Physics 417: Problem Set 4 (*Due in class Wednesday 10/9*)

Problem 1: Time evolution and energy eigenstates

Consider a three-state system with orthonormal basis kets $|1\rangle$, $|2\rangle$, $|3\rangle$, and suppose the Hamiltonian acts as:

$$\begin{aligned} H|1\rangle &= E_0|1\rangle + A|3\rangle \\ H|2\rangle &= E_1|2\rangle \\ H|3\rangle &= E_0|3\rangle + A|1\rangle \end{aligned} \tag{1}$$

for some real numbers E_0, E_1 and A.

(a) Find the matrix representation of H.

(b) Find the eigenvalues and eigenvectors of H.

(c) If the state of the system at time $t = 0$ is $|\psi, t = 0\rangle = |2\rangle$, what is $|\psi, t\rangle$ at a later time t?

(d) If the state of the system at time $t = 0$ is $|\psi, t = 0\rangle = |3\rangle$, what is $|\psi, t\rangle$ at a later time t?

Problem 2: Time evolution of gaussian wavepackets

In the previous problem set, we considered a Gaussian wavepacket with width σ_x in position space, mean position $\bar{x}$ and mean momentum $\bar{k}$. Suppose a free particle of mass m is initially in the state described by the Gaussian wavepacket with $\bar{x} = 0$ and general $\bar{k}$, so

$$\psi(x, t = 0) = \mathcal{N} e^{-x^2/4\sigma_x^2} e^{i\bar{k}x} \tag{2}$$

Feel free to use computer algebra programs such as Maple or Mathematica, as long as you still show me your steps.

(a) By decomposing the initial state into Fourier modes, show that the time-evolution of the state is given by

$$\psi(x, t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \frac{1}{\sqrt{1 + i\hat{t}}} \times \exp\left(-\frac{(x - 2i\bar{k}\sigma^2)^2}{4\sigma^2(1 + i\hat{t})} - \bar{k}^2\sigma^2\right) \tag{3}$$

where $\hat{t} \equiv \frac{\hbar t}{2m\sigma^2}$.

(b) Compute $|\psi(x,t)|^2$ and sketch it as a function of x for $t=0$ and a couple of different values of $t>0$. Illustrate how the wave packet advances according to the classical laws of physics but spreads in time.

(c) How long does it take a macroscopic object (say with mass 1 g and width 0.1 cm) to spread an appreciable amount (say double its position-space width) if it is described by a Gaussian wavepacket?

EXTRA CREDIT WORTH 1/2 OF A PROBLEM: Compute $\langle X\rangle$, $\langle P\rangle$, $\langle(\Delta X)^2\rangle$, $\langle(\Delta P)^2\rangle$ and describe how they depend on time. What happens to the uncertainty product $\langle(\Delta X)^2\rangle\langle(\Delta P)^2\rangle$ as a function of time? For what times is it minimized?

Problem 3: Quantum virial theorem

Show that for a particle moving in one dimension with general potential $V(x)$ and kinetic energy $T = P^2/2m$, that the following always holds in a stationary state:

$$\frac{1}{2}\langle XV'(X)\rangle = \langle T\rangle \tag{4}$$

(*Hint: Use the fact that* $\frac{d}{dt}\langle Q\rangle = 0$ *in a stationary state for any operator* Q*, and apply it to* $Q = XP$*.*)

Problem 4: Minimum energy

Show that for any potential $V(x)$, if $E < V_{min}$ then the corresponding solution to the time-independent Schrodinger equation is not normalizable. (*Hint: the Schrodinger equation is* $\psi''(x) = \frac{2m}{\hbar^2}(V(x)-E)\psi$*. What does this say about the sign of* $\psi^*\psi''$*? Use this in your solution.*)

Problem 5: Particle in a box

(a) Find $\psi(x,t)$ (expressing your answers as infinite sums is sufficient!) and the probability of measuring E_n at $t>0$ for a particle in a box $0 \le x \le L$ for each of the following initial states (all of these states are zero for x outside the box):

(i) $\psi(x,0) = A_1 \sin\frac{3\pi x}{L}\cos\frac{\pi x}{L}$

(ii) $\psi(x,0) = A_2 \sin^2\frac{\pi x}{L}$

(iii) $\psi(x,0) = A_3 x(x-L)$

(b) Suppose the particle is in the ground state and the box is suddenly enlarged to $0 \le x \le 2L$ without disturbing the wavefunction. If the energy is measured shortly afterwards, what value of the energy is most likely to be found? How does this compare with the original ground state energy E_1?