Rutgers University
Department of Physics & Astronomy

01:750:271 Honors Physics I

Lecture 3

3. Vectors

Goals:

- To define vector components and add vectors.
- To introduce and manipulate unit vectors.
- To define and understand scalar product.
- To define and understand vector product.

Vectors and scalars.

- **Vectors:** quantities which indicate both magnitude and direction.

  Examples: displacemement, velocity, acceleration

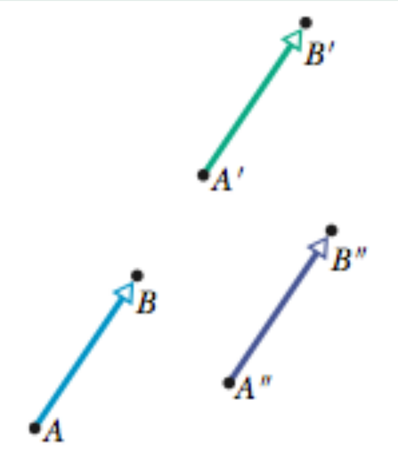
- **Scalars:** quantities which indicate only magnitude.

  Examples: time, speed, mass

- **Vectors** are represented by **arrows**:

  (i) The length of the arrow signifies magnitude.

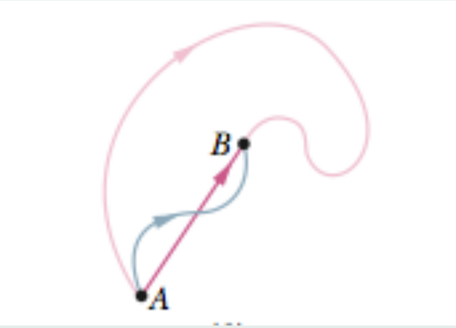
  (ii) The head of the arrow signifies direction.



Displacement vector for a particle travelling from A to B on a straight path

Note: All three vectors are identical because they have the same direction and magnitude.

A shift preserving both direction and magnitude does not change the vector. (**Translation.**)



Displacement vector for a particle travelling on a curved path.

Note: independent of the path from A to B.

- **Notation:**

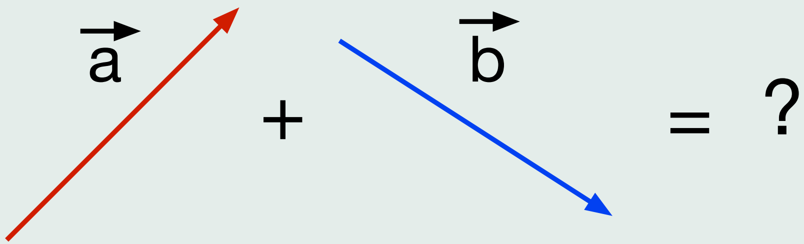
$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \quad \text{or} \quad \vec{a}, \vec{b}, \vec{c}, \ldots$$

The **magnitude** of a vector $\vec{a}$:

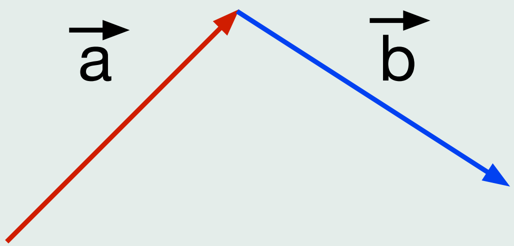
$$a \quad \text{or} \quad |\vec{a}|$$

Adding vectors geometrically

- What is the **sum** of two vectors?

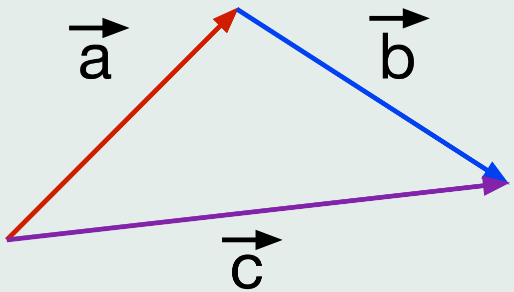


- **Step 1.** Draw the vectors **head to tail**



Home Page
Title Page
Page 6 of 37
Go Back
Full Screen
Close
Quit

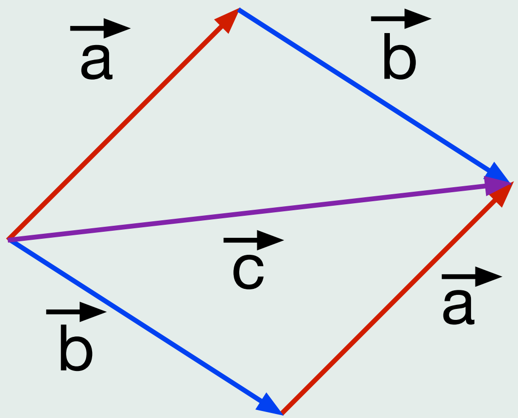
- **Step 2.** The **vector sum** of $\vec{a}$ and $\vec{b}$ is the vector $\vec{c}$ pointing from the **tail** of $\vec{a}$ to the **head** of $\vec{b}$.



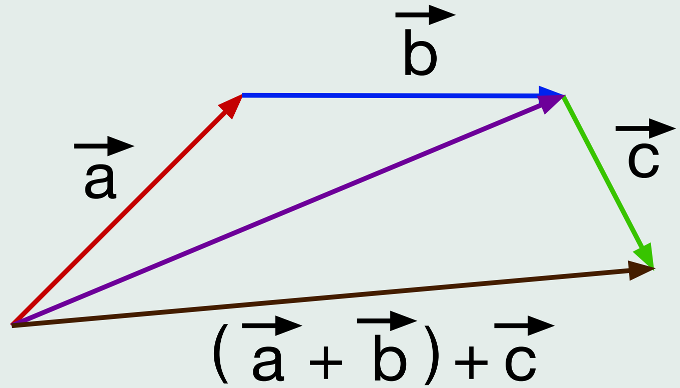
Mathematical formula:

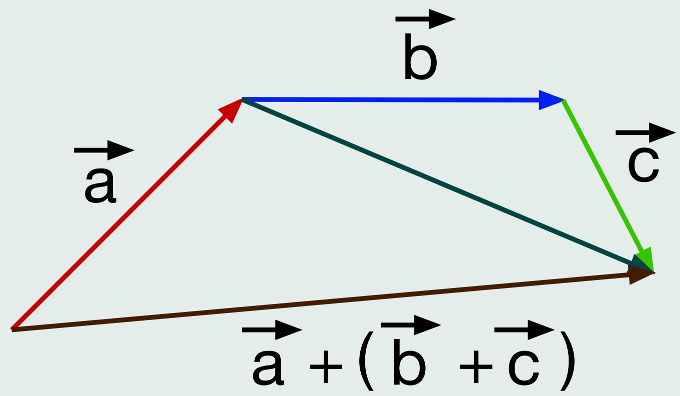
$$\boxed{\vec{a} + \vec{b} = \vec{c}}$$

- **Commutativity:** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

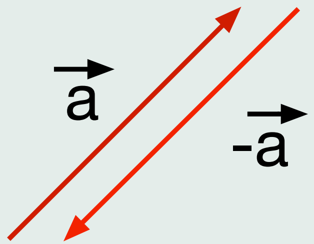


- **Associativity:** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$



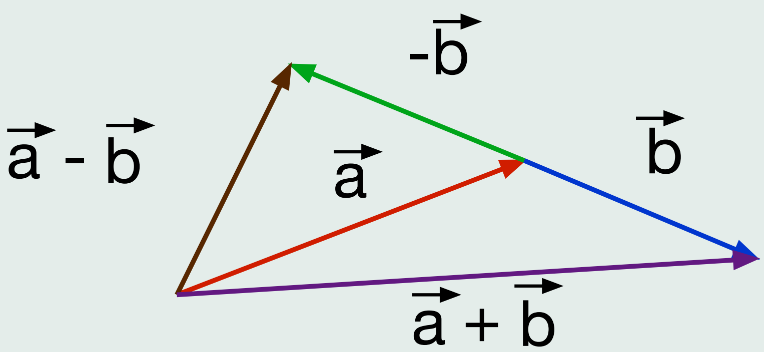


- **Inverse:** $\vec{a} + (-\vec{a}) = 0$



Note: $-\vec{a}$ has the same magnitude as $\vec{a}$, but it points in opposite direction

- **Vector subtraction:** $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.



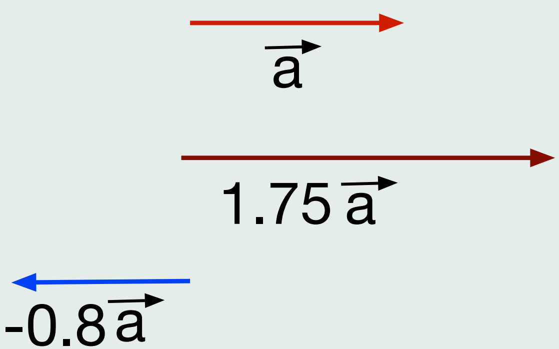
Multiplying vectors by scalars

- If $\vec{a}$ **vector**, $s \neq 0$ number then

$$s\vec{a} = \textbf{vector} \text{ with magnitude } |s\vec{a}| = s|\vec{a}|$$

$$\text{and direction} \begin{cases} \text{same as } \vec{a} & \text{if } s > 0 \\ \text{opposite to } \vec{a} & \text{if } s < 0 \end{cases}$$

Example



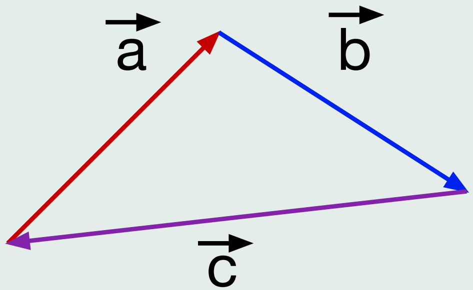
Note:

$$0\,\vec{a} = 0.$$

Home Page
Title Page
Page 11 of 37
Go Back
Full Screen
Close
Quit

i-Clicker

Which of the following statements is false for the three vectors below?



$A)\ \vec{a}+\vec{b}+\vec{c}=0$

$B)\ \vec{c}+\vec{b}=-\vec{a}$

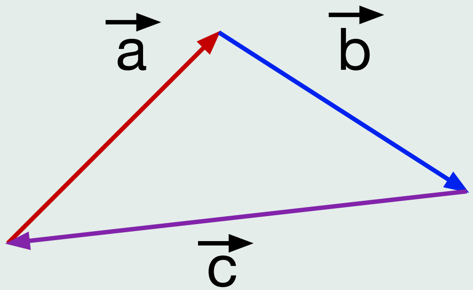
$C)\ |\vec{c}|<|\vec{a}|+|\vec{b}|$

$D)\ |\vec{c}|=|\vec{a}|+|\vec{b}|$

$E)$ None of the above.

Answer

Which of the following statements is false for the three vectors below?



$A)\ \vec{a} + \vec{b} + \vec{c} = 0$

$B)\ \vec{c} + \vec{b} = -\vec{a}$

$C)\ |\vec{c}| < |\vec{a}| + |\vec{b}|$

$D)\ |\vec{c}| = |\vec{a}| + |\vec{b}|$

$E)$ None of the above.

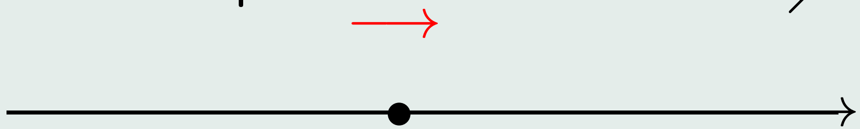
Triangle inequality: $|\vec{c}| < |\vec{a}| + |\vec{b}|$ since $\vec{a}, \vec{b}, \vec{c}$ not colinear.

Components of vectors

- **Axis** = line equipped with a **preferred direction**, also called **orientation**.

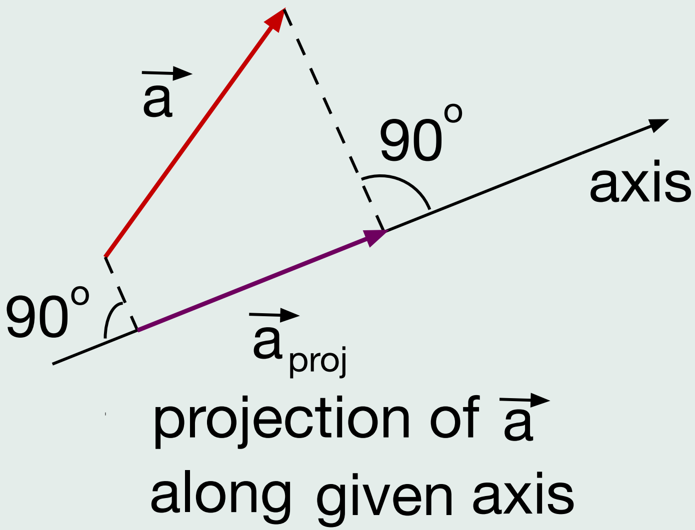
Example: one dimensional motion

positive direction $x \nearrow$



O = origin: $x = 0$

- **Projection:** suppose a $\vec{a}$ and a given axis are in **the same plane**



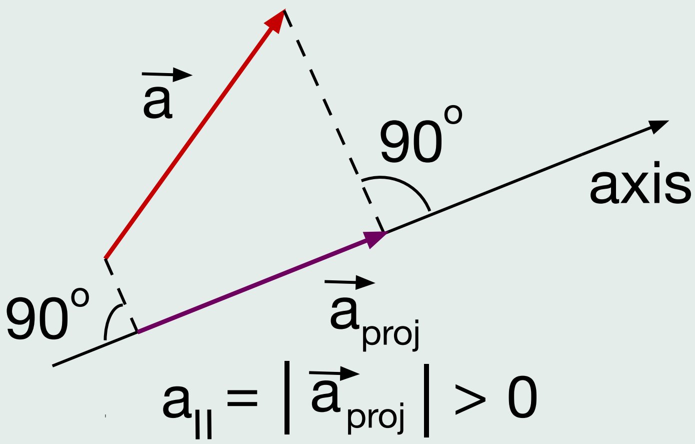
Note:

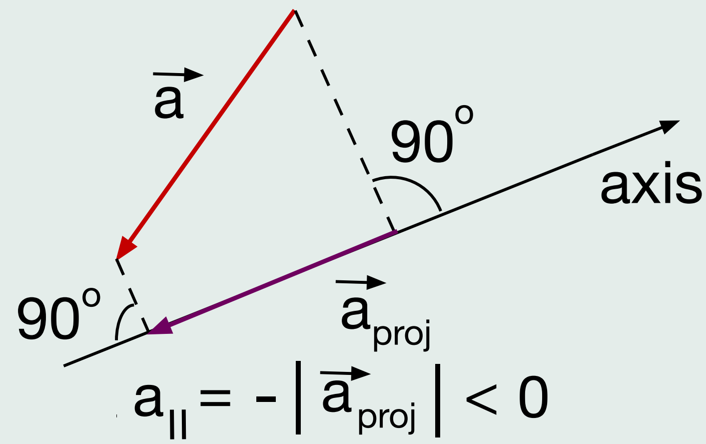
- $\vec{a}_{\text{proj}}$ is a **vector** along the given axis.
- $\vec{a}_{\text{proj}}$ is **not** the component of $\vec{a}$ along the given axis. (as stated in the textbook.)

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 15 of 37
Go Back
Full Screen
Close
Quit

- The **component** of $\vec{a}$ along given axis is a **number**

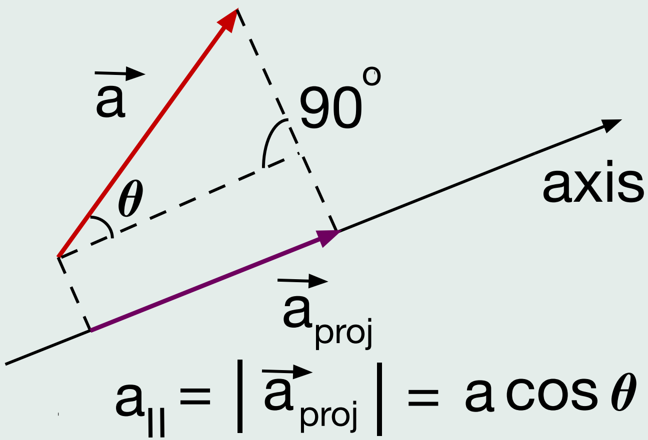
$$a_{\parallel} = \begin{cases} |\vec{a}_{\mathsf{proj}}| & \text{if } \vec{a}_{\mathsf{proj}} \text{ points in the positive direction} \\ -|\vec{a}_{\mathsf{proj}}| & \text{if } \vec{a}_{\mathsf{proj}} \text{ points in the negative direction} \end{cases}$$





- **Right triangle rule**

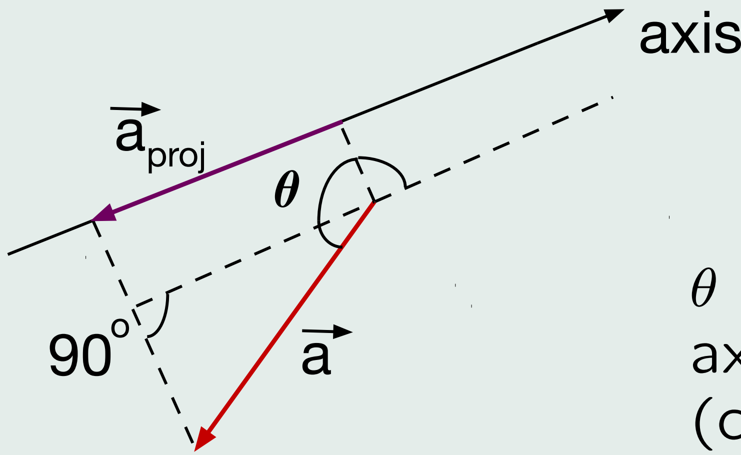
$$a_{\|} = a \cos\theta$$



θ = angle between the axis and the vector (counterclockwise)

$$\left|\vec{a}_{proj}\right| = a\cos(\theta-\pi) = -a\cos\theta$$

$$a_{||} = -\left|\vec{a}_{proj}\right| = a\cos\theta$$



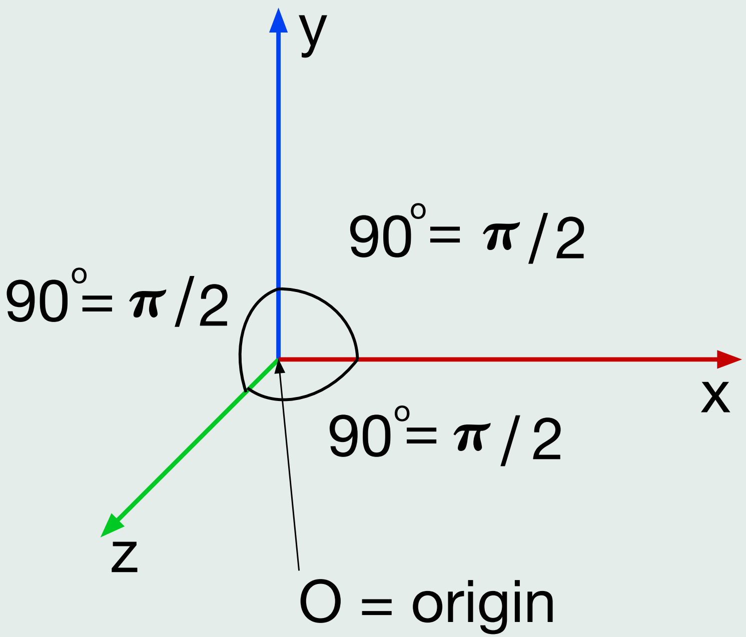
θ = angle between the axis and the vector (counterclockwise)

Summary:

- The **projection** of $\vec{a}$ is the **vector** $\vec{a}_{\text{proj}}$.
- The **component** of $\vec{a}$ is the **number**

$$a_{\parallel} = a \cos \theta$$

- **Right handed coordinate system**:
  three **mutually orthogonal** axes meeting at a point O called **origin**.

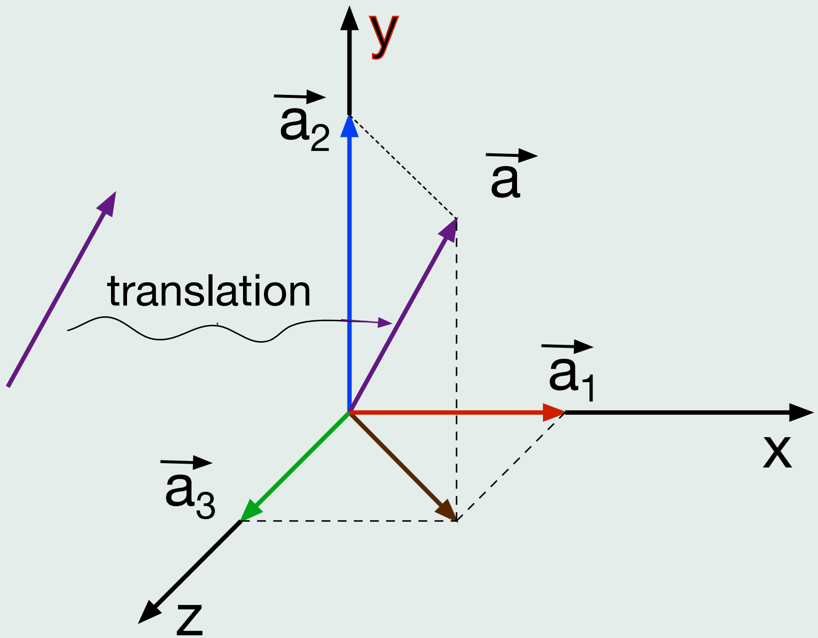


The x and y axes are in the page.

The z-axis sticks out of the page.

x, y, z: **coordinates**

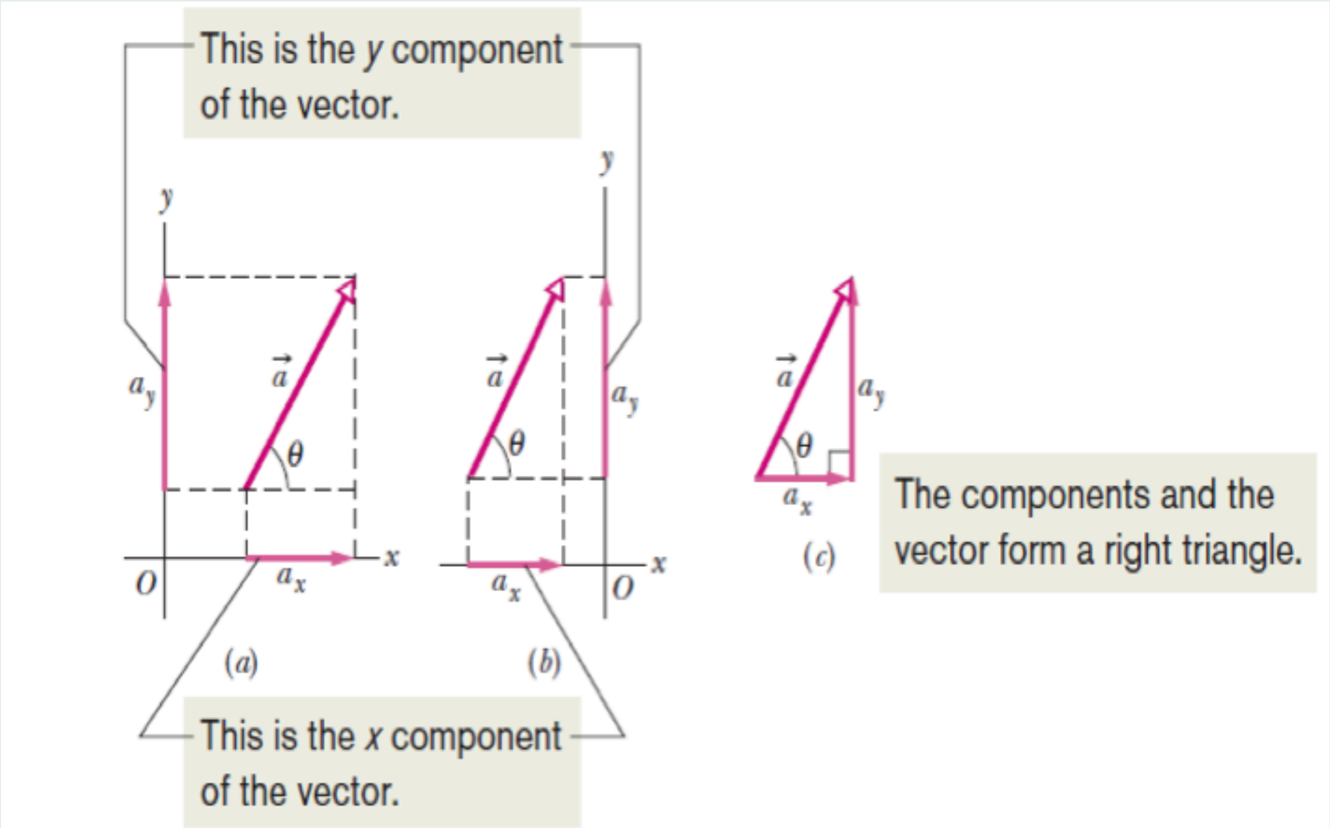
- The **components** of $\vec{a}$ along the three axes



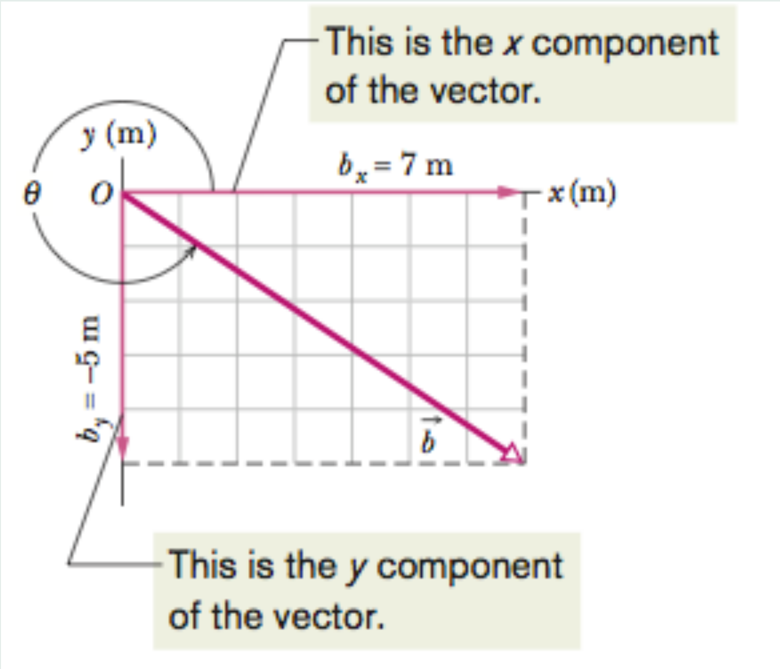
$\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$: the **projections** of $\vec{a}$ on the x, y, z axes. (**vectors**)

a_x, a_y, a_z: the **components** of $\vec{a}$ along the x, y, z axes (**numbers**)

- **Planar vectors** in x, y plane



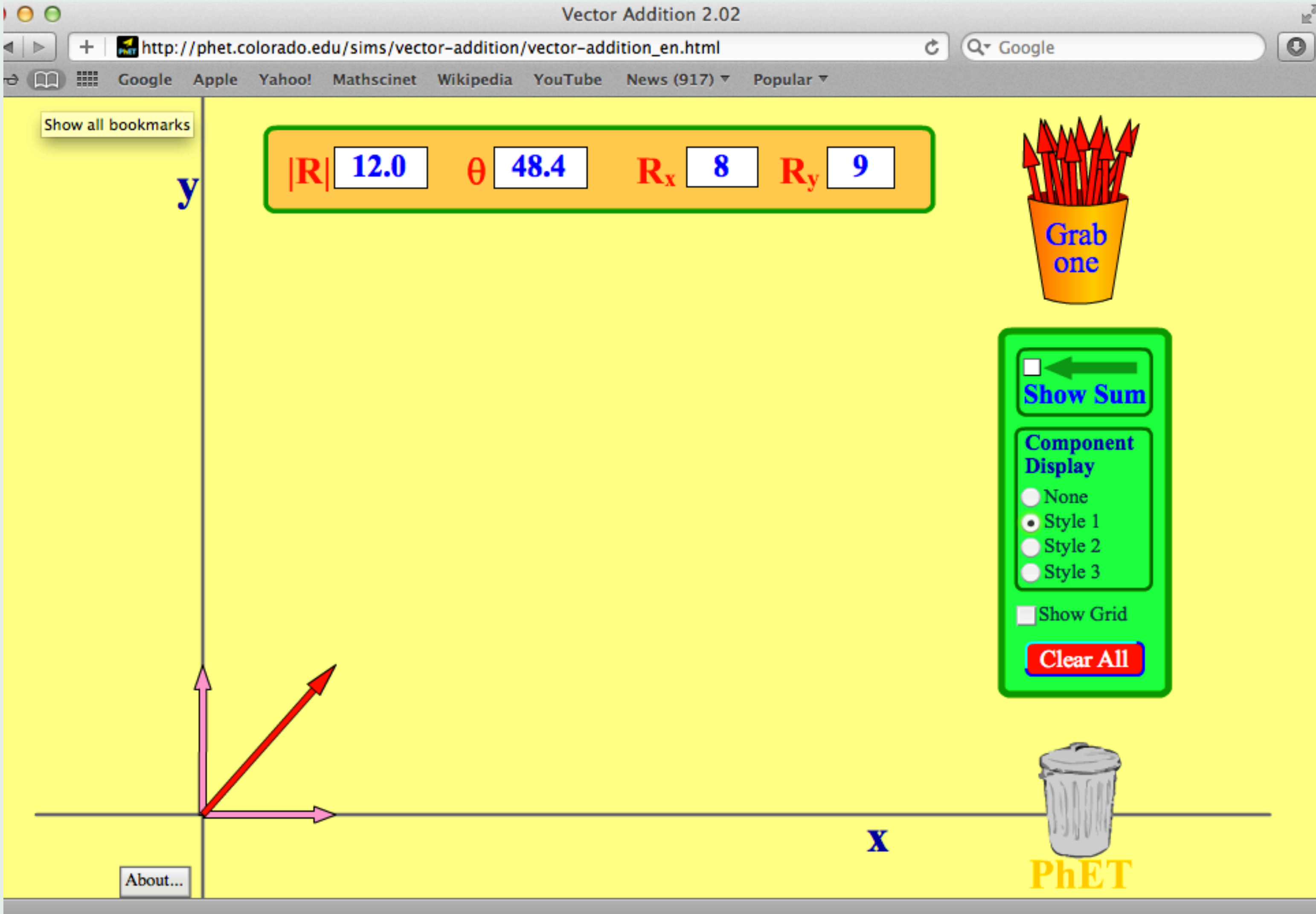
- The **right triangle rules** for planar vectors



$$a_x = a\cos\theta$$
$$a_y = a\sin\theta$$

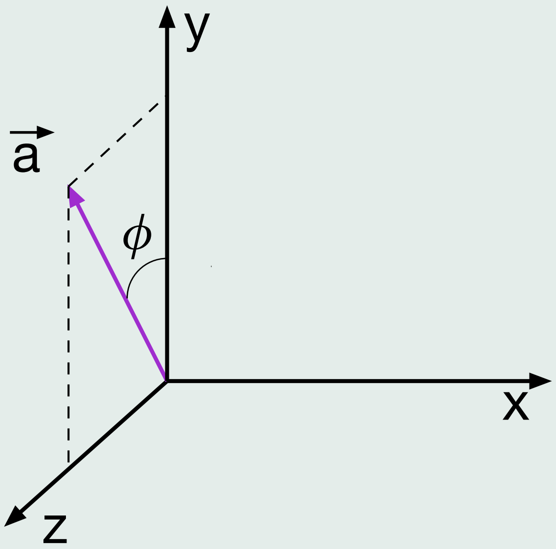
$$a = \sqrt{a_x^2 + a_y^2}$$

$$\tan\theta = \frac{a_y}{a_x}$$
$$(\text{if } \ a_x \neq 0).$$


Vector Addition 2.02
http://phet.colorado.edu/sims/vector-addition/vector-addition_en.html
Google
Google
Apple
Yahoo!
Mathscinet
Wikipedia
YouTube
News (917)
Popular
Show all bookmarks
|R| 12.0
θ 48.4
Rx 8
Ry 9
y
Grab one
Show Sum
Component Display
None
Style 1
Style 2
Style 3
Show Grid
Clear All
x
About...
PhET

i-Clicker

A vector $\vec{a}$ is contained in the (y, z) plane such that the angle between $\vec{a}$ and the y axis is ϕ. What are the components of $\vec{a}$?



$A)\ a_x = a\cos\phi,\ a_y = a\sin\phi,\ a_z = 0$

$B)\ a_x = a\cos\phi,\ a_y = 0,\ a_z = a\sin\phi$

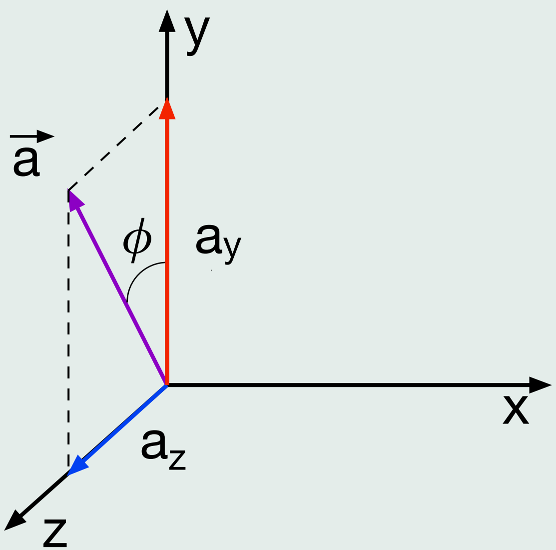
$C)\ a_x = 0,\ a_y = a\sin\phi,\ a_z = a\cos\phi$

$D)\ a_x = 0,\ a_y = a\cos\phi,\ a_z = a\sin\phi$

$E)\ a_x = a\sin\phi,\ a_y = 0,\ a_z = a\cos\phi$

Answer

A vector $\vec{a}$ is contained in the (y, z) plane such that the angle between $\vec{a}$ and the y axis is ϕ. What are the components of $\vec{a}$?



$A)\ a_x = a\cos\phi,\ a_y = a\sin\phi,\ a_z = 0$

$B)\ a_x = a\cos\phi,\ a_y = 0,\ a_z = a\sin\phi$

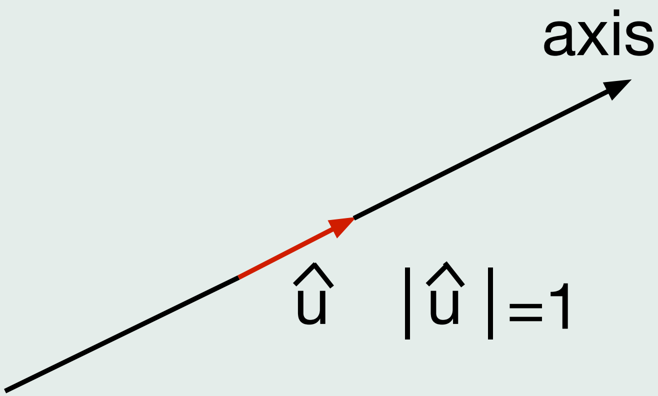
$C)\ a_x = 0,\ a_y = a\sin\phi,\ a_z = a\cos\phi$

$D)\ a_x = 0,\ a_y = a\cos\phi,\ a_z = a\sin\phi$

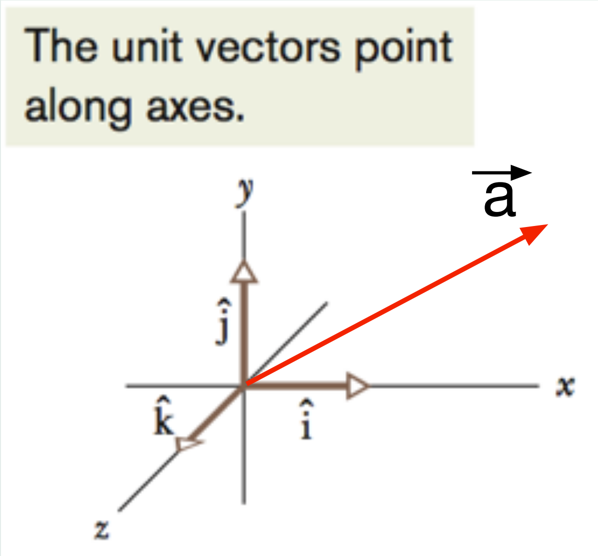
$E)\ a_x = a\sin\phi,\ a_y = 0,\ a_z = a\cos\phi$

Unit vectors

- **Unit vector** = vector of magnitude **1** pointing in the positive direction along an axis



- **Unit vectors** for a right handed coordinate system



If $\vec{a}$ has **components** a_x, a_y, a_z, its **projections** are

$$\begin{aligned}\vec{a}_1 &= a_x\widehat{i}\\ \vec{a}_2 &= a_y\widehat{j}\\ \vec{a}_3 &= a_z\widehat{k}\end{aligned}$$

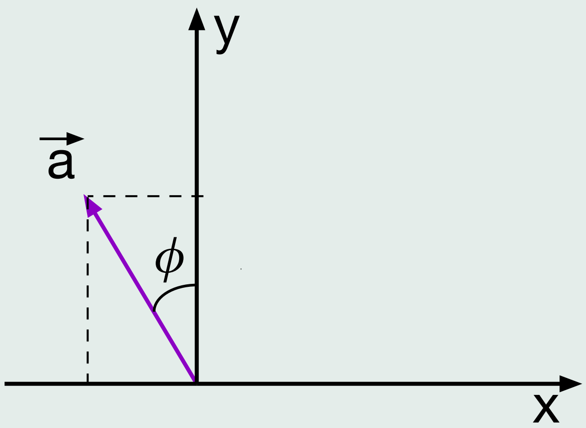
$$\vec{a} = a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k}$$

- **Two vectors are equal if and only if their components are equal.**

$$\boxed{\vec{a} = \vec{b} \quad \Leftrightarrow \; a_x = b_x, \;\; a_y = b_y, \;\; a_z = b_z.}$$

i-Clicker

Which of the following expressions is correct for the vector $\vec{a}$ shown below?



$A)\ \vec{a} = a\cos\phi\widehat{i} + a\sin\phi\widehat{j}$

$B)\ \vec{a} = a\sin\phi\widehat{i} + a\cos\phi\widehat{j}$

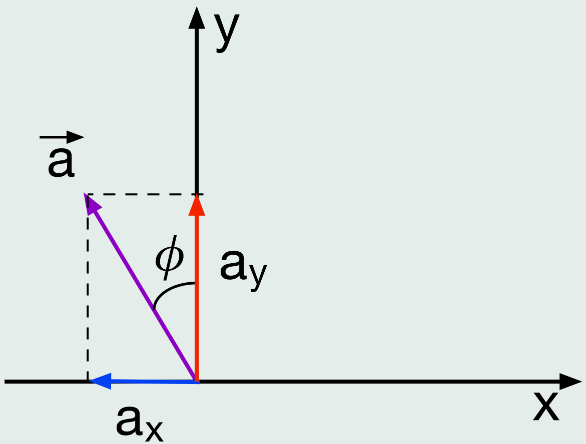
$C)\ \vec{a} = -a\sin\phi\widehat{i} + a\cos\phi\widehat{j}$

$D)\ \vec{a} = a\cos\phi\widehat{i} - a\sin\phi\widehat{j}$

$E)$ None of the above.

Answer

Which of the following expressions is correct for the vector $\vec{a}$ shown below?



$A)\ \vec{a} = a\cos\phi\widehat{i} + a\sin\phi\widehat{j}$

$B)\ \vec{a} = a\sin\phi\widehat{i} + a\cos\phi\widehat{j}$

$C)\ \vec{a} = -a\sin\phi\widehat{i} + a\cos\phi\widehat{j}$

$D)\ \vec{a} = a\cos\phi\widehat{i} - a\sin\phi\widehat{j}$

$E)$ None of the above.

Adding vectors by components

- For any two vectors:

$$\vec{a} = a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k} \qquad \vec{b} = b_x\widehat{i} + b_y\widehat{j} + b_z\widehat{k}$$

we have:

$$\vec{a} + \vec{b} = (a_x + b_x)\widehat{i} + (a_y + b_y)\widehat{j} + (a_z + b_z)\widehat{k}$$

$$\vec{a} - \vec{b} = (a_x - b_x)\widehat{i} + (a_y - b_y)\widehat{j} + (a_z - b_z)\widehat{k}$$

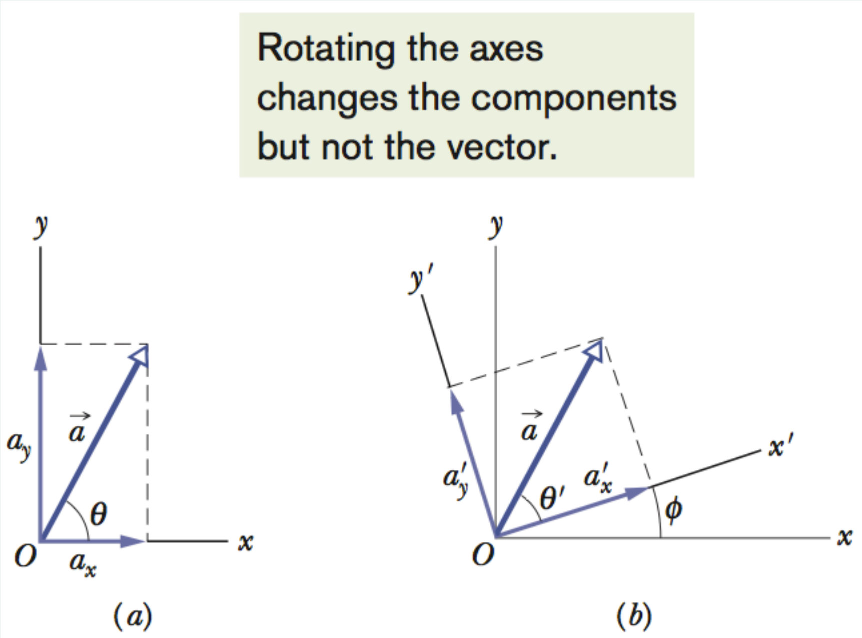
More generally, if s, t are scalars,

$$s\vec{a} + t\vec{b} = (sa_x + tb_x)\widehat{i} + (sa_y + tb_y)\widehat{j} + (sa_z + tb_z)\widehat{k}$$

Vectors and the laws of physics

- Relations among vectors do not depend on the choice of a coordinate system.

- Relations in physics are also independent of the choice of a co-ordinate system.



$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(a'_x)^2 + (a'_y)^2} \qquad \theta = \theta' + \phi$$

Multiplying vectors

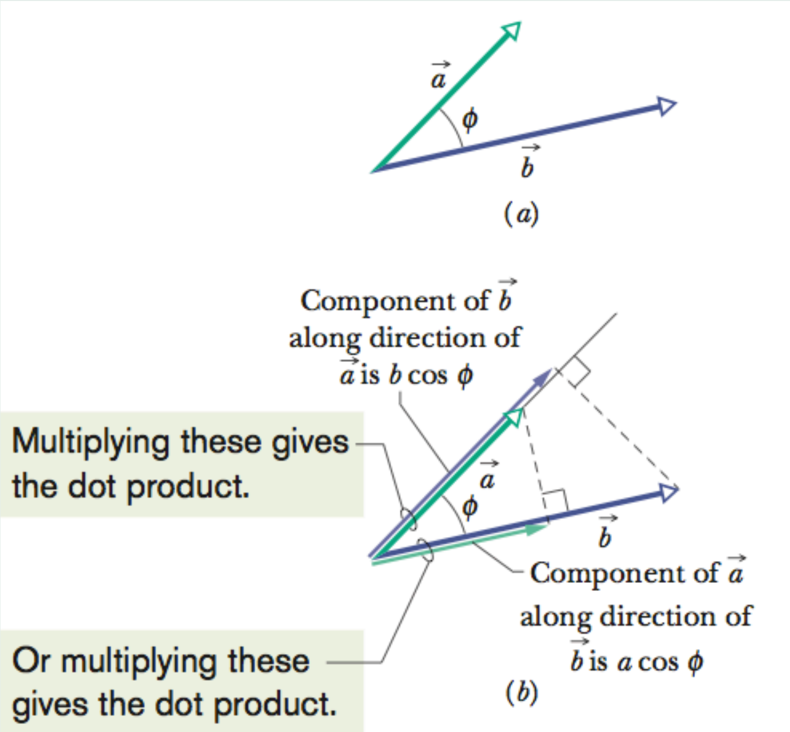
- **Scalar (dot) product**

Associates to any two vectors $\vec{a}$, $\vec{b}$ the **number**

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\phi$$
$$= \vec{b}\cdot\vec{a}$$

commutative

Order is irrelevant!



- **Scalar product in unit vector notation**

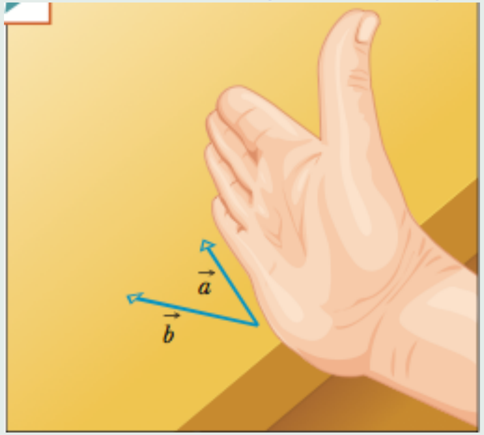
$$\widehat{i} \cdot \widehat{i} = \widehat{j} \cdot \widehat{j} = \widehat{k} \cdot \widehat{k} = 1$$

$$\widehat{i} \cdot \widehat{j} = \widehat{j} \cdot \widehat{k} = \widehat{k} \cdot \widehat{i} = 0$$

$$\vec{a} = a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k} \qquad \vec{b} = b_x\widehat{i} + b_y\widehat{j} + b_z\widehat{k}$$

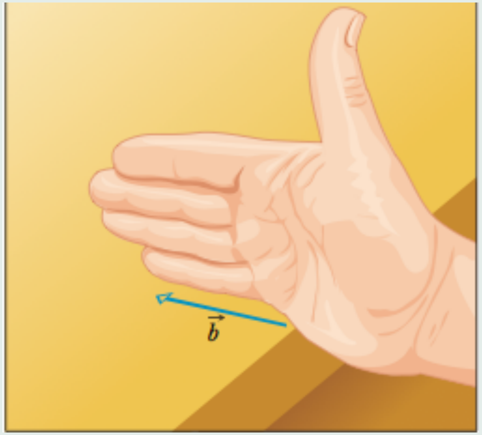
$$\begin{aligned}\vec{a} \cdot \vec{b} &= \left(a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k}\right) \cdot \left(b_x\widehat{i} + b_y\widehat{j} + b_z\widehat{k}\right) \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

- **Vector (cross) product**



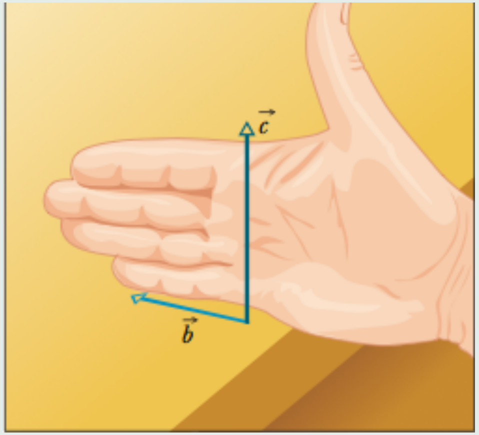
$\vec{c} \perp$ to the plane of the two vectors

direction of $\vec{c}$: right hand rule



$$\vec{a} \times \vec{b} = \vec{c}$$

$$|\vec{c}| = |\vec{a}|\,|\vec{b}| \sin\phi$$



$$\vec{b} \times \vec{a} = -\vec{c}$$

Anti-commutative

Order is relevant !

- **Vector product in unit vector notation**

$$\widehat{i} \times \widehat{j} = -\widehat{j} \times \widehat{i} = \widehat{k} \qquad \widehat{j} \times \widehat{k} = -\widehat{k} \times \widehat{j} = \widehat{i}$$

$$\widehat{k} \times \widehat{i} = -\widehat{i} \times \widehat{k} = \widehat{j}$$

$$\widehat{i} \times \widehat{i} = \widehat{j} \times \widehat{j} = \widehat{k} \times \widehat{k} = 0$$

$$\vec{a} = a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k} \qquad \vec{b} = b_x\widehat{i} + b_y\widehat{j} + b_z\widehat{k}$$

$$\begin{aligned}\vec{a} \times \vec{b} &= (a_x\widehat{i} + a_y\widehat{j} + a_z\widehat{k}) \times (b_x\widehat{i} + b_y\widehat{j} + b_z\widehat{k}) \\ &= (a_y b_z - b_y a_z)\widehat{i} + (a_z b_x - b_z a_x)\widehat{j} + (a_x b_y - b_x a_y)\widehat{k}\end{aligned}$$