Physics 602 **Homework Set 6** **Spring, 2008**

Due date: Wednesday April 23

Reading: T&H Ch. 6.1-5, 7.1-6

1) (10 points) In class we used a "rigid-ion" picture to derive an electron-phonon coupling

$$D_q = -i\sqrt{\frac{N}{2M\omega_q}}\, q\, \widetilde{V}_a(q)\ .$$

If $\widetilde{V}_a(q) \to$ constant as $q \to 0$, then $D(q) \propto q$. However, this might not always be the case. Let's consider a model in which $\widetilde{V}_a(q) = \widetilde{V}_a^{\mathrm{bare}}(q)/\epsilon(q)$ with $V_a^{\mathrm{bare}}(r) = Ze^2/r$ (Ze is the charge of the ion) and with (i) $\epsilon(q) = 1 + k_s^2/q^2$, or (ii) $\epsilon(q) = \epsilon_\infty$ (constant). For example, these would be appropriate for Mg ions in metallic magnesium or in MgO, respectively. Show that, for optical phonons, $D(q) \propto q$ in the former case and q^{-1} in the latter case (the so-called "Frölich electron-phonon interaction").

2) (10 points) Cooper pair problem for the case of triplet spin pairing: Consider the operator $\mathcal{A} = \sum_{\mathbf{k}} a(\mathbf{k}) c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\uparrow}$. Show that the even part of $a(\mathbf{k})$ gives no contribution (consider the anticommutator of the c's), so that you can assume $a(\mathbf{k}) = -a(-\mathbf{k})$. From this, show that interaction energy of the Cooper pair, given by applying $\mathcal{A}$ on the ordinary ground state, is zero, so no binding is obtained.

3) (20 points)
   a) Introducing the pair creation operator
   $$b_k^\dagger \;=\; c^\dagger_{k\uparrow}\, c^\dagger_{-k\downarrow}\ \ ,$$
   calculate $[\, b_{\mathbf{k}}\, ,\, b_{\mathbf{k}'}\,]$, and $[\, b_{\mathbf{k}}\, ,\, b^\dagger_{\mathbf{k}'}\,]$, and show that these two operators do not quite obey Bose commutation relations.
   b) Do Taylor and Heinonen problem 2.4, showing that the γ and $\gamma^\dagger$ operators defined there do obey good Fermion anticommutation relations.

4) (30 points) Here are several miscellaneous details connected with my BCS lecture notes:
   a) Show that $\langle\Psi_0|(\hat{N} - \bar{N})^2|\Psi_0\rangle = \sum_k u_k^2 v_k^2$, where $\bar{N} = \langle\Psi_0|\hat{N}|\Psi_0\rangle$. Here $|\Psi_0\rangle$ is the BCS ground state.
   b) If Ψ_0 is constructed from the state with all u_k and all v_k real and positive, while ${\Psi_0}'$ is constructed from the state with all u_k real and all v_k having phase ϕ (that is, $v_k = e^{i\phi} \times$ (real positive)), show that $|\langle{\Psi_0}'|\Psi_0\rangle| \neq 1$. Indeed, show that it approaches zero in the thermodynamic limit (dense k) for any $\phi \neq 0$.
   c) In class we defined the states $|\Psi_0\rangle$, $|\Psi_k\rangle$, $|\Psi_{-k}\rangle$, and $|\Psi_{k,-k}\rangle$. Show that the energy of the second, third, and fourth of these are E_k, E_k, and $2E_k$, respectively, relative to the first one, where $E_k = \sqrt{\Delta^2 + \tilde{\epsilon_k}^2}$.