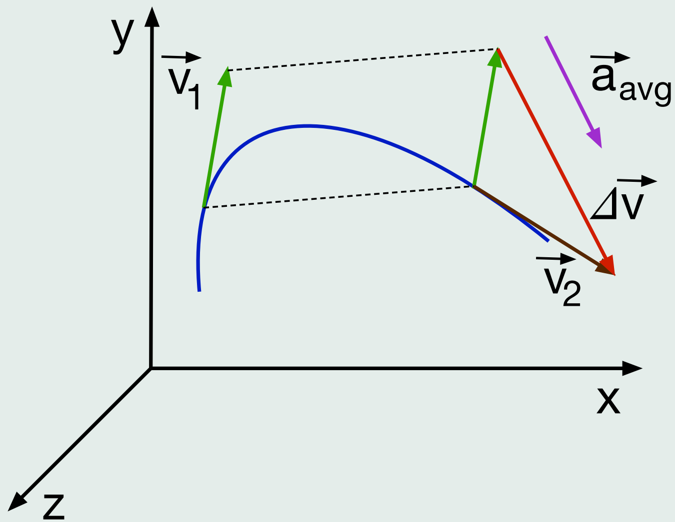
$$v_x = \frac{dx}{dt}\ , v_y = \frac{dy}{dt},\ \ v_z = \frac{dz}{dt}$$



Average and Instantaneous Acceleration

- **Average acceleration** over a time interval Δt

$$\vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t}$$

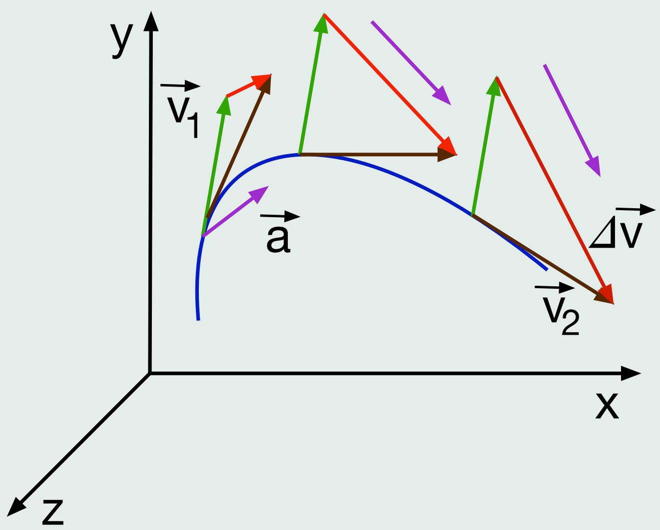


Note:

Since $\Delta t > 0 \Rightarrow \vec{a}_{\text{avg}}$ is always parallel with $\Delta \vec{v}$ and points in the same direction.

- **Instantaneous acceleration** at time t

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \left(\frac{dv_x}{dt}\right)\widehat{i} + \left(\frac{dv_y}{dt}\right)\widehat{j} + \left(\frac{dv_z}{dt}\right)\widehat{k}$$



- **Acceleration** is a **vector** which measures the **change** of the velocity **vector**.

- **Components** of $\vec{a}$:

$$a_x = \frac{dv_x}{dt}\ , a_y = \frac{dv_y}{dt},\ \ a_z = \frac{dv_z}{dt}$$

- The acceleration is non-zero if

either

the **magnitude** of $\vec{v}$ **changes**.

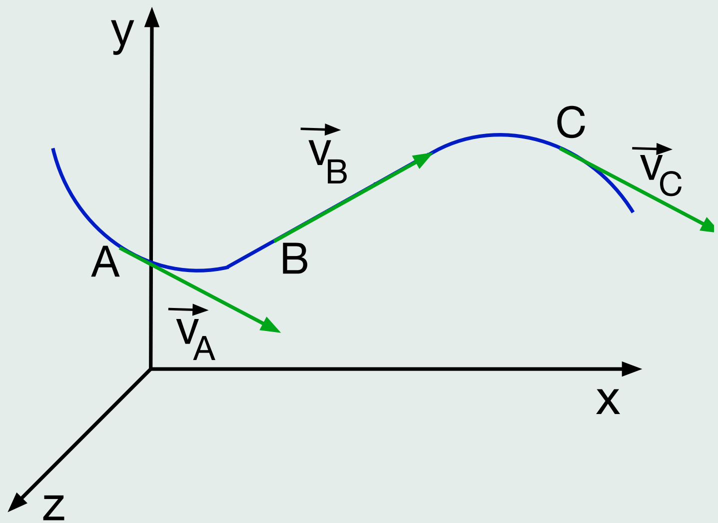
or

the **direction** of $\vec{v}$ **changes**.

- The acceleration is 0 if and only if **both** the **magnitude** and the **direction** of $\vec{v}$ are **constant.**

i-Clicker

A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration $\vec{a}$ zero?



A) Everywhere

B) At A

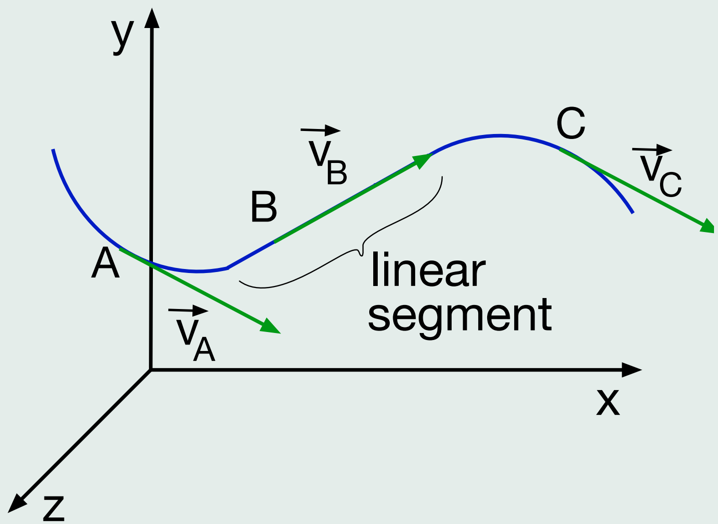
C) At C

D) At B

E) Nowhere

Answer

A particle moves on a path as shown below such that the **magnitude** of its velocity vector is **constant**. Where is the instantaneous acceleration $\vec{a}$ zero?



$A)$ Everywhere

$B)$ At A

$C)$ At C

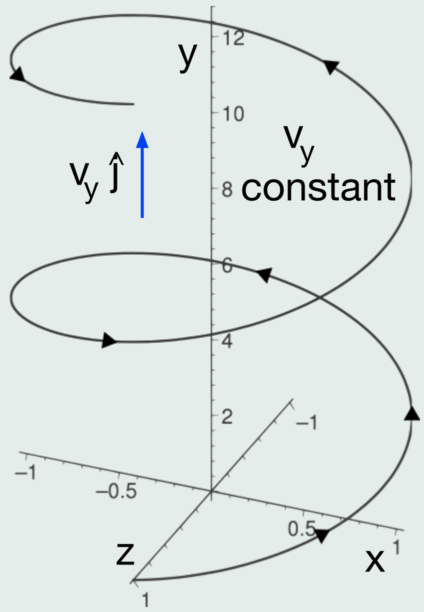
$D)$ At B

$E)$ Nowhere

$|\vec{v}|$ constant $\Rightarrow$ $\vec{v}$ constant on linear segment; $\vec{v}$ **not** constant on curved segments; changes direction.

i-Clicker

A particle moves on a helix as shown below such that the y-component of its velocity is constant. Which of the following statements is false?



$A)\ a_x \neq 0$

$B)\ a_z \neq 0$

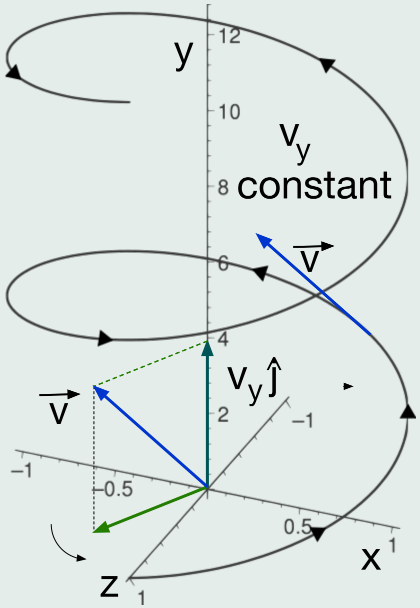
$C)\ a_y = 0$

$D)\ a_y \neq 0$

$E)$ None of the above.

Answer

A particle moves on a helix as shown below such that the y-component of its velocity is constant. Which of the following statements is false?



$A)\ a_x \neq 0$

$B)\ a_z \neq 0$

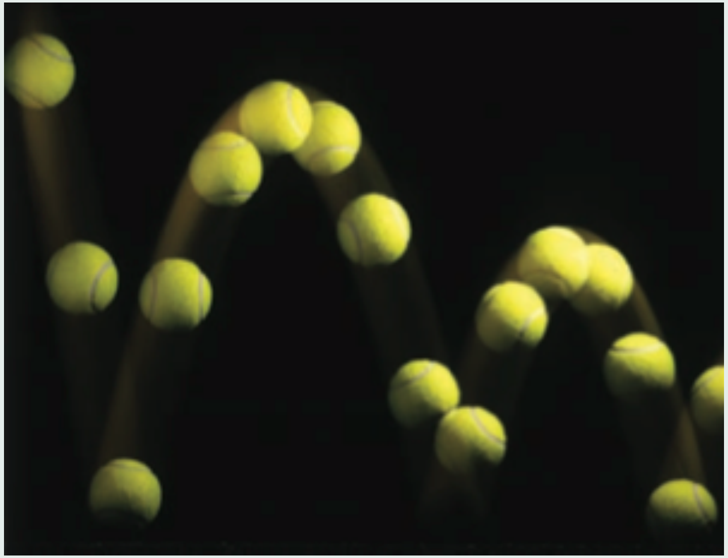
$C)\ a_y = 0$

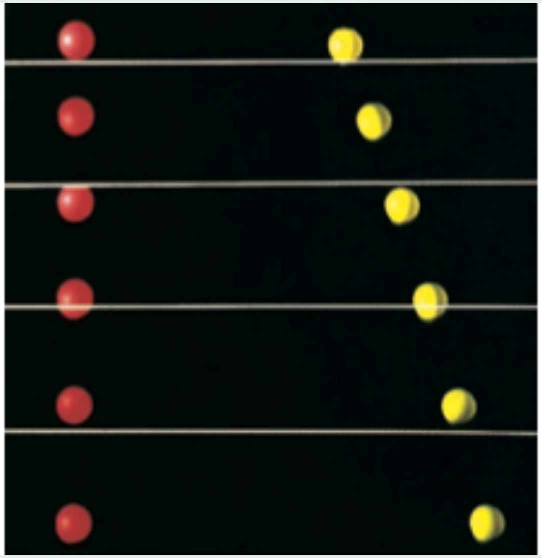
$D)\ a_y \neq 0$

$E)$ None of the above.

Projectile Motion

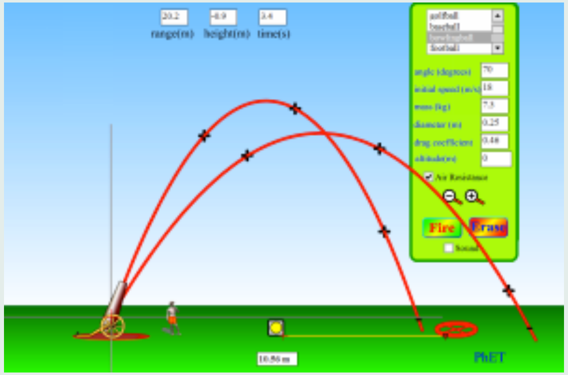
• **Projectile Motion:** motion in a vertical plane with constant acceleration equal to the free fall acceleration.





Examples: tennis, baseball, basketball, football . . .

How hard and at what angle should the quarterback throw?



How should the gun shoot in order to hit the target?

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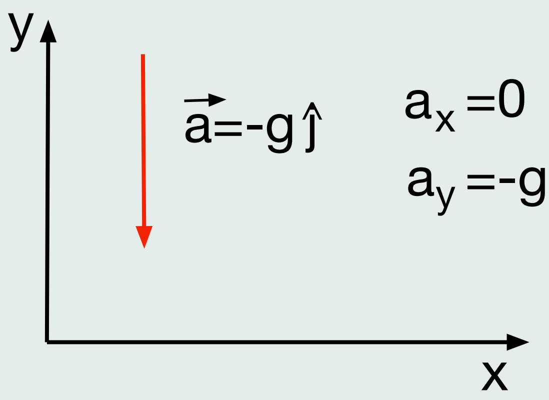
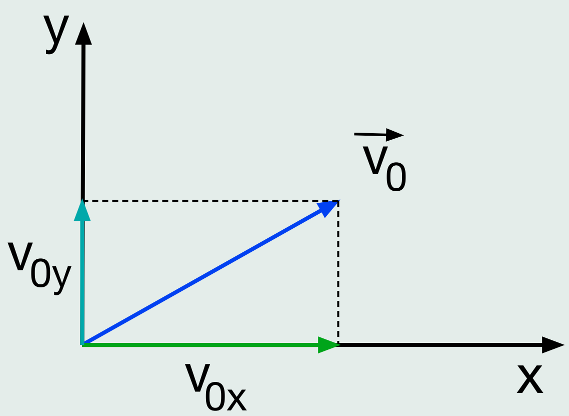
- Set up coordinate system



Acceleration



Initial velocity

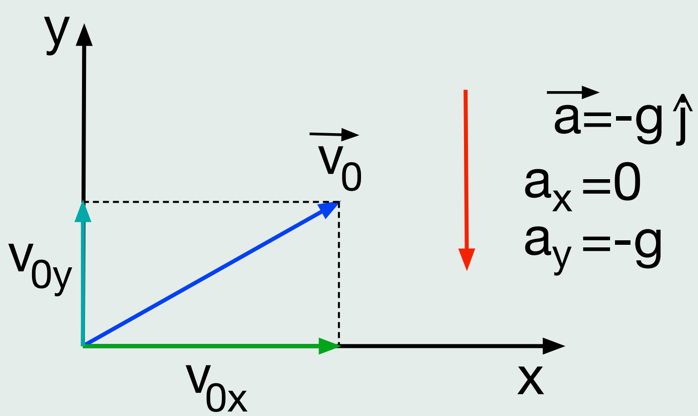
Note: no correlation between v_{0x}, v_{0y}; can take any values independently from each other.

i-Clicker

Recall that

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Then what is the time dependence of v_x ?



$A)\ v_x = 0$

$B)\ v_x = v_{0x}$

$C)\ v_x = v_{0x} - gt$

$D)\ v_x = v_{0x} + gt$

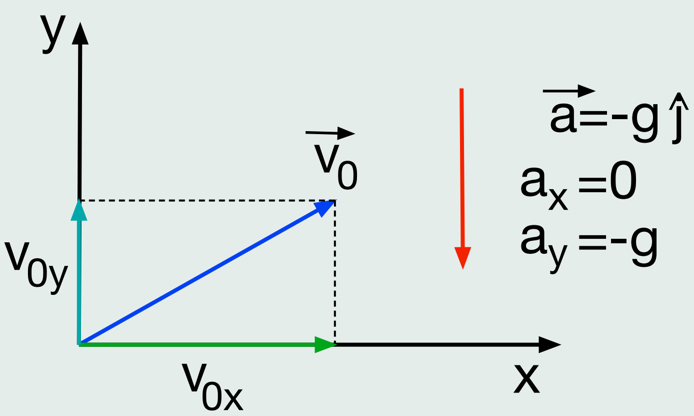
$E)\ v_x = v_{0y}$

Answer

Recall that

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Then what is the time dependence of v_x ?



$A)\ v_x = 0$

$B)\ v_x = v_{0x}$

$C)\ v_x = v_{0x} - gt$

$D)\ v_x = v_{0x} + gt$

$E)\ v_x = v_{0y}$

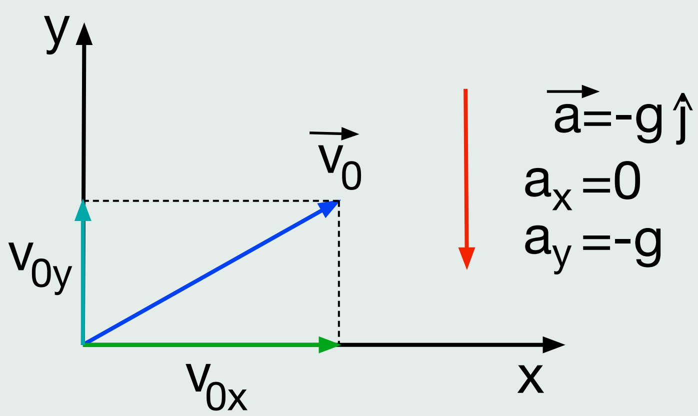
$$\frac{dv_x}{dt} = a_x = 0 \ \Rightarrow \ v_x = \text{constant} = v_{0x}$$

i-Clicker

Recall that

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Then what is the time dependence of v_y ?



$A)\ v_y = 0$

$B)\ v_y = v_{0y}$

$C)\ v_y = v_{0y} - gt$

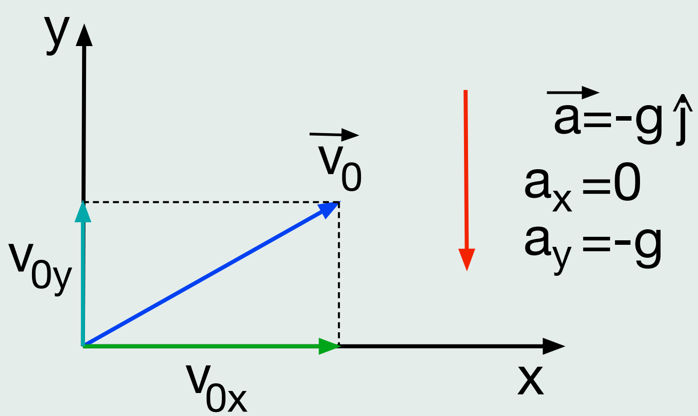
$D)\ v_y = v_{0y} + gt$

$E)\ v_y = v_{0x}$

Recall that

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Then what is the time dependence of v_y ?



$A)\ v_y = 0$

$B)\ v_y = v_{0y}$

$C)\ v_y = v_{0y} - gt$

$D)\ v_y = v_{0y} + gt$

$E)\ v_y = v_{0x}$

$$\frac{dv_y}{dt} = a_y = -g \ \Rightarrow \ v_y = v_{0y} - gt$$

Is there any correlation between vertical and horizontal motion?

No!

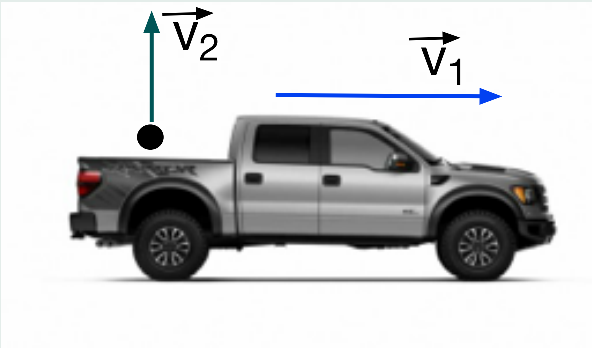
$$v_x = v_{0x}, \qquad v_y = v_{0y} - gt$$

v_{0x}, v_{0y} are completely independent.

Can decompose projectile motion into vertical and horizontal parts.

i-Clicker

A ball is thrown straight up with velocity vector $\vec{v}_2$ from a truck moving with constant velocity $\vec{v}_1$. For which values of v_1, v_2 will the ball land back in the truck at a later time?



$A)$ $v_1 < v_2$

$B)$ $v_1 > v_2$

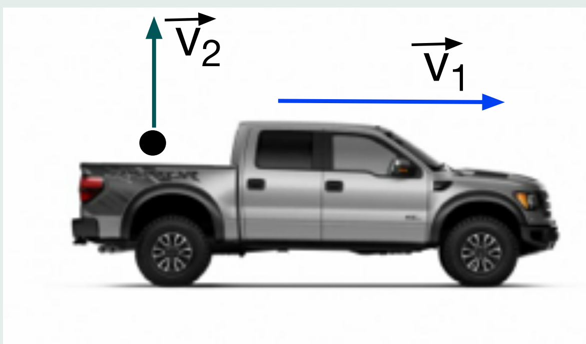
$C)$ $v_1 = v_2$

$D)$ It will not happen for any values of v_1, v_2.

$E)$ For any values of v_1, v_2

Answer

A ball is thrown straight up with velocity vector $\vec{v}_2$ from a truck moving with constant velocity $\vec{v}_1$. For which values of v_1, v_2 will the ball land back in the truck at a later time?



$A)$ $v_1 < v_2$

$B)$ $v_1 > v_2$

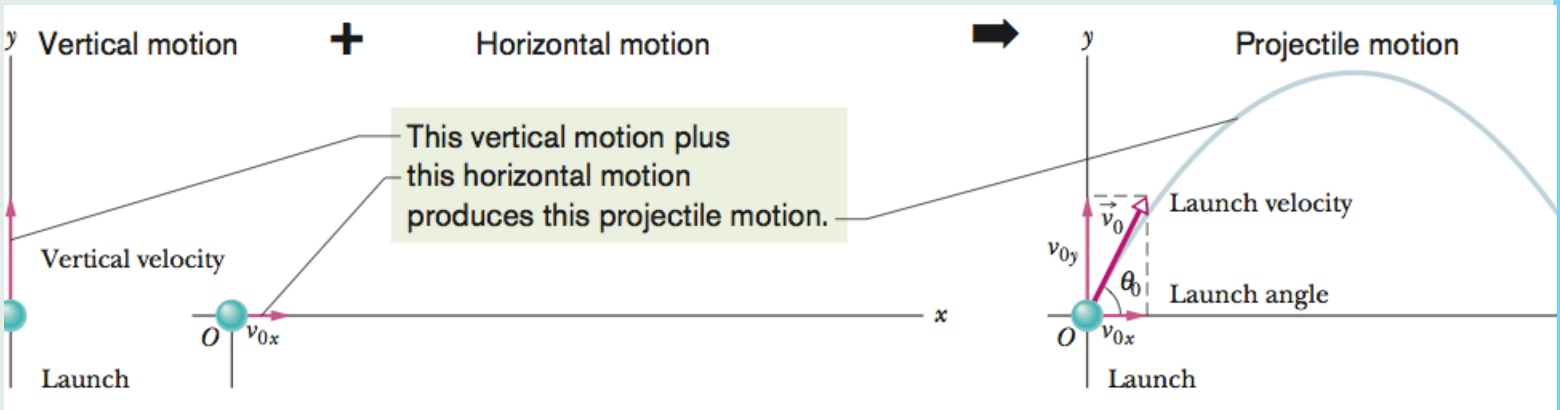
$C)$ $v_1 = v_2$

$D)$ It will not happen for any values of v_1, v_2.

$E)$ For any values of v_1, v_2

The horizontal component v_{0x} for the ball is equal to v_1. Since $a_x = 0$, $v_x = v_1$ at any time $t > 0$.

• Projectile motion analysed



Initial velocity

$$\vec{v}_0 = v_{0x}\widehat{i} + v_{0y}\widehat{j}, \qquad v_{0x} = v_0 \cos\theta_0, \qquad v_{0y} = v_0 \sin\theta_0$$

Horizontal motion

$$a_x = 0$$

$$\begin{aligned} v_x &= v_{0x} \\ &= v_0 \cos\theta_0 \end{aligned}$$

$$\begin{aligned} x - x_0 &= v_{0x} t \\ &= \left(v_0 \cos\theta_0\right) t \end{aligned}$$

Vertical motion

$$a_y = -g$$

$$\begin{aligned} v_y &= v_{0y} - gt \\ &= v_0 \sin\theta_0 - gt \end{aligned}$$

$$\begin{aligned} y - y_0 &= v_{0y} t - gt^2/2 \\ &= \left(v_0 \sin\theta_0\right) t - gt^2/2 \end{aligned}$$

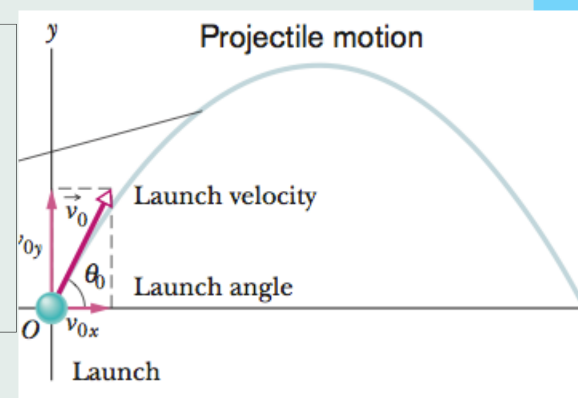
$$v_y^2 = \left(v_0 \sin\theta_0\right)^2 - 2g(y - y_0)$$

- **Trajectory** $(\theta_0 \neq \pi/2)$

$$x - x_0 = \big(v_0 \cos\theta_0\big)t, \qquad y - y_0 = \big(v_0 \sin\theta_0\big)t - gt^2/2$$

$$\Downarrow \ \text{Eliminate t} : t = \frac{x - x_0}{v_0 \cos\theta_0}$$

$$y = y_0 + \big(\tan\theta_0\big)(x - x_0) - \frac{g(x - x_0)^2}{2\big(v_0 \cos\theta_0\big)^2}$$



Parabola

- **Horizontal range**

  The **horizontal** distance R travelled until returning to initial height.

$$y - y_0 = (\tan\theta_0)(x - x_0) - \frac{g(x - x_0)^2}{2\left(v_0 \cos\theta_0\right)^2}$$

$$y = y_0 \quad \Rightarrow \quad (\tan\theta_0)\, R - \frac{gR^2}{2\left(v_0 \cos\theta_0\right)^2}$$

$$\boxed{R = \frac{v_0^2}{g}\sin(2\theta_0)} \qquad \left(\text{using } 2\sin\theta_0\cos\theta_0 = \sin(2\theta_0)\right)$$

- **Maximum horizontal range**

For fixed v_0, when is R maximum?

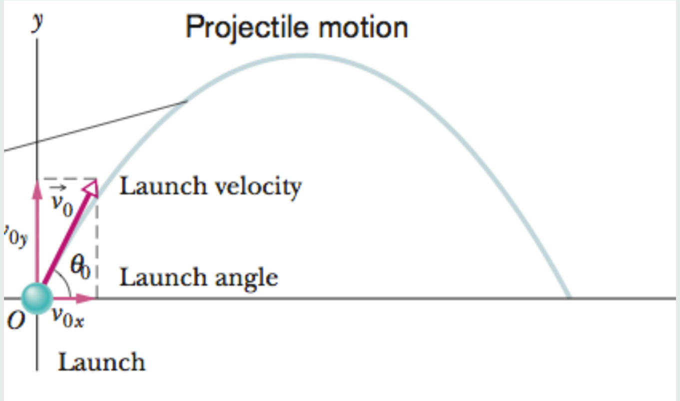
$$0 \leq \sin(2\theta_0) \leq 1, \qquad \sin(2\theta_0) = 1 \iff 2\theta_0 = \pi/2$$

Hence R is maximum for $\theta_0 = \pi/4$ and

$$R_{max} = \frac{v_0^2}{g}$$

i-Clicker

A projectile is launched from the origin as shown below. How long will it take for it to reach maximum height?



$A)$ $(v_0/g)\sin\theta_0$

$B)$ $(v_0/g)\cos\theta_0$

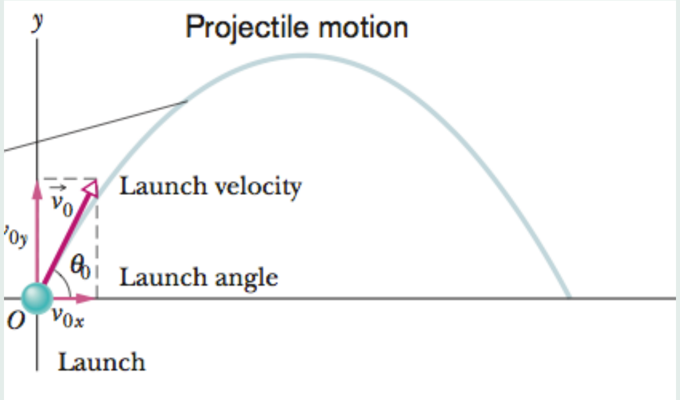
$C)$ v_0/g

D $(v_0/g)\tan\theta_0$

$E)$ None of the above.

Answer

A projectile is launched from the origin as shown below. How long will it take for it to reach maximum height?



A) $(v_0/g)\sin\theta_0$

B) $(v_0/g)\cos\theta_0$

C) v_0/g

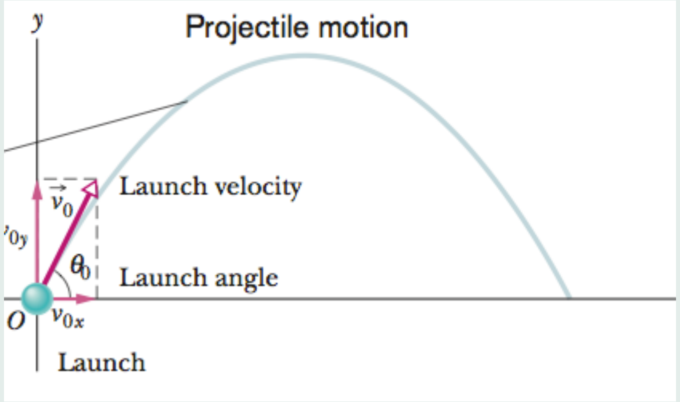
D $(v_0/g)\tan\theta_0$

E) None of the above.

$v_y = v_0 \sin\theta_0 - gt$. Max height: $v_y = 0 \Rightarrow t = (v_0/g)\sin\theta_0$

i-Clicker

A projectile is launched from the origin as shown below. What is its maximum height?



$A)$ $(v_0^2/g)\sin\theta_0$

$B)$ $(v_0^2/2g)\cos\theta_0$

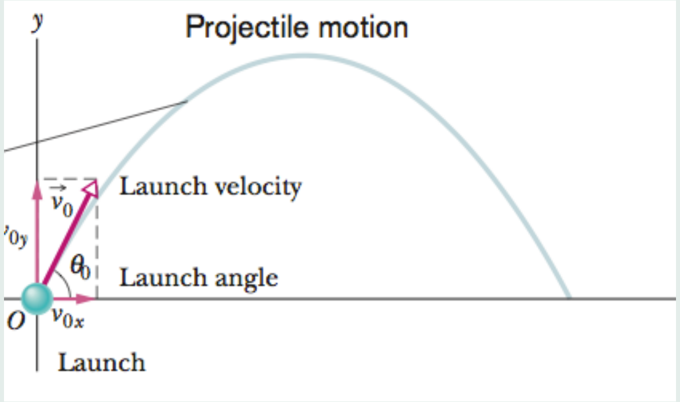
$C)$ $v_0^2/2g$

D $(v_0 \sin\theta_0)^2/2g$

$E)$ None of the above.

Answer

A projectile is launched from the origin as shown below. What is its maximum height?



$A)$ $(v_0^2/g)\sin\theta_0$

$B)$ $(v_0^2/2g)\cos\theta_0$

$C)$ $v_0^2/2g$

D $\left(v_0\sin\theta_0\right)^2/2g$

$E)$ None of the above.

$$v_y^2 = (v_0\sin\theta_0)^2 - 2gy;\ v_y = 0 \Rightarrow y = \left(v_0\sin\theta_0\right)^2/2g$$