Physics 417: Problem Set 6 (*DUE ON THURSDAY 10/24*)

Problem 1: Griffiths 2.44

Problem 2: Probability current density

Recall the formula for the probability current density we derived in class:

$$j(x,t) = \frac{\hbar}{2mi}\left(\psi(x,t)^*\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial\psi(x,t)^*}{\partial x}\right) \tag{1}$$

This gives the probability flux (probability per unit time) flowing past a point x at time t.

(a) Show that in general,

$$\int_{-\infty}^{\infty} dx\, j(x,t) = \langle v \rangle \tag{2}$$

(b) Consider the time evolved Gaussian wavepacket of problem 4.2. Starting from eq (3) of problem set 4, compute the probability current density for this wavepacket.

[*The answer is:*

$$j(x,t) = \frac{\hbar(x\hat{t} + 2\bar{k}\sigma^2)}{2m\sqrt{2\pi}(1+\hat{t}^2)^{3/2}\sigma^3} e^{\frac{-(x-2\bar{k}\hat{t}\sigma^2)^2}{2(1+\hat{t}^2)\sigma^2}} \tag{3}$$

Remember, Mathematica is your friend.]

(c) Plot $j(x,t)$ vs x for the Gaussian wavepacket for different values of t. Comment on your result. In particular, why is $j(x,t)$ positive in some places and negative in others?

(d) Compute $\int_{-\infty}^{\infty} dx\, j(x,t)$ explicitly for the Gaussian wavepacket and verify the general result (2).

Problem 3: Quantum mechanics in 2D

In this problem we will work out the separation of variables for the 2D Schrodinger equation with circular symmetry. This has many of the same concepts as in 3D, but is much simpler. To begin, we need the Laplacian in polar coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} \tag{4}$$

Now consider a circularly symmetric potential in 2D: $V(\vec{x}) = V(r)$.

(a) Using the ansatz $\psi(r,\theta) = R(r)\Theta(\theta)$ and the method of separation of variables, show that the 2D Schrodinger equation decomposes into:

$$\begin{aligned} &\Theta''(\theta) = -\ell^2\Theta(\theta) \\ &R''(r) + \frac{1}{r}R'(r) + \left(\frac{2m}{\hbar^2}(E - V(r)) - \frac{\ell^2}{r^2}\right)R(r) = 0 \end{aligned} \tag{5}$$

(b) Solve the angular equation to obtain the 2D analogues of spherical harmonics. What are the allowed values of ℓ? (Don't work too hard here, it's trivial, just like in 3D...)

(c) What is the transformation (analogous to $R(r) = u(r)/r$ in 3D) that turns the radial equation into a 1D Schrodinger problem? What is the effective potential?

(d) Consider a particle in an infinite circular well. Explain how the Schrodinger problem is very similar to the 3D infinite spherical well. What is the one qualitative difference in the energy levels between 2D and 3D?

Problem 4: Practice with spherical harmonics

(a) Griffiths 4.3

(b) Griffiths 4.5

EXTRA CREDIT WORTH 1/2 PROBLEM: Use the fact that spherical harmonics form a complete orthogonal basis for functions on the sphere, together with the integral representation of the spherical Bessel functions,

$$j_\ell(x) = \frac{1}{2}(-i)^\ell \int_{-1}^{1} dy\, e^{ixy} P_\ell(y) \tag{6}$$

and the orthonormality condition [4.34], to derive the *partial wave decomposition* for plane waves:

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta) \tag{7}$$

where $z = r\cos\theta$. This formula plays a key role in 3D scattering theory.